Covectors

**Definition.** Let \( V \) be a finite-dimensional vector space. A **covector** on \( V \) is a real-valued linear functional on \( V \), that is, a linear map \( \omega : V \to \mathbb{R} \).

- The space of all covectors on \( V \) is itself a real vector space under the obvious operations of pairwise addition and scalar multiplication. It is denoted by \( V^* \) and called the **dual space** to \( V \).

**Proposition 6.1.** Let \( V \) be a finite-dimensional vector space. If \((E_1, \cdots, E_n)\) is any basis for \( V \), then the covectors \((\varepsilon^1, \cdots, \varepsilon^n)\) defined by

\[
\varepsilon^i(E_j) = \delta^i_j = \begin{cases} 
1, & \text{if } i = j, \\
0, & \text{if } i \neq j,
\end{cases}
\]

form a basis for \( V^* \), called the **dual basis** to \((E_i)\). Therefore, \( \dim V^* = \dim V \).

**Example.** If \((e_i)\) denote the standard basis for \( \mathbb{R}^n \), we denote the dual basis by \((e^1, \cdots, e^n)\) (note the upper indices), and call it the **standard dual basis**.

- The basis vectors are the linear functions from \( \mathbb{R}^n \) to \( \mathbb{R} \) given by

\[
e^j(v) = e^j(v^1, \cdots, v^n) = v^j.
\]

- In other words, \( e^j \) is just the linear functional that picks out the \( j \)th component of a vector.

- In matrix notation, a linear map from \( \mathbb{R}^n \) to \( \mathbb{R} \) is represented by a \( 1 \times n \) matrix; i.e. a row matrix.

- The basis covectors can therefore also be thought of as the linear functionals represented by the row matrix

\[
e^1 = (1 \ 0 \cdots 0), \cdots, e^n = (0 \cdots 0 \ 1).
\]

- If \((E_i)\) is a basis for \( V \) and \((\varepsilon^j)\) is its dual basis, then for any matrix \( X = X^iE_i \in V \), we have

\[
\varepsilon^i(X) = X^i\varepsilon^j(E_i) = X^i\delta^j_i = X^j.
\]

Thus as in the case of \( \mathbb{R}^n \), \( e^j \) picks out the \( j \)th component of a vector w.r.t. the basis \((E_i)\).

- More generally, Proposition 6.1 shows that we can express an arbitrary covector \( \omega \in V^* \) in terms of the dual basis as

\[
(6.1) \quad \omega = \omega_j\varepsilon^j,
\]

where the components \( \omega_j \) are determined by

\[
(6.2) \quad \omega_j = \omega(E_j).
\]

- We will write basis covectors with upper indices, and components of a covector with lower indices, because this helps to ensure that mathematically meaningful expressions such as (6.1) will always follow our index conventions: Any index that is to be summed over in a given term appears exactly twice, once as a subscript and once as a superscript.
Proposition 6.2. The dual map satisfies the following properties.

(a) \((A \circ B)^* = B^* \circ A^*\).
(b) \((\text{Id}_V)^*; V^* \to V^*\) is the identity map of \(V^*\).

- Apart from the fact that \(\dim V^* = \dim V\), the second important fact is the following characterization of the second dual space \(V^{**} = (V^*)^*\).
- For each vector space \(V\) there is a natural basis-independent map \(\xi : V \to V^{**}\), defined as follows.
  For each vector \(X \in V\), define a linear functional \(\xi(X) : V^* \to \mathbb{R}\) by
  \[\xi(X)(\omega) = \omega(X), \quad \forall \omega \in V^*.\]

Proposition 6.4. For a finite-dimensional vector space \(V\), the map \(\xi : N \to V^{**}\) is an isomorphism.

Proof. Since \(V\) and \(V^*\) have the same dimension, it suffices to claim: \(\xi\) is injective.

