Riemannian Covering.

**Definition.** Suppose $\widetilde{X}$ and $X$ are connected smooth mfd’s. A map $\pi : \widetilde{X} \to X$ is called a smooth covering map if

1. $\widetilde{X}$ is path connected and locally path connected,
2. $\pi$ is surjective and continuous, and
3. each point $p \in X$ has a nbhd $U$ that is evenly covered by $\pi$, meaning that $U$ is connected and each component of $\pi^{-1}(U)$ is mapped diffeomorphically onto $U$ by $\pi$.

$X$ is called the base of the covering, and $\widetilde{X}$ is called a covering space of $X$.

**Definition 2.17.** Let $(M,g)$ and $(\widetilde{M},h)$ be two Riemannian manifolds. A map $\pi : \widetilde{M} \to M$ is a Riemannian covering map if

(i) $\pi$ is a smooth covering map,
(ii) $\pi$ is a local isometry.

**Proposition 2.18.** Let $\pi : \widetilde{M} \to M$ be a smooth covering map. For any Riemannian metric $g$ on $M$, there exist a unique Riemannian metric $h$ on $\widetilde{M}$, such that $\pi$ is a Riemannian covering map.

**Proof.** If such a metric $h$ exists, it has to satisfy for $q \in \widetilde{M}$ and $X, Y \in T_q \widetilde{M}$

(*) \[ h_q(X,Y) = g_{\pi(q)}(\pi_* X, \pi_* Y). \]

Conversely, since $\pi_*$ is a vector space isomorphism, the formula (*) defines a scalar product $h_q$ on $T_q \widetilde{M}$

This scalar product depends smoothly on $q$, since $p$ is a local diffeomorphism. \hfill \Box

**Definition.** Let $\widetilde{M}$ and $M$ be smooth manifolds, and let $\pi : \widetilde{M} \to M$ be a covering map. A covering transformation (or deck transformation) of $\pi$ is a diffeomorphism $\varphi : \widetilde{M} \to \widetilde{M}$ such that $\pi \circ \varphi = \pi$;

\[
\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\varphi} & \widetilde{M} \\
\downarrow & & \downarrow \\
M & = & M
\end{array}
\]

**Definition.** The set $C_*(\widetilde{M})$ of all covering transformations, called the covering group of $\pi$, is a group under the composition, acting on $\widetilde{M}$ on the left.

**Proposition 2.18*.** If $g$ is a Riemannian metric on $M$, then $\widetilde{g} = \pi^* g$ is a Riemannian metric on $\widetilde{M}$ that is invariant under all covering transformation. In this case $\pi$ is a Riemannian covering.
• On the contrary, a Riemannian metric $h$ on $\tilde{M}$ on $M$ does not automatically yield a metric $g$ on $M$ such that $\pi$ is a Riemannian covering map.

**Example 2.19.** Equip the sphere $S^2$ with a Riemannian metric which can be written in spherical coordinates as

$$h = d\theta^2 + a^2(\theta)d\psi^2.$$ 

The metric induces on $\mathbb{RP}^2$ a metric $g$ such that the canonical projection from $(S^2, h)$ to $(\mathbb{RP}^2, g)$ is a Riemannian covering map if

$$h_q(X, Y) = g_{\pi(q)}(\pi_*X, \pi_*Y), \quad \forall X, Y \in T_qS^2.$$ 

Since $\pi(x) = \pi(y)$ iff $x = y$ or $x = -y$,

$$g_{\pi(q)} \text{ exists} \iff h_q(X, Y) = h_{-q}(X, Y), \quad \forall X, Y \in T_qS^2. \iff a(\pi + \theta) = a(\theta).$$

**Proposition 1.** Let $\pi : \tilde{M} \to M$ be a smooth covering map. With the discrete topology, the covering group $\mathbb{C}_*(\tilde{M})$ is a zero-dimensional Lie group acting smoothly, freely, and properly on $\tilde{M}$.

**Proposition 2.20*.** Given a covering map $\pi : \tilde{M} \to M$ where $(M, g)$ is a Riemannian manifold and where all the deck transformations are isometries, there is a Riemannian metric $h$ down on $M$ such that $\pi$ is everywhere a local isometry.

**Proposition 2.20.** Let $(\tilde{M}, h)$ be a Riemannian manifold and $G$ is a free and proper group of isometries of $(\tilde{M}, h)$. Then there exists on the quotient manifold $M = \tilde{M}/G$ a unique Riemannian metric $g$ such that the canonical projection $\pi : \tilde{M} \to M$ is a Riemannian covering map.

**Proof.** Let $p \in M$ and $q \in \pi^{-1}(p)$. Since $\pi \circ f = \pi$ and $\pi$ is a local diffeomorphism, we let

$$(\pi_{p,q})_* : F_pM \to F_q\tilde{M}$$

and define a scalar product $g_p$ on $T_pM$ by, for $X, Y \in T_pM$:

$$(*) \quad g_p(X, Y) = h_q((\pi_{p,q})_*X, (\pi_{p,q})_*Y).$$

**Claim:** This quantity does not depend on the choice of $q$ in the fiber $\pi^{-1}(p)$. Indeed, for $q, q' \in \pi^{-1}(p)$, there exists an isometry $f \in G$ such that $f(q) = q'$ and

$$(\pi_{p,q})_* = f_*(\pi_{p,q'})_*.$$ 

Hence we have constructed a metric $(*)$ on $M$ such that $\pi$ is a Riemannian covering map.

