Sobolev’s Inequality, Poincaré Inequality and Compactness

I. Sobolev inequality and Sobolev Embedding Theorems

Theorem 1 (Sobolev’s embedding theorem). Given the bounded, open set \( \Omega \subset \mathbb{R}^n \) with \( n \geq 3 \) and \( 1 \leq p < n \), then

\[
W^{1,p}_0(\Omega) \subset L^{\frac{np}{n-p}}(\Omega)
\]

and \( W^{1,p}_0(\Omega) \) is continuously embedded in the space \( L^{\frac{np}{n-p}}(\Omega) \). This means that the following estimate

\[
\|f\|_{L^{\frac{np}{n-p}}(\Omega)} \leq C \|Df\|_{L^p(\Omega)}, \quad \forall f \in W^{1,p}_0(\Omega).
\]

holds true with a constant \( C = C(n, p) \in (0, +\infty) \); here we denote the weak gradient by

\[
Df = (D_{e_1}f, \cdots, D_{e_n}f) \in L^p(\Omega) \times \cdots \times L^p(\Omega).
\]

Proof (L. Nirenberg). (i) It suffices to prove the inequality (1) for all \( f \in C_0^\infty(\Omega) \). In this context we need the generalized Hölder inequality, namely, if \( f_j \in L^{p_j}(\Omega), j = 1, \cdots, m \), such that \( p_1^{-1} + \cdots + p_m^{-1} = 1 \), then there holds

\[
\int_\Omega f_1(x) \cdots f_m(x) \, dx \leq \|f_1\|_{L^{p_1}(\Omega)} \cdots \|f_m\|_{L^{p_m}(\Omega)},
\]

which can be easily deduced from Hölder’s inequality by induction.

(ii) At first, we deduce the estimate (1) in the case \( p = 1 \).

Noting that \( f \in C_0^\infty(\Omega) \), we have the following representation for all \( x \in \mathbb{R}^n \):

\[
f(x) = \int_{-\infty}^{x_i} D^{e_i} f(x_1, \cdots, x_{i-1}, t, x_{i+1}, \cdots, x_n) \, dt.
\]

This implies

\[
|f(x)| \leq \int_{-\infty}^{x_i} |D^{e_i} f| \, dt \leq \int_{-\infty}^{\infty} |D^{e_i} f| \, dx_i,
\]

and consequently

\[
|f(x)|^{\frac{np}{n-p}} \leq \left( \prod_{i=1}^{n} \int_{-\infty}^{\infty} |D^{e_i} f| \, dx_i \right)^{\frac{1}{p}}.
\]

We integrate this inequality successively with respect to the variables \( x_1, \cdots, x_n \), using each time the generalized Hölder inequality with \( p_1 = \cdots = p_m = n - 1 \) and \( m = n - 1 \).

\[
\int_{-\infty}^{\infty} |f(x)|^{\frac{np}{n-p}} \, dx_1 \leq \left( \int_{-\infty}^{\infty} |D^{e_1} f| \, dx_1 \right)^{\frac{1}{p_1}} \int_{-\infty}^{\infty} \left( \prod_{i=2}^{n} \int_{-\infty}^{\infty} |D^{e_i} f| \, dx_i \right)^{\frac{1}{p_i-1}} \, dx_1
\]

\[
\leq \left( \int_{-\infty}^{\infty} |D^{e_1} f| \, dx_1 \right)^{\frac{1}{p_1}} \prod_{i=2}^{n} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |D^{e_i} f| \, dx_i \right)^{\frac{1}{p_i-1}}.
\]
A similar integration over the variables \(x_2, \ldots, x_n\) yields
\[
\int_{\mathbb{R}^n} |f(x)| \prod_{i=1}^{n} |D^{e_i} f_i| \, dx \leq \left( \prod_{i=1}^{n} \int_{\mathbb{R}^n} |D^{e_i} f| \, dx_i \right)^{\frac{1}{n}},
\]
and finally
\[
\|f\|_{n^{-1}} \leq \left( \prod_{i=1}^{n} \int_{\mathbb{R}^n} |D^{e_i} f| \, dx_i \right)^{\frac{1}{n}} \leq \int \left( \sum_{i=1}^{n} |D^{e_i} f| \right) \, dx \leq \frac{1}{\sqrt{n}} \int_{\Omega} |Df| \, dx = \frac{1}{\sqrt{n}} \|Df\|_1, \quad \forall f \in C_0^\infty(\Omega). (3)
\]