Indeed, suppose \(X \in V\) is not zero. Extend \(X\) to a basis \((E_1, E_2, \cdots, E_n)\) for \(V\), \(X = E_1\) and let \((\varepsilon^1, \varepsilon^2, \cdots, \varepsilon^n)\) denote the dual basis for \(V^*\). Then
\[\xi(X)(\varepsilon^1) = \varepsilon^1(X) = \varepsilon^1(E_1) = 1 \neq 0,\]
so \(\xi(X) \neq 0\). \(\square\)

- The preceding proposition shows that when \(V\) is finite-dimensional, we can unambiguously identify \(V^{**}\) with \(V\) itself, because the map \(\xi\) is canonically defined, without reference to any basis.
- It is important to observe that although \(V^*\) is also isomorphic to \(V\), there is no canonical isomorphism \(V \cong V^*\).
- Because of Proposition 6.4, the real number \(\omega(X)\) obtained by applying a covector \(\omega\) to a vector \(X\) is sometimes denoted by either of the more symmetric-looking notations
  \[\langle \omega, X \rangle, \quad (X, \omega);\]
both expressions can be thought of either as the action of the covector \(\omega \in V^*\) on the vector \(X \in V\), or as the action of the covector \(\xi(X) \in V^{**}\) on the element \(\omega \in V^*\).
- Whenever one of the arguments is a vector and the other a covector, the notation \(\langle \omega, X \rangle\) is always to be interpreted as the actual pairing between vectors and covectors, not as an inner product.
Tangent Covectors on Manifolds

Definition. Let $M$ be a smooth manifold. For each $p \in M$, we define the cotangent space at $p$, denoted by $T_p^* M$, to be the dual space to $T_p M$:

$$T_p^* M = (T_p M)^*.$$ 

Elements of $T_p^* M$ are called tangent covectors at $p$.

- If $(x^i)$ are smooth local coordinates on an open subset $U \subset M$, then $\forall p \in U$, the coordinate basis $(\frac{\partial}{\partial x^i})_p$ give rise to a dual basis for $T_p^* M$, which we denote for the moment by $(\lambda^i)_p$.
- Any covector $\omega \in T_p^* M$ can thus be written uniquely as $\omega = \omega_i \lambda^i|_p$, where

$$\omega_i = \omega \left( \frac{\partial}{\partial x^i} \right)_p.$$

- Let $(\tilde{x}^j)$ be another set of smooth local coordinates whose domain contains $p$, and let $(\tilde{\lambda}^j)_p$ denote the basis for $T_p^* M$ dual to $(\frac{\partial}{\partial \tilde{x}^j})_p$.
- We can compute the components of the same covector $\omega$ w.r.t. the new coordinate system as follows.
  - First recall that the coordinate vector fields transform as follows:

$$\left( \frac{\partial}{\partial x^i} \right)_p = \left( \frac{\partial \tilde{x}^j}{\partial x^i}(p) \right) \left( \frac{\partial}{\partial \tilde{x}^j} \right)_p \quad (6.4)$$

  - Writing $\omega$ in both systems as

$$\omega = \omega_i \lambda^i|_p = \tilde{\omega}_j \tilde{\lambda}^j|_p,$$

  we can use (6.4) to compute the components $\omega_i$ in terms of $\tilde{\omega}_j$:

$$\omega_i = \omega \left( \frac{\partial}{\partial x^i} \right)_p = \omega \left( \frac{\partial \tilde{x}^j}{\partial x^i}(p) \frac{\partial}{\partial \tilde{x}^j} \right)_p = \frac{\partial \tilde{x}^j}{\partial x^i}(p) \tilde{\omega}_j \quad (6.5)$$

- In the early days of smooth manifold theory, before most of the abstract coordinate-free definition we are using were developed, mathematicians tended to think of a tangent vector at a point $p$ as an assignment of an $n$-tuples $(X^1, \ldots, X^n)$ and $(\tilde{X}^1, \ldots, \tilde{X}^n)$ assigned to two different systems $(x^i)$ and $(\tilde{x}^j)$ were related by the transformation law:

$$\tilde{X}^j = \frac{\partial \tilde{x}^j}{\partial x^i}(p) X^i \quad (6.6)$$
• Similarly, a tangent covector was thought of as \( n \)-tuple \((\omega_1, \cdots, \omega_n)\) that transform, by virtue of \((6.5)\), according to the following slightly different rule:

\[
(6.7) \quad \omega_i = \frac{\partial \tilde{x}^j}{\partial x^i}(p)\tilde{\omega}_j.
\]

• Thus it becomes customary to call the tangent covectors **covariant vectors** because their components transform in the same way as ("vary with") the coordinate partial derivatives, with the Jacobian matrix \( \left( \frac{\partial \tilde{x}^j}{\partial x^i} \right) \) multiplying the objects associated with the "new" coordinates \((\tilde{x}^j)\) to obtain those associated with the "old" coordinates \((x^i)\).

