Appendix

Proof of Lemma 1:

First, if \( y_s \) is the stock after trade in a steady state, then \( P(y_s) = f(y_s) \). The reason is that in the steady state we have \( t((1 - \mu)y_s) = y_s \). Hence, the consumer’s arbitrage equation \( f(y_s) - P(y_s) = \rho[f(y_s) - P(t((1 - \mu)y_s))] \) implies \( P(y_s) = f(y_s) \).

To prove that a steady state must exist, we first show that if \( P \) and \( f \) cross, then there always is a steady state. Then we establish that \( P \) and \( f \) necessarily cross.

We claim that if \( S = \{ q : P(q) = f(q) \} \) is non-empty then \( q' = \max S \) is a steady state. To see this, observe that \( q' \) is the maximum state to have acceptance price \( P(q') \). Indeed, if there existed \( q'' > q' \) with \( P(q'') = P(q') \), then from the definition of \( q' \) we must have \( f(q'') < f(q') \).

Furthermore, since \( t(\cdot) \) is non-decreasing, and since \( P(\cdot) \) is non-increasing, we have \( P(t((1 - \mu)q'')) \leq P(t((1 - \mu)q')) \). This implies the contradiction \( P(q'') = (1 - \rho)f(q'') + \rho P(t((1 - \mu)q'')) < (1 - \rho)f(q') + \rho P(t((1 - \mu)q')) = P(q') \). Hence following the offer \( P(q') \), all \( q \leq q' \) accept and all \( q > q' \) reject. Now from \( P(q') = (1 - \rho)f(q') + \rho P(t((1 - \mu)q')) \) and \( P(q') = f(q') \) we have \( P(t((1 - \mu)q')) = P(q') \) leads to a state after trade equal to \( q' \), i.e. \( t((1 - \mu)q') = q' \).

Suppose now that there is a stationary equilibrium which does not have any steady state. We claim that this implies \( P(q) < \overline{\pi} \) for \( q \in [0, \hat{q}] \) and \( P(q) > \underline{\pi} \) for \( q \in (\hat{q}, 1] \). To see this, note that since there is no steady state, it follows from the previous paragraph that the set \( S \) is empty, i.e. \( P(q) \neq \overline{\pi} \) for any \( q \in [0, \hat{q}] \) and \( P(q) \neq \underline{\pi} \) for any \( q \in (\hat{q}, 1] \). An argument similar to Fudenberg, Levine and Tirole (1985, Lemma 2) establishes that \( P(q) \geq \overline{\pi} \) for all \( q \in [0, 1] \). Hence we necessarily have \( P(q) > \underline{\pi} \) for all \( q \in (\hat{q}, 1] \). Furthermore, we cannot have \( P(q) > \overline{\pi} \) for some \( q \in [0, \hat{q}] \). Otherwise, since \( P(\cdot) \) is non-increasing, we would have \( P(0) > \overline{\pi} \). But then \( P(0) = (1 - \rho)\overline{\pi} + \rho P(t(0)) \) implies \( P(t(0)) > P(0) \). This is a contradiction, as \( t(0) \geq 0 \) and \( P(\cdot) \) is a non-increasing function.

Next, we show that \( P(q) < \overline{\pi} \) for \( q \in [0, \hat{q}] \) and \( P(q) > \underline{\pi} \) for \( q \in (\hat{q}, 1] \) imply that the total stock is increasing for \( q \in [0, \hat{q}] \) and decreasing for \( q \in (\hat{q}, 1] \), and that this yields a contradiction.

By the consumer’s arbitrage equation, we have \( \overline{\pi} - P(q) = \rho[\overline{\pi} - P(t((1 - \mu)q))] \) for all \( q \in [0, \hat{q}] \). Since \( \rho \in (0, 1) \) for all \( z > 0 \) and \( \overline{\pi} - P(q) > 0 \), we have \( \overline{\pi} - P(q) < \overline{\pi} - P(t(1 - \mu)q) \) which implies \( P(t((1 - \mu)q)) < P(q) \) for all \( q \in [0, \hat{q}] \). Since \( P(\cdot) \) is decreasing, we have \( t((1 - \mu)q) > q \) for all \( q \in [0, \hat{q}] \).