(iii) We now consider the case \(1 < p < n\).
- Here we insert \(|f|^\gamma\) with \(\gamma > 1\) into (3) and obtain the following relation with the aid of Hölder’s inequality and the condition \(p^{-1} + q^{-1} = 1\):
\[
\|\|f\|^\gamma\|_{\frac{n}{n-p}} \leq \frac{1}{\sqrt{n}} \int_{\Omega} |D|f|^\gamma| \, dx = \frac{\gamma}{\sqrt{n}} \int_{\Omega} |f|^{\gamma-1} |Df| \, dx \leq \frac{\gamma}{\sqrt{n}} \|f|^{\gamma-1}\|_q \|Df\|_p,
\]
and consequently
\[
\|\|f\|_{n^{-1}} \leq \frac{\gamma}{\sqrt{n}} \|\|f\|^{\gamma-1}\|_q \|Df\|_p.
\]
Choosing
\[
\gamma = \frac{(n-1)p}{n-p} = \frac{np - p}{n-p},
\]
we infer
\[
\gamma_n = \frac{\gamma}{n-1} = (\gamma - 1)q = \frac{np}{n-p}.
\]
Finally, we arrive at
\[
\|\|f\|_{\frac{np}{n-p}} \leq \gamma \sqrt{n} \|Df\|_p, \quad \forall f \in C_0^\infty(\Omega).
\]
with the constant \(C = \frac{np - p}{\sqrt{n(n-p)}}\).

(iv) If now \(u \in W^{1,p}_0(\Omega)\), we approximate \(u\) in \(W^{1,p}\)-norm by \(C_0^\infty\) functions \(u_m\), and apply (1) to the difference \(u_\ell - u_m\). It follows that \(\{u_m\}\) is a Cauchy sequence in \(L^{\frac{np}{n-p}}\). Thus \(u\) itself is contained in the same space and satisfies (1). \(\square\)
**Corollary.** If \( kp < n \), the Sobolev space \( W^{k,p}_0(\Omega) \) is continuously embedded in \( L^{\frac{nkp}{n-p}}(\Omega) \). That is, there exists a number \( C \) depending only on \( k \), \( p \) and \( n \) such that

\[
\| u \|_{L^{\frac{nkp}{n-p}}(\Omega)} \leq C \| u \|_{W^{k,p}_0(\Omega)}, \quad \forall f \in W^{1,p}_0(\Omega).
\]

**Proof.** Suppose \( kp < n \) and \( u \in W^{k,p}_0(\Omega) \). Let \( p^* = \frac{np}{n-p} \). Then, since \( D^\alpha u \in L^p(\Omega) \) for all \( |\alpha| \leq k \), the Sobolev inequality implies

\[
\| D^\beta u \|_{L^{p^*}(\Omega)} \leq C \| u \|_{W^{k,p}(\Omega)}, \quad \text{if } |\beta| \leq k - 1,
\]

and hence \( u \in W^{k-1,p^*}(\Omega) \).

- Similarly, we find \( u \in W^{k-2,p^{**}}(\Omega) \), where

\[
p^{**} = \frac{1}{p^*} - \frac{1}{n} = \frac{1}{p} - \frac{2}{n},
\]

and

\[
\| D^\gamma u \|_{L^{p^{**}}(\Omega)} \leq C \| u \|_{W^{k,p^*}(\Omega)}, \quad \text{if } |\gamma| \leq k - 2,
\]

- Proceeding thusly, we finally obtain, after \( k \) steps, that (4) holds and \( u \in W^{0,q}(\Omega) = L^q(\Omega) \), for

\[
\frac{1}{q} = \frac{1}{p} - \frac{k}{n}.
\]

- Equipped with the extension operator \( E \), we extend the embedding theorem from the Sobolev spaces \( W^{k,p}_0(\Omega) \) to the spaces \( W^{k,p}(\Omega) \), if \( \Omega \) is a \( C^k \)-domain.

- Namely, if \( u \in W^{k,p}(\Omega) \), we consider \( E u \in W^{k,p}(\Omega') \), for some domain \( \Omega' \) containing \( \Omega \), which then is contained in \( L^{\frac{nkp}{n-p}}(\Omega') \), if \( kp < n \).

And hence \( u \in L^{\frac{nkp}{n-p}}(\Omega) \), by restriction from \( \Omega' \) to \( \Omega \).