**Claim:** Uniqueness. If $\pi$ is Riemannian covering map, then $g$ must satisfy $(*)$. \qed

**Example 2.21.** $\mathbb{RP}^n \cong S^n/\{\pm 1\}$.

Equip $\mathbb{RP}^n$ with a canonical metric such that the projection $p : S^n \to \mathbb{RP}^n$ is a Riemannian covering map, with $G = \{\text{Id}, -\text{Id}\}$ as the isometry group.
Flat tori. $T^n \cong \mathbb{R}^n/\Gamma$, $\Gamma$: a lattice.

Let $\{\vec{e}_1, \cdots, \vec{e}_n\}$ be any basis of $\mathbb{R}^n$. Obtain torus $T^n$ by making the quotient of $\mathbb{R}^n$ by the set of translations

$$(v_1, \cdots, v_n) \rightarrow (v_1, \cdots, v_n) + \vec{e}_i, \quad i = 1, \cdots, n.$$ 

Equivalently, for the lattice $\Gamma$ associated with this basis, namely

$$\Gamma = \{k_1\vec{e}_1 + \cdots + k_n\vec{e}_n : k_i \in \mathbb{Z}, \quad i = 1, \cdots, n\},$$

we consider $\mathbb{R}^n/\Gamma$ by

$$\pi(x_1\vec{e}_1 + \cdots x_n\vec{e}_n) = (e^{2i\pi x_1}, \cdots, e^{2i\pi x_n}).$$

Any translation is an isometry of the canonical Euclidean structure of $\mathbb{R}^n$. Hence we can equip $\mathbb{R}^n/\Gamma$ with a Riemannian metric $g_\Gamma$ for which $\pi: \mathbb{R}^n \to \mathbb{R}^n/\Gamma$ is a Riemannian covering map.

- This metric $g_\Gamma$ is locally Euclidean for any choice of $\Gamma$; this is why the resulting Riemannian manifold is called a flat torus.

- A local chart for $\mathbb{R}^n/\Gamma$ around $(e^{2i\pi x_1}, \cdots, e^{2i\pi x_n})$ is given by

$$\Phi: \left(\frac{x_1 - 1}{2}, \frac{x_1 + 1}{2}\right) \times \cdots \times \left(\frac{x_n - 1}{2}, \frac{x_n + 1}{2}\right) \to \mathbb{R}^n/\Gamma$$

where

$$\Phi(y_1, \cdots, y_n) = \text{class in } \mathbb{R}^n/\Gamma \text{ of } (e^{2i\pi y_1}, \cdots, e^{2i\pi y_n}).$$

- If $\langle \cdot, \cdot \rangle_\mathbb{R}$ is the Euclidean scalar product on $\mathbb{R}^n$ and $(\frac{\partial}{\partial x_j})$ are the coordinate vector fields on $\mathbb{R}^n/\Gamma$ associated with the chart $\Phi$, the metric is given in this chart by

$$g_\Gamma = \sum_{ij=1}^{n} \langle \vec{e}_i, \vec{e}_j \rangle_\mathbb{R} dx_i dx_j.$$ 

**Theorem 2.23.** The metrics $g_\Gamma$ and $g'_\Gamma$ are isometric iff there exist an isometry of $\mathbb{R}^n$ which sends the lattice $\Gamma$ on the lattice $\Gamma'$.

**Proof.** ($\Leftarrow$) If there exists an isometry $F : \mathbb{R}^n \to \mathbb{R}^n$ with $F(\Gamma) = \Gamma'$, then $F$ goes to the quotient and gives an isometry $f$ between $\mathbb{R}^n/\Gamma$ and $\mathbb{R}^n/\Gamma'$:

$$\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{F} & \mathbb{R}^n \\
\pi \downarrow & & \downarrow \pi' \\
\mathbb{R}^n/\Gamma & \xrightarrow{f} & \mathbb{R}^n/\Gamma'
\end{array}$$

($\Rightarrow$) If $f$ is an isometry from $\mathbb{R}^n/\Gamma$ to $\mathbb{R}^n/\Gamma'$, since $\mathbb{R}^n$ is simply connected, there is a map $F$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that

$$f \circ \pi = \pi' \circ F.$$ 

The map $F$ is an isometry and by construction, $F(\Gamma) = \Gamma'$. □
Classification of 2-dimensional flat torus

**Definition.** Two metrics $g_1$ and $g_2$ on a manifold are **homothetic** if there exists $\lambda \in \mathbb{R}$ such that $g_1 = \lambda g_2$.

- We want to classify the flat metrics on $T^2$ up to isometries and homotheties.
  The classification is equivalent to the classification of the lattice of $\mathbb{R}^2$ up to isometries and dilations.

- Let $a_1$ be the shortest non-zero vector of $\Gamma$.
  - After using if necessary a dilation and a rotation, we can assume that $a_1 = (1, 0)$.
- Let $a_2$ be the shortest vector of $\Gamma \setminus \mathbb{Z} \cdot a_1$.
  - The vectors $a_1$ and $a_2$ generate $\Gamma$.
  - Since $\Gamma$ is symmetry w.r.t.the $y$-axis, we can assume that $a_2$ lies in the first quadrant, and then the $x$-coordinate of $a_2$ is smaller than $1/2$
    (otherwise the length of $a_2 - a_1$ would be shorter than $a_2$).

○ The class of the lattice $\Gamma$ is hence determined by the position of $a_2$ in the domain

$$M = \{(x, y) : x^2 + y^2 \geq 1, 0 \leq x \leq 1, y > 0\},$$

and two lattices corresponding to two different points of $M$ belong to two different classes.