— Analogously, tangent vectors were called **contravariant vectors**, because their components transform in the opposite way.
The Cotangent Bundle

Definition. The disjoint union
\[ T^* M = \coprod_{p \in M} T_p^* M \]
is called the cotangent bundle of \( M \). It has a natural projection map
\[ \pi : T^* M \to M \]
sending \( \omega \in T^* M \) to \( p \in M \).

- As above, given any smooth local coordinates \((x^i)\) on \( U \subset M \), for each \( p \in U \) we denote the basis for \( T_p^* M \) dual to \( \frac{\partial}{\partial x^i} \bigg|_p \) by \( (\lambda^i)^\prime_p \).

This defines \( n \) maps \( \lambda^1, \cdots, \lambda^n : U \to T^* M \), called coordinate covector fields.

Proposition 6.5. Let \( M \) be a smooth manifold and let \( T^* M \) be its cotangent bundle. With the standard projection map and the natural vector space structure on each fiber, \( T^* M \) has a unique smooth manifold structure making it into a rank-\( n \) vector bundle over \( M \) for which all coordinate covector fields are smooth local sections.

Proof. Given a smooth chart \((U, \varphi)\) on \( M \), with coordinate functions \((x^i)\), define \( \Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n \) by
\[ \Phi(\xi^i \lambda^i)^\prime_p = (p, (\xi_1, \cdots, \xi_n)) \]
where \( \lambda^i \) is the \( i \)th coordinate covector field associated with \((x^i)\).

- Suppose \((\tilde{U}, \tilde{\varphi})\) is another smooth chart on \( M \), with coordinate functions \((\tilde{x}^i)\), and let \( \tilde{\Phi} : \pi^{-1}(\tilde{U}) \to \tilde{U} \times \mathbb{R}^n \) be defined analogously.

- On \( \pi^{-1}(U \cap V) \), it follows from (6.5) that
\[ \Phi \circ \tilde{\Phi}^{-1}(p, (\tilde{\xi}_1, \cdots, \tilde{\xi}_n)) = \left( p, \left( \frac{\partial \tilde{x}^j}{\partial x^i}(p)\tilde{\xi}_j, \cdots, \frac{\partial \tilde{x}^j}{\partial x^n}(p)\tilde{\xi}_j \right) \right) \]
The GL\((n, \mathbb{R})\)-valued function \( (\frac{\partial \tilde{x}^j}{\partial x^i})(p) \) is smooth, so it follows that \( T^* M \) has a smooth structure making it into a smooth vector bundle for which the map \( \Phi \) are smooth local trivilizations. □

- As in the case of the tangent bundle, smooth local coordinates for \( M \) yield smooth local coordinates for its cotangent bundle.

- If \((x^i)\) are smooth coordinates on an open set \( U \subset M \), then the map \( \pi^{-1}(U) \) to \( \mathbb{R}^{2n} \) given by
\[ \xi_i \lambda^i \bigg|_i \mapsto (x^1(p), \cdots, x^n(p), \xi_1, \cdots, \xi_n) \]
is a smooth coordinate chart for \( T^* M \).

- We will call \((x^i, \xi_i)\) the standard coordinates for \( T^* M \) associated with \((x^i)\).
Definition. A section of $T^*M$ is called a **covector field** or a (differential) 1-form.

- In any smooth local coordinates on an open set $U \subset M$, a covector field $\omega$ can be written in terms of the coordinate covector fields $(\lambda^i)$ as $\omega = \omega_i \lambda^i$ for $n$ functions $\omega_i : U \to \mathbb{R}$ called the **component functions** of $\omega$.
  They are characterized by
  \[ \omega_i(p) = \omega^p \left( \frac{\partial}{\partial x^i} \right)_p. \]

Lemma 6.6 (Smoothness Criteria for Covector Fields). Let $M$ be a smooth manifold, and let $\omega : M \to T^*M$ be a rough section.

1. If $\omega = \omega_i \lambda^i$ is the coordinate representation for $\omega$ in any smooth chart $(U, x^i)$ for $M$, then $\omega$ is smooth iff its component functions are smooth.
2. $\omega$ is smooth iff for every vector field $X$ on an open subset $U \subset M$, the function $\langle \omega, X \rangle : U \to \mathbb{R}$ defined by
   \[ \langle \omega, X \rangle(p) = \langle \omega^p, X_p \rangle = \omega^p(X_p) \]
   is smooth.