Since \( E u = u \) on \( \Omega \) and \( \| E u \|_{W^{k,p}(\Omega')} \leq c \| u \|_{W^{k,p}(\Omega)} \) depending on \( \Omega \), we have thus proved the following version of the Sobolev embedding theorem:

**Sobolev Embedding Theorem*.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded \( C^k \)-domain. If \( kp < n \), the Sobolev space \( W^{k,p}(\Omega) \) is continuously embedded in \( L^{\frac{nkp}{n-p}}(\Omega) \). That is, there exists a number \( C \) depending only on \( k \), \( p \), \( n \) and \( \Omega \) such that

\[
\| u \|_{L^{\frac{nkp}{n-p}}(\Omega)} \leq C \| u \|_{W^{k,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega).
\]
II. Poincaré Inequality.

Poincaré Inequality. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. There exists a positive constant $C_p$ such that, for every $u \in W_0^{1,p}(\Omega)$,

$$\|u\|_{L^p(\Omega)} \leq C_p \|\nabla u\|_{L^p(\Omega)}.$$  \hfill (4)

**Proof.** First we prove the formula for $u \in C^\infty_0(\Omega)$; then, if $u \in W_0^{1,p}(\Omega)$, select a sequence $\{u_k\} \subset C^\infty_0(\Omega)$ converging to $u$ in $W_0^{1,p}$-norm as $k \to \infty$, i.e. 

$$\|u_k - v\|_{L^p(\Omega)} \to 0, \quad \|\nabla u_k - \nabla v\|_{L^p(\Omega)} \to 0$$

In particular

$$\|u_k\|_{L^p(\Omega)} \to \|v\|_{L^p(\Omega)}, \quad \|\nabla u_k\|_{L^p(\Omega)} \to \|\nabla v\|_{L^p(\Omega)}.$$ 

Since (4) holds for every $v_k$, we have

$$\|u_k\|_{L^p(\Omega)} \leq C_p \|\nabla u_k\|_{L^p(\Omega)}.$$ 

Letting $k \to \infty$, we obtain (4) for $u$. Thus, it is enough to prove (4) for $u \in C^\infty_0(\Omega)$.

### To this purpose, from the divergence theorem, we may write

$$\int_{\Omega \cap \{v > 0\}} \text{div} (v^p x) \, dx = 0, \quad \int_{\Omega \cap \{v < 0\}} \text{div} (v^p x) \, dx = 0.$$  \hfill (5)

since $v = 0$ on $\partial \Omega$. Now

$$\text{div} (v^p x) = pv \nabla v \cdot x + n v^p$$

so that (5) yields

$$\int_{\Omega \cap \{v > 0\}} v^p \, dx = -\frac{p}{n} \int_{\Omega \cap \{v > 0\}} v^{p-1} \nabla v \cdot x \, dx,$$

$$\int_{\Omega \cap \{v < 0\}} v^p \, dx = -\frac{p}{n} \int_{\Omega \cap \{v < 0\}} v^{p-1} \nabla v \cdot x \, dx.$$ 

Since $\Omega$ is bounded, we have $\max_{x \in \Omega} |x| = M < \infty$; therefore, using the Schwartz’s inequality, we get

$$\int_{\Omega \cap \{v > 0\}} v^p \, dx = \left| \frac{p}{n} \int_{\Omega \cap \{v > 0\}} v^{p-1} \nabla v \cdot x \, dx \right| 
\leq \frac{pM}{n} \left( \int_{\Omega \cap \{v > 0\}} |v|^{p-1} \, dx \right)^{1/q} \left( \int_{\Omega \cap \{v > 0\}} |\nabla v|^p \, dx \right)^{1/p},$$

$$= \frac{pM}{n} \left( \int_{\Omega \cap \{v > 0\}} |v|^q \, dx \right)^{1/q} \|\nabla v\|_{L^p(\Omega \cap \{v > 0\})}.$$ 

Analogous estimate can be derived for the integral over $\Omega \cap \{v < 0\}$. From this it follows (4) with $C_p = pM/n$. \hfill \Box

Inequality (4) implies that in $W_0^{1,p}(\Omega)$, the norm $\|u\|_{W_0^{1,p}}$ is equivalent to $\|\nabla\|_{L^p}$. Indeed,

$$\|u\|_{W_0^{1,p}} = (\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p)^{1/p},$$

and from (4)

$$\|\nabla u\|_{L^p} \leq \|u\|_{W_0^{1,p}} \leq (C_p^p + 1)^{1/p} \|\nabla u\|_{L^p}.$$
III. Compactness Theorem of Rellich and Kondrachov

We call the Banach space \((\mathcal{B}_1, \| \cdot \|_1)\) is \textbf{compactly embedded} into the Banach space \((\mathcal{B}_2, \| \cdot \|_2)\) if the injective mapping \(I_1 : \mathcal{B}_1 \to \mathcal{B}_2\) is compact; this means that bounded sets in \(\mathcal{B}_1\) are mapped onto precompact sets in \(\mathcal{B}_2\).