A similar argument also establishes that \( t((1 - \mu)(\hat{q} + \epsilon)) < \hat{q} + \epsilon \) for all \( \epsilon \in (0, 1 - \hat{q}] \). Hence,
lim_{ε→0} t((1 - μ)(q + ε)) ≤ q < t((1 - μ)q). Since T(·) is upper hemi-continuous, lim_{ε→0} t((1 - μ)(q + ε)) ∈ T((1 - μ)q). This contradicts the definition of t(·) = \min T(·).

Q.E.D.

Lemma 4 If f(q) = f(q') and P(q) = P(q') for some q < q' then q cannot be a steady state.

Proof: If q were a steady state, then when the state after depreciation is (1 - μ)q the monopolist will charge P(q). All buyers in \((1 - μ)q, q']\) will accept this, contradicting that t((1 - μ)q) = q. Q.E.D.

Proof of Lemma 2:

First, we show that y ∈ (q, 1) cannot be a steady state. Suppose not; then Lemma 1 implies that P(y) = \underline{v}. Since P(·) is a decreasing function, we have P(q) ≤ \underline{v} for q ≥ y. Similar to Fudenberg, Levine, and Tirole (1985), we have P(q) ≥ \underline{v} for all q ∈ [0, 1]. Hence, P(q) = \underline{v} for all q ∈ [y, 1]. Lemma 4 then shows that y cannot be a steady state.

To prove (ii), suppose that \hat{q} is a steady state so that P(\hat{q}) = \underline{v}. Then since P(·) is decreasing, and since (as shown in the proof of Lemma 1) P(y) ≤ \underline{v} for y ≤ \hat{q}, we have P(y) = \underline{v} for all y ∈ [0, \hat{q}]. Hence Lemma 4 shows that y < \hat{q} cannot be a steady state. We conclude that the only possible steady state other than \hat{q} is 1.

To prove (iii) suppose that 0 < y^* < \hat{q} is a steady state. Then we must have P(y) = \underline{v} for all y ∈ [0, y^*], so Lemma 4 implies that no y < y^* can be a steady state, and that P(q) < \underline{v} for q ∈ (y^*, \hat{q}) (otherwise y^* could not be a steady state). Thus by Lemma 1 no y ∈ (y^*, \hat{q}] can be a steady state, leaving y = 1 as the only other possible steady state. Suppose y = 1 were not a steady state, so P(1) > \underline{v}. This would imply P(q) > \underline{v} for all q ∈ (\hat{q}, 1]. We can then use the argument at the end of the proof of Lemma 1 to obtain a contradiction.

To prove (i), note that if there is no steady state in [0, \hat{q}), then by the first paragraph y = 1 is the only possible steady state. It then follows from Lemma 1 that y = 1 must be a steady state. Q.E.D.

Proof of Theorem 1:

We start by defining \bar{μ}. To this effect, let \bar{x}_0 = 1 - μ, \bar{x}_1 = (1 - μ)\bar{q}, \bar{x}_2 = \frac{\bar{v} - \bar{u}}{\underline{v} - \bar{u}}, and for k ≥ 3 let

\[ \bar{x}_k = (1 - μ)^{-1} \left( \bar{x}_{k-1} - (\bar{x}_{k-2} - \bar{x}_{k-1}) \frac{\bar{v}}{\rho^{k-2}(\underline{v} - \bar{u})} \right). \]  

(A-1)

Let Λ = \{μ ≥ 0 | \exists m < \infty s.t. \bar{x}_{m+1} < 0 ≤ \bar{x}_m and s.t. \{\bar{x}_k\}_{k=0}^m is a strictly decreasing sequence\}, and define \bar{μ} = \sup Λ. Note that 0 ∈ Λ and that Λ is open in \(\mathbb{R}_+\). It follows that \bar{μ} > 0. Now let
\( \mu_2 \) be the solution to \((1 - \mu)\hat{q} = \frac{\varphi - \nu}{\varphi - \frac{\nu}{2}} \). Then for \( \mu \geq \mu_2 \) we have \( \bar{x}_2 \geq \bar{x}_1 \), so \( \bar{\mu} \leq \mu_2 < 1 \).