Definition. We denote the real vector space of all smooth covector fields on $M$ by $T^*(M)$.

- As smooth sections of a vector bundle, elements of $T^*(M)$ can be multiplied by smooth real-valued function:
  - If $f \in C^\infty(M)$ and $\omega \in T^*(M)$, the covector field $f\omega$ is defined by
    \[ (f\omega)_p = f(p)\omega^p. \]
  - Like the space of smooth vector fields, $T^*(M)$ is a module over $C^\infty(M)$.

- Geometrically, we think of a vector field on $M$ as a rule that attaches an arrow to each point of $M$.

- What kind of geometric picture can we form of a covector field?
  - The key idea is that a nonzero linear functional $\omega_p \in T^*_pM$ is completely determined by two pieces of data:
    1. its kernel, which is a codimension-1 linear subspace of $T_pM$ (a hyperplane), and
    2. the set of vectors $X$ for which $\omega^p(X) = 1$, which is an affine hyperplane parallel to the kernel.

- Thus you can visualize a **covector field** as defining a pair of affine hyperplanes in each tangent space, one through the origin and another parallel to it, and varying continuously from point to point.
  - At points where the covector field takes on the value zero, one of the hyperplanes goes off to infinity.
The Differential of a Function

- In elementary calculus, the gradient of a smooth real-valued function $f$ on $\mathbb{R}^n$ is defined as the vector field whose components are the partial derivatives of $f$. In our notation, this would read

$$\text{grad } f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}.$$ 

Unfortunately, in this form, the gradient does not make coordinate independent sense.

Example. Let $f(x, y) = x^2$ on $\mathbb{R}^2$, and let $X$ be the vector field

$$X = \text{grad } f = 2x \frac{\partial}{\partial x}.$$ 

Compute the coordinate expression of $X$ in polar coordinates of $X$ in polar coordinates (on some open set on which they are defined) and show that grad $f$ is not equal to

$$\frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta}.$$ 

- Although the first partial derivatives of a smooth function cannot be interpreted in a coordinate-independent way as the components of a vector field, it turns out that they can be interpreted as the components of a covector field.
- This is the most important application of covector fields.

Definition. Let $f$ be a smooth real-valued function on a smooth manifold $M$. We define a covector field $df$, called the differential of $f$, by

$$df_p(X_p) = X_p f, \ \forall X_p \in T_p M.$$ 

Lemma 6.7. The differential of a smooth function is a smooth covector field.

Proof. (1) It is straightforward to verify that at each point $p \in M$, $df_p(X_p)$ depends linearly on $X_p$, so that $df_p$ is indeed a covector at $p$.

(2) To see that $df$ is smooth, we use Lemma 6.6 (b). For any smooth vector field $X$ on an open subset $U \subset M$, the function $\langle df, X \rangle$ is smooth because it is equal to $X f$. □
To see what $df$ looks like more concretely, we need to compute its coordinate representations.

Let $(x^i)$ be smooth coordinates on an open subset $U \subset M$, and let $(\lambda^i)$ be the corresponding coordinate coframe on $U$.

Writing $df$ in coordinates as $df_p = A_i(p)\lambda^i\bigg|_p$ for some functions $A_i : U \to \mathbb{R}$, the definition of $df$ implies

$$A_i(p) = df_p \left( \frac{\partial}{\partial x^i} \bigg|_p \right) = \frac{\partial}{\partial x^i} \bigg|_p f = \frac{\partial f}{\partial x^i}(p).$$

This yields the following formula for the coordinate representation of $df$:

$$(6.9) \quad df_p = \frac{\partial f}{\partial x^i}(p)\lambda^i\bigg|_p.$$

Thus the component functions of $df$ in any smooth coordinate chart are the partial derivatives of $f$ w.r.t. those coordinates.

Because of this, we can think of $df$ as an analogue of the classical gradient, reinterpreted in a way that makes coordinate-independent sense on a manifold.