**Compactness Theorem of Rellich and Kondrachov.**

Let \(\Omega\) denote a bounded, domain in \(\mathbb{R}^n\).

(i.1) Let \(1 \leq p < n\). Then for all \(1 \leq q < \frac{np}{n-p}\), \(W^{1,p}_0(\Omega)\) is compactly embedded into \(L^q(\Omega)\). This means for each sequence \(\{f_k\}_{k=1,2,\ldots} \subset W^{1,p}_0(\Omega)\) with \(\|f\|_{W^{1,p}(\Omega)} \leq s \in [0, \infty)\) we can select a subsequence \(\{f_{k_\ell}\}_{\ell=1,2,\ldots}\) and element \(f \in L^q(\Omega)\) satisfying \(\lim_{\ell \to \infty} \|f_{k_\ell} - f\| = 0\).

(ii.1) \(W^{1,2}_0(\Omega)\) is compactly embedded into \(L^2(\Omega)\).

(ii.2) If \(\Omega\) is Lipschitz, then \(W^{1,2}(\Omega)\) is compactly embedded into \(L^q(\Omega)\).

To prove (i.1) and (i.2), we shall use the following result.

**Interpolation Inequality.** If the exponents \(1 \leq p \leq q \leq r\) fulfill

\[
\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}
\] with \(\lambda \in [0, 1]\),

then

\[
\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}, \quad \forall f \in L^r(\Omega).
\]

**Proof of Interpolation Inequality.** Noting

\[
1 = \frac{\lambda q}{p} + \frac{(1-\lambda)q}{r} = \left(\frac{p}{\lambda q}\right)^{-1} + \left(\frac{r}{(1-\lambda)q}\right)^{-1}
\]

we obtain

\[
\|f\|_q = \left(\int_{\Omega} |f|^\lambda \|f\|^{(1-\lambda)q} \, dx\right)^{1/q} \\
\leq \left(\int_{\Omega} |f|^p \, dx\right)^{\frac{\lambda}{p}} \left(\int_{\Omega} |f|^{r} \, dx\right)^{\frac{1-\lambda}{r}} = \|f\|_p^\lambda \|f\|_r^{1-\lambda}. \quad \square
\]

**Proof of (i.1).** Step 1. We start with an arbitrary sequence \(\{f_k\}\) with

\[
\|f\|_{W^{1,p}(\Omega)} \leq s \in [0, \infty),
\]

and make the transition to a sequence \(\{g_k\}_{k=1,2,\ldots} \subset C_0(\Omega)\) with the property

\[
\|g_k - f_k\|_{W^{1,p}(\Omega)} \leq \frac{1}{k}.
\]

The latter satisfies the restriction

\[
\|g_k\|_{W^{1,p}(\Omega)} \leq 1 + s, \quad \forall k \in \mathbb{N}.
\]
If we manage to select a subsequence \( \{g_{k_\ell}\} \) convergent in \( L^1(\Omega) \) from the sequence \( \{g_k\} \), then the sequence \( \{f_{k_\ell}\} \) is convergent in \( L^1(\Omega) \) as well; here we observe
\[
\|g_k - f_k\|_{L^1(\Omega)} \leq c\|g_k - f_k\|_{W^{1,p}(\Omega)} \leq \frac{c}{k}.
\]

**Step 2.** In order to show that the sequence \( \{f_{k_\ell}\} \) converges even in the space \( L^q(\Omega) \) with \( 1 < q < \frac{np}{n-p} \), we apply the interpolation inequality by choosing \( \lambda \in (0, 1) \) with the property
\[
\frac{1}{q} = \lambda + (1 - \lambda)\frac{n-p}{np}.
\]
The interpolation inequality and the Sobolev inequality yield the estimate
\[
\|f\|_q \leq \|f\|_1^{1-\lambda} \|f\|_{np/(n-p)}^\lambda \leq \|f\|_1^{1-\lambda} (C\|Df\|_p)^{\lambda}, \quad \forall f \in W_0^{1,p}(\Omega).
\]
Therefore, we have
\[
\|f_{k_\ell} - f_{k_m}\|_q \leq \overline{C}\|f_{k_\ell} - f_{k_m}\|_1^{\lambda} \to 0, \quad \text{as } \ell, m \to \infty
\]

**Step 3.** It still remains to select a subsequence in \( L^1(\Omega) \) from the sequence \( \{g_k\}_{k=1,2,...} \subset C_0^\infty(\Omega) \).