First, we show that if \( \mu' \in \Lambda \) and \( \mu < \mu' \) then \( \mu \in \Lambda \), i.e. \( \Lambda = [0, \bar{\mu}] \). We will prove this by showing that \( \Delta_k = (\bar{x}_{k-1} - \bar{x}_k) \) is decreasing in \( \mu \). By definition we have \( \bar{x}_0 = 1 - \mu, \bar{x}_1 = (1 - \mu)\hat{q}, \bar{x}_2 = \frac{\varphi - \nu}{\varphi - \frac{\nu}{2}} \), and

\[
\bar{x}_k = (1 - \mu)^{-1}(\bar{x}_{k-1} - a_{k-2}\Delta_{k-1}),
\]

where \( a_k = \frac{\varphi}{\rho^{(\bar{\gamma} - \frac{\nu}{2})}} \). Thus \( \frac{d\Delta_k}{d\mu} < 0 \) for \( k \leq 2 \) and

\[
\frac{d\bar{x}_3}{d\mu} = (1 - \mu)^{-2}(a_1\bar{x}_1 + \bar{x}_2 - 2a_1\Delta_2) = (1 - \mu)^{-2}(a_1\bar{x}_1 - \bar{x}_2 + 2(1 - \mu)\bar{x}_3).
\]

Thus \( \frac{d\bar{x}_3}{d\mu} > 0 \) and \( \frac{d\Delta_k}{d\mu} < -2(1 - \mu)^{-1}\bar{x}_3 \). We now prove by induction that if \( \frac{d\bar{x}_{k-1}}{d\mu} > 0 \) and \( \frac{d\Delta_{k-1}}{d\mu} < -(k - 2)(1 - \mu)^{-1}\bar{x}_{k-1} \) then \( \frac{d\bar{x}_k}{d\mu} > 0 \) and \( \frac{d\Delta_k}{d\mu} < -(k - 1)(1 - \mu)^{-1}\bar{x}_k \). Indeed, \( \Delta_k = (1 - (1 - \mu)^{-1})\bar{x}_{k-1} + (1 - \mu)^{-1}a_{k-2}\Delta_{k-1} \), so

\[
\frac{d\Delta_k}{d\mu} < -(1 - \mu)^{-2}\bar{x}_{k-1} + (k - 1)(1 - \mu)^{-2}a_{k-2}\Delta_{k-1} + (1 - \mu)^{-1}a_{k-2}\frac{d\Delta_{k-1}}{d\mu}
\leq -(1 - \mu)^{-2}\left(\bar{x}_{k-1} - (k - 1)a_{k-2}\Delta_{k-1} + (k - 2)a_{k-2}\bar{x}_{k-1}\right)
\lesssim -(1 - \mu)^{-2}(k - 1)\left(\bar{x}_{k-1} - a_{k-2}\Delta_{k-1}\right)
\lesssim -(k - 1)(1 - \mu)^{-1}\bar{x}_k,
\]

where the third inequality follows from \( a_{k-2} > 1 \).

Next, we show that for \( \mu \in \Lambda \) there exists a stationary equilibrium with unique steady state \( y_s = 1 \). For \( \mu \in \Lambda \) define the triplet \( \{P(\cdot), R(\cdot), t(\cdot)\} \) as follows.

\[
P(q) = \begin{cases} 
p_m & \text{for } q \in [0, \bar{y}_m], 
p_k & \text{for } q \in (\bar{y}_{k+1}, \bar{y}_k) \text{ and } k \in \{m - 1, \ldots, 0\} 
\end{cases} \tag{A-2}
\]

\[
t(q) = \begin{cases} 
\bar{y}_{m-1} & \text{for } q \in [0, \bar{x}_m], 
\bar{y}_{k-1} & \text{for } q \in (\bar{x}_{k+1}, \bar{x}_k), k \in \{m - 1, \ldots, 2\} 
1 & \text{for } q \in (\bar{x}_2, 1 - \mu) 
\end{cases} \tag{A-3}
\]