If we apply (6.9) to the special case in which $f$ is one of the coordinate functions $x^j : U \to \mathbb{R}$, we obtain

$$dx^j\bigg|_p = \frac{\partial x^j}{\partial x^i}\lambda^i\bigg|_p = \delta^i_j \lambda^i\bigg|_p = \lambda^j\bigg|_p.$$

In other words, the coordinate covector field $\lambda^j$ is $dx^j$. Therefore, (6.9) can be rewritten as

$$df_p = \frac{\partial f}{\partial x^i}(p)dx^i\bigg|_p,$$

or as an equation between covector fields instead of covectors:

$$(6.10) \quad df = \frac{\partial f}{\partial x^i}dx^i.$$

In particular, in the 1-dimensional case, this reduces to

$$df = \frac{df}{dx}dx.$$

Example 6.8. If $f(x, y) = x^2y \cos x$ on $\mathbb{R}^2$, then

$$df = \frac{\partial(x^2y \cos x)}{\partial x}dx + \frac{\partial(x^2y \cos x)}{\partial y}dy$$

$$= (2xy \cos x - x^2 y \sin x)dx + (x^2 \cos x)dy.$$
It is important to observe that for a smooth real-valued function $f : M \to \mathbb{R}$, we have now defined two different kinds of derivatives of $f$ at a point $p \in M$.

1. The pushforward $f_*$ is defined as a linear map from $T_pM$ to $T_{f(p)}\mathbb{R}$.
2. The differential $df_p$ as a covector at $p$; i.e., a linear map from $T_pM$ to $\mathbb{R}$.

These are really the same at $p$, once we take into account the canonical identification between $\mathbb{R}$ and its tangent space at any point; one easy way to see this is to note that both are represented in coordinates by the row matrix whose components are the partial derivatives of $f$.

**Proposition 6.11 (Derivative of a Function Along a Curve).** Suppose $M$ is a smooth manifold, $\gamma : M \to \mathbb{R}$ is a smooth curve, and $f : M \to \mathbb{R}$ is a smooth function. Then the derivative of the real-valued function $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is given by

$$(f \circ \gamma)'(t) = df_{\gamma(t)}(\gamma'(t)).$$

**Proof.** Directly from the definition, for any $t_0 \in J$,

$$df_{\gamma(t_0)}(\gamma'(t_0)) = \gamma'(t_0)f = (\gamma_*, \frac{d}{dt} \bigg|_{t_0})f = \frac{d}{dt} \bigg|_{t_0} (f \circ \gamma) = (f \circ \gamma)'(t_0).$$

If $\gamma$ is smooth curve in $M$, we have two different meanings for the expression $(f \circ \gamma)'(t)$.

1. $f \circ \gamma$ can be interpreted as a smooth curve in $\mathbb{R}$. Thus $(f \circ \gamma)'(t)$ is its tangent vector at the point $f \circ \gamma(t)$, an element of the tangent space $T_{f \circ \gamma(t)}\mathbb{R}$. This tangent vector is equal to $f_*(\gamma'(t))$.
2. $f \circ \gamma$ can also be considered simply as a real-valued function of one real variable, and then $(f \circ \gamma)'(t)$ is just its ordinary derivative. Proposition 6.11 shows that this derivative is equal to the real number $df_{\gamma(t)}(\gamma'(t))$. 

Pullbacks

- Let $F : M \to N$ be a smooth map and $p \in M$ be arbitrary. The pushforward map
  
  $F_* : T_pM \to T_{F(p)}N$

  yields a dual linear map
  
  $F^* : T^*_{F(p)}N \to T^*_pM$,

  which is characterized by
  
  $(F^*\omega)(X) = \omega(F_*X), \forall \omega \in T^*_{F(p)}N, X \in T_pM$.

- When we introduced the pushforward map, we made a point of noting that vector fields do not pushforward to vector fields, except in the special case of a diffeomorphism.

- The surprising thing about pullbacks is that smooth vector fields always pull back to smooth covector fields.

**Definition.** Given a smooth map $G : M \to N$ and a smooth covector field $\omega$ on $N$, define a covector field $G^*\omega$ on $M$ by

\begin{equation}
G^*\omega_p = G^*(\omega_{G(p)}).
\end{equation}

- Observe that there is no ambiguity about what point to pull back from, in contrast to the vector field case.