Therefore, we take an arbitrary \( \varepsilon \in (0, 1) \) and consider the sequence of functions
\[
g_{k,\varepsilon}(x) := \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \rho\left(\frac{x - y}{\varepsilon}\right) g_k(y) dy = \int_{\mathbb{R}^n} \rho(z) g_k(x - \varepsilon z) \in C_0^\infty(\Theta),
\]
where
\[
\Theta = \{x \in \mathbb{R}^n : \text{dist}(x, \Omega) < 1\}.
\]
For each fixed \( \varepsilon \in (0, 1] \), the sequence of functions \( \{g_{k,\varepsilon}\} \) is **uniformly bounded** and **equicontinuous**, since we have the following estimates for all \( x \in \Theta \):
\[
|g_{k,\varepsilon}(x)| \leq \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \rho\left(\frac{x - y}{\varepsilon}\right) |g_k(y)| dy \leq \frac{C_0}{\varepsilon^n} \sup_{|x| \leq 1} \rho(x)
\]
and
\[
|Dg_{k,\varepsilon}(x)| \leq \frac{1}{\varepsilon^{n+1}} \int_{\mathbb{R}^n} |D\rho\left(\frac{x - y}{\varepsilon}\right)||g_k(y)| dy
\leq \varepsilon^{-(n+1)} \sup_{|x| \leq 1} |D\rho(z)| \int_{\mathbb{R}^n} |g_k(y)| dy
\leq \frac{C_0}{\varepsilon^{n+1}} \sup_{|x| \leq 1} |D\rho(z)|.
\]

**Step 4.** For each \( \varepsilon > 0 \), the Arzelà-Ascoli theorem thus yields a subsequence \( \{g_{k_{\varepsilon,\varepsilon}}\} \) of the sequence \( \{g_{k,\varepsilon}\} \) converging uniformly in the set \( \overline{\Omega} \).

- We now set \( \varepsilon_m = \frac{1}{m} \) with \( m \in \mathbb{N} \). Using the Cantor’s diagonal procedure we select a subsequence \( \{g_{k_m}\} \) of the sequence \( \{g_k\} \) such that, for each fixed \( m \in \mathbb{N} \), the sequence \( \{g_{k_{m,\varepsilon_m}}\} \) converges uniformly in the set \( \overline{\Omega} \).
Step 5. We have the inequality
\[ |g_k(z) - g_{k,\varepsilon}(x)| \leq \int_{|z| \leq 1} \rho(z)|g_k(x) - g_k(x - \varepsilon z)|\,dz \]
\[ \leq \int_{|z| \leq 1} \rho(z) \int_0^\varepsilon |Dg_k(x - tz)|\,dt\,dz, \]
for all \( x \in \Omega \), which implies the estimate
\[ (5) \quad \int_\Omega |g_k(z) - g_{k,\varepsilon}(x)|\,dx \leq \varepsilon \int_\Omega |Dg_k(x - tz)|\,dx \leq C_1 \varepsilon, \quad \forall k \in \mathbb{N}. \]
Choosing an arbitrary number \( \varepsilon > 0 \), we obtain the relation
\[ \|g_{k_1} - g_{k_2}\|_{L^1(\Omega)} \]
\[ \leq \|g_{k_1} - g_{k_1,\varepsilon m}\|_{L^1(\Omega)} + \|g_{k_1,\varepsilon m} - g_{k_2,\varepsilon m}\|_{L^1(\Omega)} + \|g_{k_2,\varepsilon m} - g_{k_2}\|_{L^1(\Omega)} \]
\[ \leq (2C_1 + |\Omega|)\varepsilon, \quad \forall \ell_1, \ell_2 \geq \text{some constant } \ell_0(\varepsilon). \]
Consequently, \( \{g_k\} \) represents a Cauchy sequence in \( L^1(\Omega) \) and hence possesses a limit in \( L^1(\Omega) \).

