Covering maps

**Definition.** Suppose $\tilde{X}$ and $X$ are topological spaces. A map $\pi : \tilde{X} \to X$ is called a **covering map** if

1. $\tilde{X}$ is path connected and locally path connected,
2. $\pi$ is surjective and continuous, and
3. each point $p \in X$ has a nbhd $D$ that is evenly covered by $\pi$, meaning that $U$ is connected and each component of $\pi^{-1}(U)$ is mapped homeomorphically onto $U$ by $\pi$.

$X$ is called the **base** of the covering, and $\tilde{X}$ is called a **covering space** of $X$.

**Definition.** Suppose $\tilde{X}$ and $X$ are connected smooth mfds. A map $\pi : \tilde{X} \to X$ is called a **smooth covering map** if

1. $\tilde{X}$ is path connected and locally path connected,
2. $\pi$ is surjective and continuous, and
3. each point $p \in X$ has a nbhd $D$ that is evenly covered by $\pi$, meaning that $U$ is connected and each component of $\pi^{-1}(U)$ is mapped diffeomorphically onto $U$ by $\pi$.

$X$ is called the **base** of the covering, and $\tilde{X}$ is called a **covering space** of $X$.

**Definition.** Any finite or countably infinite group with the discrete topology is a zero-dimensional Lie group. We call any such a group a **discrete group**.

**Lemma 1.** Suppose a discrete group $\Gamma$ acts continuously on a manifold $\tilde{M}$. The action is proper iff the following condition holds:

\[(\star) \quad \text{"Any two points } p, p' \in \tilde{M} \text{ have neighborhoods } U, U' \text{ such that the set } \{ \varphi \in \Gamma : (\varphi \cdot U) \cap U' \neq \emptyset \} \text{ is finite."} \]

**Proof.** $(\Rightarrow)$ First suppose that the action is proper. Let $p, p' \in \tilde{M}$ be arbitrary, and let $U, U'$ be precompact nbhds of $p$ and $p'$, respectively.

$-$ If $(\star)$ does not hold, then there exist infinitely many distinct elements $\varphi_i \in \Gamma$ and points $p_i \in U$ such that $\varphi_i \cdot p_i \in U'$.

$-$ Since $U$ and $U'$ are compact, by passing to a subsequence, we may assume that the sequences $\{p_i\}$ and $\{\varphi \cdot p_i\}$ converge.

$-$ Since $G$ acts on $M$ properly, $\{\varphi_i\}$ has a convergent subsequence. (Why?) But this is impossible, because $\{\varphi_i\}$ has an infinite sequence of distinct points in a discrete space.

$(\Leftarrow)$ Assume that $(1)$ holds. Suppose $\{(\varphi_i, p_i)\}$ is a sequence in $\Gamma \times \tilde{M}$ such that $p_i \to p$ and $\varphi_i \cdot p_i \to p'$.

$-$ Let $U$ and $U'$ be nbhds of $p$ and $p'$ respectively, satisfying property $(\star)$.

$-$ For all sufficiently large $i$, $p_i \in U$ and $\varphi_i \cdot p_i \in U'$.

$-$ Since there are only finitely many $\varphi \in \Gamma$ for which $(\varphi \cdot U) \cap U' \neq \emptyset$, $\exists \varphi \in \Gamma$ such that $\varphi_i = \varphi$ for infinitely many $i$; in particular, some subsequence of $\{\varphi_i\}$ converges.

$-$ Hence the action is proper. (Why?) $\square$

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Lemma 2. For any continuous action of a Lie group $G$ on a manifold $M$, the quotient map $\pi : M \to M/G$ is open.

Proof. For any open set $U \subset M$, $\pi^{-1}(\pi(U)) = \bigcup_{g \in G} \theta_g(U)$.
Since $\theta_g$ is a homeomorphism, $\theta_g(U)$ is open $\forall g \in U$, and therefore $\pi^{-1}(\pi(U))$ is open in $M$. Because $\pi$ is a quotient map, this implies that $\pi(U)$ is open in $M/G$. □

Theorem 3. Suppose $\widetilde{M}$ is a connected smooth manifold, and $\Gamma$ is a discrete group acting smoothly, freely and properly on $\widetilde{M}$. Then the quotient space $\widetilde{M}/\Gamma$ is a topological manifold and has a unique smooth structure such that $\pi : \widetilde{M} \to \widetilde{M}/\Gamma$ is a smooth covering map.

Proof. It follows from the quotient manifold theorem that $\widetilde{M}/\Gamma$ has a unique smooth structure such that $\pi$ is a smooth submersion.

— Because a smooth covering map is in particular a submersion, any other smooth manifold structure on $\widetilde{M}$ making $\pi$ into a smooth covering map must be equal to this one.

— Because $\dim \widetilde{M}/\Gamma = \dim \widetilde{M} - \dim \Gamma = \dim \widetilde{M}$, $\pi$ is a local diffeomorphism.

• Thus it remains only to show that $\pi$ is a covering map.