\[
R(q) = \begin{cases} 
(\bar{y}_{m-1} - q)P(\bar{y}_{m-1}) + \delta R(\bar{x}_{m-1}) & \text{for } q \in [0, \bar{x}_m], 
(\bar{y}_{k-1} - q)P(\bar{y}_{k-1}) + \delta R(\bar{x}_{k-1}) & \text{for } q \in (\bar{x}_{k+1}, \bar{x}_k], k \in \{m - 1, \ldots, 2\} 
(1 - q)p + \frac{\delta \nu}{1 - \mu} & \text{for } q \in (\bar{x}_2, 1 - \mu) 
\end{cases} \tag{A-4}
\]

where \( p_k = \sigma - \rho^k(\sigma - \nu) \) and \( \bar{y}_k = \frac{\bar{x}_k}{1 - \mu} \), for \( k = 0, \ldots, m \). The sequence \( \{\bar{x}_k\}_{k=2}^{m+1} \) has been defined so that when the state is \( \bar{x}_k \) the monopolist is indifferent between selecting \( \bar{y}_{k-1} \) and \( \bar{y}_{k-2} \), i.e.
defining $\pi(y; x) = P(y)(y - x) + \delta R((1 - \mu)y)$ we have $\pi(\bar{y}_{k-1}; \bar{x}_k) = \pi(\bar{y}_{k-2}; \bar{x}_k)$. Let $h_k(x) = \pi(\bar{y}_{k-1}; x) - \pi(\bar{y}_{k-2}; x)$; then $h_k(\bar{x}_k) = 0$ and $\frac{\partial h_k}{\partial x} = -P(\bar{y}_{k-1}) + P(\bar{y}_{k-2}) < 0$. It follows that for $x < \bar{x}_k$ we have $\pi(\bar{y}_{k-1}; x) > \pi(\bar{y}_{k-2}; x)$, and for $x > \bar{x}_k$ we have $\pi(\bar{y}_{k-1}; x) < \pi(\bar{y}_{k-2}; x)$. Since $\pi(y; x)$ is strictly increasing in $y$ on any of the intervals $[0, \bar{y}_m], (\bar{y}_m, \bar{y}_{m-1}], \ldots, (\bar{y}_1, 1]$ it follows that that $t(x)$ is the smallest solution to (3). It is straightforward to verify that (4) is also satisfied, so \{P(\cdot), R(\cdot), t(\cdot)\} is a stationary triplet for $\mu \in \Lambda$.

Finally, consider any stationary equilibrium whose unique steady state is $y_s = 1$, and let \{P_0(\cdot), R_0(\cdot), t_0(\cdot)\} be the associated stationary triplet. We claim that \{P_0(\cdot), R_0(\cdot), t_0(\cdot)\} = \{P(\cdot), R(\cdot), t(\cdot)\}.

First, we show that $P_0(y) = P(y)$ for all $y \in (\hat{q}, 1]$. Indeed, since $\hat{q}$ is not a steady state, it must be that $P(\hat{q}) < \pi$. Otherwise, we would have $\pi(q) = \pi(q) = \pi$ for all $q \leq \hat{q}$. But as shown in the proof of Lemma 1, this would imply that $\hat{q}$ is a steady state. Now $P(\hat{q}) < \pi$ and the existence of $\varepsilon > 0$ s.t. $P(q) > \gamma$ for all $q \in (\hat{q}, \hat{q} + \varepsilon)$ would yield the same contradiction as in the last two paragraphs of the proof of Lemma 1. We conclude that $P_0(q) = \gamma$ for all $q \in (\hat{q}, 1]$.

Next, we claim that $t_0(\bar{x}_1) = 1$. Indeed, $P(\hat{q}) < \pi$ and (4) applied to $q = \hat{q}$ imply that $P_0(t_0(\bar{x}_1)) < P_0(\hat{q})$. Since $P_0(\cdot)$ is non-increasing, we therefore must have $t_0(\bar{x}_1) > \hat{q}$. Finally, since $P_0(q) = \gamma$ for all $q > \hat{q}$, (3) implies $t_0(\bar{x}_1) = 1$.