Proof of (ii.1). (a) We again start with a sequence \( \{f_k\} \) with \( \|f_k\|_{W^{1,2}(\Omega)} \leq s \in [0, \infty) \), and make the transition to a sequence \( \{g_k\}_{k=1,2,\ldots} \subset C_0^\infty(\Omega) \) with the property \( \|g_k - f_k\|_{W^{1,2}(\Omega)} \leq \frac{1}{k} \). Thus \( \{g_k\} \) satisfies the restriction
\[ \|g_k\|_{W^{1,2}(\Omega)} \leq 1 + s, \quad \forall k \in \mathbb{N}. \]
To prove Proposition 2, it suffices to select a subsequence \( \{g_{k_i}\} \) convergent in \( L^2(\Omega) \) from the sequence \( \{g_k\} \).

(b) For this purpose, we again take an arbitrary \( \varepsilon \in (0, 1) \) and consider the sequence of functions
\[ g_{k,\varepsilon}(x) := \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \rho \left( \frac{x - y}{\varepsilon} \right) g_k(y)\,dy = \int_{\mathbb{R}^n} \rho(\varepsilon)g_k(x - \varepsilon z) \in C_0^\infty(\Theta), \]
where
\[ \Theta = \{ x \in \mathbb{R}^n : \text{dist}(x, \Omega) < 1 \}. \]
As shown in Step 3 of the proof of Proposition 1, the sequence of functions \( \{g_{k,\varepsilon}\} \) is uniformly bounded and equicontinuous, for each fixed \( \varepsilon \in (0, 1) \).

– Then, as in Step 4 of the proof of Proposition 1, the Arzelá-Ascoli theorem and Cantor’s digonal process yield a subsequence \( \{g_{k_i}\} \) of the sequence \( \{g_k\} \) such that, for each fixed \( m \in \mathbb{N} \), the sequence \( \{g_{k_i,1/m}\} \) converges uniformly in the set \( \overline{\Omega} \).

(c) To show that \( \{g_{k_i}\} \) convergent in \( L^2(\Omega) \), the crucial step is to establish, analogously to (5),
\[ (7) \quad \|g_k(z) - g_{k,\varepsilon}(x)\|_{L^2(\Omega)} \leq \varepsilon \|Dg_k\|_{L^2(\Omega)}, \quad \forall k \in \mathbb{N}. \]
In fact, we have
\[
|g_k(z) - g_{k,\varepsilon}(x)|^2 \leq \left( \int_{|z| \leq 1} \rho(z)|g_k(x) - g_k(x - \varepsilon z)|dz \right)^2 \\
\leq \left[ \int_{|z| \leq 1} \rho(z) \left( \int_0^\varepsilon |Dg_k(x - tz)|dt \right)dz \right]^2,
\]
for all \( x \in \Omega \), which implies the estimate
\[
\int_\Omega |g_k(z) - g_{k,\varepsilon}(x)|^2 dx \leq \int_\Omega \left[ \int_{|z| \leq 1} \rho(z) \left( \int_0^\varepsilon |Dg_k(x - tz)|dt \right)dz \right]^2 dx \\
= \left( \int_{|z| \leq 1} \rho(z)dz \right) \left( \int_{|z| \leq 1} \rho(z)^{1/2} \left( \int_0^\varepsilon |Dg_k(x - tz)|dt \right)dz \right)^2 \\
\leq \varepsilon^2 \int_\Omega |Dg_k(x)|dx, \quad \forall k \in \mathbb{N}.
\]

Analogously to (6), we use the triangle inequality for \( L_2 \)-norm and (7) to conclude that \( \{g_{k,\varepsilon}\} \) is a Cauchy sequence in \( L_2(\Omega) \) and hence possesses a limit in \( L_2(\Omega) \). □

**Proof of (i.2) and (ii.2).** If \( 1 < p < \infty \) and \( \{f_k\} \) is a bounded sequence in \( W^{1,p}(\Omega) \), we consider \( E f_k \in W^{1,p}_0(\Omega') \), for some domain \( \Omega' \) containing \( \Omega \), i.e. \( Eu = u \) on \( \Omega \) and
\[
\|Ef_k\|_{W^{1,p}(\Omega')} \leq c\|f_k\|_{W^{1,p}(\Omega)}, \quad \text{for some constant } c \text{ depending on } \Omega.
\]