⊙ Let $p \in \widetilde{M}$. By Lemma 1, there exists a nbhd $U_0$ of $p$ in $\widetilde{M}$ such that $(\varphi \cdot U_0) \cap U_0 = \emptyset$ for all $\varphi \in \Gamma$ except possibly finitely many elements $\varphi_1, \ldots, \varphi_k$.

— Reordering if necessary, we may assume that $\varphi_1 = e$ and $\varphi_i \neq e$ for $2 \leq i \leq k$.

— Shrinking $U_0$ if necessary, we may assume that $U_0$ is connected and $\varphi_i^{-1} \cdot p \notin U_0$ (which implies $p \notin \varphi_i \cdot U_0$) for $i = 2, \ldots, k$.

— Replacing $U_0$ by

$$U = U_0 \setminus (\varphi_2 \cdot U_0 \cup \cdots \varphi_k \cdot U_0),$$

we obtain a nbhd $U$ of $p$ satisfying

$$(**): (\varphi \cdot U) \cap U = \emptyset, \ \forall \varphi \in \Gamma \text{ except } \varphi = e.$$  

⊙ Let $V = \pi(U)$, which is open in $\widetilde{M}/\Gamma$ by Lemma 2.
Because $\pi^{-1}(V)$ is the union of the disjoint connected open sets $\varphi \cdot U$ for $\varphi \in \Gamma$, to show that $\pi$ is a covering map we need only show that $\pi$ is a homeomorphism from each such set onto $V$.

— For each $\varphi \in \Gamma$, the following diagram commute:

$$
\begin{array}{ccc}
U & \xrightarrow{\varphi} & \varphi \cdot U \\
\pi & & \downarrow \pi \\
V & \xrightarrow{\pi} & V
\end{array}
$$

Since $\varphi : U \to \varphi \cdot U$ is a homeomorphism (in fact, a diffeomorphism), it suffices to show that $\pi : U \to V$ is a homeomorphism.

(1) We already know that it is surjective, continuous, and open.

(2) To see that it is injective, suppose $\pi(q) = \pi(q')$ for $q, q' \in U$, which means that $q' = \varphi \cdot q$ for some $\varphi \in \Gamma$. By (**), this can happen only if $\varphi = e$, which is to say that $q = q'$.

This completes the proof that $\pi$ is a smooth covering map. □
Examples (Proper Discrete Group Actions)

(a) The discrete Lie group \( \mathbb{Z}^n \) acts smoothly and freely on \( \mathbb{R}^n \) by translation.
   — To check that the action is proper, one verify that condition (*) is satisfied by sufficiently small balls around \( p \) and \( p' \).
   — The quotient manifold \( \mathbb{R}^n / \mathbb{Z}^n \) is homeomorphic to the \( n \)-torus \( \mathbb{T}^n \), and Theorem 3 says that there is a unique smooth structure on \( \mathbb{T}^n \) making the quotient map into a smooth covering map.
   ○ To verify that this smooth structure on \( \mathbb{T}^n \) is the same as the one we defined previously, thinking of \( \mathbb{T}^n \) as the product manifold \( S^1 \times \cdots \times S^1 \), we check that the covering map
     \[
     (x^1, \cdots, x^n) \mapsto (e^{2\pi ix^1}, \cdots, e^{2\pi ix^n})
     \]
     is a local diffeomorphism w.r.t. the product smooth structure on \( \mathbb{T}^n \), and makes the same identifications as the quotient map \( \mathbb{R}^n \to \mathbb{R}^n / \mathbb{Z}^n \); thus Proposition 6 below implies \( \mathbb{R}^n / \mathbb{Z}^n \) is diffeomorphic to \( \mathbb{T}^n \).

(b) The two elements group \( \{ \pm 1 \} \) acts on \( S^n \) by multiplication.
   — This action is obviously smooth and free, and it is proper because the group is compact.
   This defines a smooth structure on \( S^n / \{ \pm 1 \} \).
   — In fact, this quotient map is diffeomorphic to \( \mathbb{R}P^n \) with the smooth structure we defined previously.
     − Indeed, let \( p : S^n \to \mathbb{R}P^n \) be the smooth covering map obtained by restricting the canonical projection \( \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^{n+1} \) to the sphere.
     − This map makes the same identification as the quotient map \( \pi : S^n \to S^n / \{ \pm 1 \} \).
     − By Proposition 6 below, therefore \( S^n / \{ \pm 1 \} \) is diffeomorphic to \( \mathbb{R}P^n \).

Proposition 6 (Uniqueness of Smooth Quotients).
Suppose \( \pi_1 : M \to N_1 \) and \( \pi_2 : M \to N_2 \) are surjective submersions that are constant on each other’s fibers.
Then there exists a unique diffeomorphism \( F : N_1 \to N_2 \) such that \( F \circ \pi_1 = \pi_2 \):

\[
\begin{array}{ccc}
\begin{array}{c}
M
\end{array} & \xrightarrow{\pi_1} & \begin{array}{c}
N_1
\end{array} \\
\downarrow & & \downarrow F \\
\begin{array}{c}
N_2
\end{array} & \xrightarrow{\pi_2} & \begin{array}{c}
M
\end{array}
\end{array}
\]