**Lemma 6.12.** Let $G : M \to N$ be a smooth map. Suppose $f \in C^\infty(N)$ and $\omega \in T^*(N)$. Then

\begin{align}
(6.13) & \quad G^*df = d(f \circ G); \\
(6.14) & \quad G^*(f\omega) = (f \circ G)G^*\omega.
\end{align}

**Proof.** To prove (6.13), we let $X_p \in T_pM$ be arbitrary and compute

\begin{align}
(G^*df)_p(X_p) &= (G^*(df_{G(p)}))(X_p) \quad \text{(by (6.12))} \\
&= df_{G(p)}(G_*X_p) \quad \text{(by definition of $G^*$)} \\
&= (G_*X_p)f \quad \text{(by definition of $df$)} \\
&= X_p(f \circ G) \quad \text{(by definition of $G_*$)} \\
&= d(f \circ G)_p(X_p) \quad \text{(by definition of $d(f \circ G)$)}.
\end{align}

Similarly, for (6.14) we compute

\begin{align}
(G^*(f\omega))_p &= (G^*(f\omega)_{G(p)}) \quad \text{(by (6.12))} \\
&= G^*(f(G(p))\omega_{G(p)}) \quad \text{(by (6.8))} \\
&= f(G(p))G^*(\omega_{G(p)}) \quad \text{(by linearity of $G^*$)} \\
&= f(G(p))(G^*\omega)_p \quad \text{(by (6.12))} \\
&= ((f \circ G)(G^*\omega))_p \quad \text{(by (6.8))}. \quad \square
\end{align}
Proposition 6.13. Suppose \( G : M \to N \) is smooth, and let \( \omega \) be a smooth covector field on \( N \). Then \( G^* \omega \) is a smooth vector field on \( M \).

Proof. Let \( p \in M \) be arbitrary, and choose smooth coordinates \((x^i)\) for \( M \) near \( p \) and \((y^j)\) for \( N \) near \( G(p) \).

Writing \( \omega \) in coordinates as \( \omega = \omega_j dy^j \) for smooth functions \( \omega_j \) defined near \( G(p) \) and using Lemma 6.12 twice, we have the following computation in a nbhd of \( p \):

\[
G^* \omega = G^* (\omega_j dy^j) = (\omega_j \circ G) d(y^j \circ G),
\]

which is smooth. \( \square \)

- In the course of the preceding proof we derived the following formula for the pullback of a covector field w.r.t. smooth coordinates \((x^i)\) on the domain and \((y^j)\) on the range:

\[
G^* \omega = G^* (\omega_j dy^j) = (\omega_j \circ G) d(y^j \circ G) = (\omega_j \circ G) dG^j,
\]

where \( G^j \) is the \( j \)th component function of \( G \) in these coordinates.

- In other words, to compute \( G^* \omega \), all we need to do is to substitute the component functions of \( G \) for the coordinate functions of \( N \) everywhere that appear in \( \omega \).

Example. Let \( G : \mathbb{R}^3 \to \mathbb{R}^2 \) be the map given by

\[
(u, v) = G(x, y, z) = (x^2y, y\sin z),
\]

and let \( \omega \in T^*(\mathbb{R}^2) \) be the covector field

\[
\omega = u\, dv + v\, du.
\]

According to (6.15), the pullback \( G^* \omega \) is given by

\[
G^* \omega = (u \circ G) d(v \circ G) + (v \circ G) d(u \circ G)
\]

\[
= (x^2) d(y \sin z) + (y \sin z) d(x^2 y)
\]

\[
= x^2 y (\sin z \, dy + y \cos z \, dz) + y \sin z (2xy \, dx + x^2 \, dy)
\]

\[
= 2x^2 y \sin z dx + 2xy^2 \sin z dy + x^2 y^2 \cos z dz.
\]

Example. Let \((r, \theta)\) be polar coordinates on the half-plane \( H = \{(x, y) : x > 0\} \).

- We can think of the change of coordinates \((x, y) = (r \cos \theta, r \sin \theta)\) as the coordinate expression for the identity map on \( H \), but using \((r, \theta)\) as coordinates for the domain and \((x, y)\) for the range.

- Then the pullback formula (6.15) tells us what we can compute the polar coordinate expression for a covector field simply by substituting \( x = r \cos \theta, y = \sin \theta \) and expanding. For example,

\[
xdy - ydx = Id^*(xdy - ydx)
\]

\[
=r \cos \theta dr (r \sin \theta) - r \sin \theta d(r \cos \theta)
\]

\[
=r \cos \theta (r \sin \theta \, dr + r \cos \theta \, d\theta) - r \sin \theta (r \cos \theta \, dr - r \sin \theta \, d\theta)
\]

\[
=(r \cos \theta \sin \theta - r \sin \theta \cos \theta) dr + (r^2 \cos^2 \theta + r^2 \sin^2 \theta) d\theta
\]

\[
=r^2 d\theta.
\]