By (8), the sequence \( \{Ef_k\} \) is also bounded in \( W^{1,p}(\Omega) \).
- Hence, if \( p < n \), then (i) implies \( \{Ef_k\} \) contains a subsequence \( \{Ef_{k_\ell}\} \) which converges in \( L^{np/(n-p)}(\Omega') \). Since \( Ef_{k_\ell} = f_{k_\ell} \) in \( \Omega \), the sequence \( f_{k_\ell} \) converges in \( L^{np/(n-p)}(\Omega) \).
- By (ii.1), \( \{Ef_k\} \) contains a subsequence \( \{Ef_{k_\ell}\} \) which converges in \( L^2(\Omega') \). Since \( Ef_{k_\ell} = f_{k_\ell} \) in \( \Omega \), the sequence \( f_{k_\ell} \) converges in \( L^2(\Omega) \). □
IV. Poincaré Inequality: revisited.

- We have already proved that there exists a positive constant $C_p$ such that, for every $u \in W_0^{k,p}(\Omega)$,
  \[ \|u\|_{L^p(\Omega)} \leq C_p \|\nabla u\|_{L^p(\Omega)}. \] (9)

- On the other hand, (9) cannot hold if $u$ = constant.

- Roughly speaking, the hypotheses that guarantee the validity of (9) require that $u$ vanishes in some “nontrivial set”. For instance, under each of the following conditions, (9) holds:
  (i) $u \in W_0^{1,p}(\Omega)$; i.e. $u$ has zero trace on a nonempty relatively open subset $\Gamma_0 \subset \partial \Omega$;
  (ii) $u \in W^{1,2}(\Omega)$ and $u = 0$ on a set $E \subset \Gamma$ with positive measure $|E| = \alpha > 0$;
  (iii) $u \in W^{1,2}(\Omega)$ and $\int_{\Omega} u = 0$, i.e. $u$ has mean value zero in $\Omega$.

**Poincaré Inequality: revisited.** Let $\Omega$ be a bounded, Lipschitz domain. Assume that $u$ satisfies one of the hypotheses (i), (ii), (iii) above. Then, there exists $C_p$ such that (8) holds.

**Proof.** Assume that one of the hypotheses (i), (ii), (iii) holds. By contradiction suppose (9) is not true. This means that $\forall j \in \mathbb{N}$, $\exists u_j$ satisfies the same hypothesis such that
  \[ \|u_j\|_{L^p(\Omega)} > j \|\nabla u_j\|_{L^p(\Omega)}. \] (10)

Normalize $u_j$ in $L^p(\Omega)$ by setting
  \[ w_j = \frac{u_j}{\|u_j\|_{L^p(\Omega)}}. \]

Then, from (10),
  \[ \|w_j\|_{L^p(\Omega)} = 1 \quad \text{and} \quad \|\nabla w_j\|_{L^p(\Omega)} < \frac{1}{j} \leq 1. \]

Thus $\{w_j\}$ is bounded in $W^{1,p}(\Omega)$ and by Rellich’s theorem there exists a sequence $\{w_{jk}\}$ and $w$ also satisfies the same hypothesis such that
  \[ \begin{cases} w_j \rightarrow w \text{ strongly in } L^p(\Omega), \\ \nabla w_j \rightharpoonup \nabla w \text{ weakly in } L^p(\Omega). \end{cases} \]

The continuity of the norm gives
  \[ \|w\|_{L^p(\Omega)} = \lim_{j \rightarrow \infty} \|w_j\|_{L^p(\Omega)} = 1. \]

On the other hand, the weak semicontinuity of the norm yields
  \[ \|\nabla w\|_{L^p(\Omega)} \leq \liminf_{j \rightarrow \infty} \|\nabla w_j\|_{L^p(\Omega)} = 0 \]
so that $\nabla w = 0$. Since $\Omega$ is connected, $w$ is constant and since $w$ satisfies one of the hypotheses (i), (ii), (iii), we infer $w = 0$, in contradiction to $\|w\|_{L^p(\Omega)} = 1$. \(\square\)

**Corollary.** If $u \in W^{1,p}(\Omega)$, let
  \[ \frac{1}{|\Omega|} \int_{\Omega} u \, dx = u_\Omega. \]

Then
  \[ \|u - u_\Omega\|_{L^p(\Omega)} \leq C_p \|\nabla u\|_{L^p(\Omega)}. \]