By the left-continuity of $t_0(\cdot)$, there exists $\varepsilon > 0$ s.t. $t_0(x) = 1$ for all $x \in (\hat{x}_1 - \varepsilon, \hat{x}_0]$. Furthermore, (4) implies $P_0(y) = \rho \pi + (1 - \rho)\gamma$ for $y \in (\hat{y}_1 - \varepsilon, \hat{y}_0]$. Now define $\bar{x}_2 = \inf\{x : t_0(x) = 1\}$. We claim that $\bar{x}_2 = \bar{x}_2$, so $P_0(\bar{x}_2 - \varepsilon) = P_0(\bar{x}_2 - \varepsilon)$ and $t_0(x) = t(x)$ for $x \in (\bar{x}_2, \bar{x}_1]$. To see this, recall that we shown above that $\pi(\bar{y}_1; x) \leq \pi(\bar{y}_0, x)$ for $x > \bar{x}_2$, and $\pi(\bar{y}_1; x) > \pi(\bar{y}_0, x)$ for $x < \bar{x}_2$. Since $\pi_0(\bar{y}_1; x) = \pi(\bar{y}_1; x)$ and $\pi_0(\bar{y}_0, x) = \pi(\bar{y}_0, x)$ we must have $\bar{x}_2 = \bar{x}_2$.

The same argument can now be applied inductively. Suppose (A-2)-(A-4) hold for all $i = 0, \ldots, k$. Then since we must have $P_0(\bar{y}_k) < \pi$, we know that $P_0(\bar{y}_k) > P_0(t_0(\bar{x}_k))$ and so $t_0(\bar{x}_k) > \bar{y}_{k+1}$. Since $\bar{x}_{k+1} = \inf\{x : t_0(x) = 1\}$ condition (A-3) implies that $t_0(\bar{x}_{k+1}) = \bar{y}_{k+1}$. Similarly to the previous paragraph, we can then use left-continuity of $t_0(\cdot)$ to define $\bar{x}_{k+2} = \inf\{x : t_0(x) = \bar{y}_k\}$, and conclude that $\bar{x}_{k+2} = \bar{x}_{k+2}$. Q.E.D.

**Proof of Corollary 1:**

By the definition of $\bar{x}_k$, for $k \geq 3$ we have $\lim_{z \to 0} \bar{x}_k = \lim_{z \to 0} (\bar{x}_{k-1} - (\bar{x}_{k-2} - \bar{x}_{k-1}) \frac{\hat{q} \gamma}{\gamma - \hat{q}})$. Hence, $\lim_{z \to 0} (\bar{x}_{k-1} - \bar{x}_k) > \lim_{z \to 0} (\bar{x}_{k-2} - \bar{x}_{k-1})$. By induction,

$$\lim_{z \to 0} (\bar{x}_{k-1} - \bar{x}_k) > \lim_{z \to 0} (\bar{x}_{k-2} - \bar{x}_{k-1}) > \hat{q} - \bar{x}_2 = \frac{(1 - \hat{q}) \gamma}{\gamma - \hat{q}}.$$
It follows that as \( z \) tends to zero, it takes the monopolist at most \( \hat{m} \leq \frac{\pi - \pi^*}{(1 - \gamma)\pi} + 1 \) steps to sell to all consumers.

**Proof of Theorem 2:**

Let \( \tilde{x}_0 = (1 - \mu)\tilde{q} \) and iteratively define the sequence \( \{\tilde{x}_k\}_{k=0}^{\infty} \) by \( \tilde{x}_k = \frac{\tilde{p}_{k-1} \tilde{x}_{k-1}}{\mu \tilde{p}_1 + (1 - \mu) \tilde{p}_{k-1}} \), where \( \tilde{p}_k = (1 - \rho^k)\tilde{p}_1 + \rho^k \tilde{p} \). Observe that \( \tilde{x}_0 < \tilde{x}_1 < \cdots \). If \( x_\infty = \lim_{k \to \infty} \tilde{x}_k \geq 1 - \mu \), then set \( m = \sup\{k : \tilde{x}_k < 1 - \mu\} \) and redefine \( \tilde{x}_{m+1} = 1 - \mu \). Let \( \tilde{y}_k = \frac{\tilde{x}_k}{1 - \mu} \), and define the triplet:

\[
P(q) = \begin{cases} 
\pi & \forall q \in [0, \tilde{q}] \\
\tilde{p}_k & \forall q \in (\tilde{y}_{k-1}, \tilde{y}_k] \text{ and } k = 1, \ldots, m + 1 
\end{cases}
\]

\[
t(q) = \begin{cases} 
\tilde{q} & \forall q \in [0, \tilde{x}_1] \\
\tilde{y}_{k-1} & \forall q \in (\tilde{x}_{k-1}, \tilde{x}_k] \text{ and } k = 2, \ldots, m + 1 
\end{cases}
\]

\[
R(q) = \begin{cases} 
\left(\frac{\delta q}{1 - \delta} + (\tilde{q} - q)\right)\pi & \forall q \in [0, \tilde{x}_1] \\
\left(\frac{\delta q}{1 - \delta} - (q - \tilde{x}_{k-1})\right)\tilde{p}_{k-1} & \forall q \in (\tilde{x}_{k-1}, \tilde{x}_k] \text{ and } k = 2, \ldots, m + 1 
\end{cases}
\]

If \( x_\infty = \lim_{k \to \infty} \tilde{x}_k < 1 - \mu \), define \( \tilde{R}(x) = \left(\frac{\mu \tilde{x}_{k-1}}{(1 - \delta)(1 - \mu)} - (x - \tilde{x}_{k-1})\right)\tilde{p}_{k-1} \), for \( x \in (\tilde{x}_{k-1}, \tilde{x}_k] \) and \( k \geq 1 \). Set \( x^* = \max\{x \in [(1 - \mu)\tilde{q}, x_\infty] : \tilde{R}(x) \geq R_1(x)\} \), where \( R_1(q) = \left(\frac{\delta q}{1 - \delta} + (1 - q)\right)\pi \), and let \( m \) be such that \( x^* \in (\tilde{x}_m, \tilde{x}_{m+1}] \). To see that \( m \) exists note that \( \tilde{R}(x_\infty) - R_1(x_\infty) = \left(\frac{\mu x^*}{(1 - \delta)(1 - \mu)} - (1 - x^*)\right)\pi < 0 \) whenever \( x_\infty < 1 - \mu \), so we have \( x^* < x_\infty \). Furthermore the definition of \( \mu \) implies that \( x^* \geq (1 - \mu)\tilde{q} \), with strict inequality when \( \mu > \frac{\mu}{\mu} \). Next, redefine \( \tilde{x}_{m+1} = x^* \), and define the triplet:

\[
P(q) = \begin{cases} 
\pi & \forall q \in [0, \tilde{q}] \\
\tilde{p}_k & \forall q \in (\tilde{y}_{k-1}, \tilde{y}_k] \text{ and } k = 1, \ldots, m + 1 \\
\tilde{q} & \forall q \in (\tilde{y}_{m+1}, 1] 
\end{cases}
\]

\[
t(q) = \begin{cases} 
\tilde{y}_{k-1} & \forall q \in (\tilde{x}_{k-1}, \tilde{x}_k] \text{ and } k = 2, \ldots, m + 1 \\
1 & \forall q \in (\tilde{x}_{m+1}, 1 - \mu] 
\end{cases}
\]

\[
R(q) = \begin{cases} 
\left(\frac{\delta q}{1 - \delta} + (\tilde{q} - q)\right)\pi & \forall q \in [0, \tilde{x}_1] \\
\left(\frac{\delta q}{1 - \delta} - (q - \tilde{x}_{k-1})\right)\tilde{p}_{k-1} & \forall q \in (\tilde{x}_{k-1}, \tilde{x}_k] \text{ and } k = 2, \ldots, m + 1 \\
\left(\frac{\delta q}{1 - \delta} + (1 - q)\right)\tilde{q} & \forall q \in (\tilde{x}_{m+1}, 1 - \mu] 
\end{cases}
\]

Let us now show that the stationary triplets defined above satisfy (3). We will consider the case \( x_\infty < 1 - \mu \); the proof for the case \( x_\infty \geq 1 - \mu \) is similar. Note that \( \pi(y; x) \) is strictly increasing in
y on any of the intervals \([0, \tilde{y}_0], (\tilde{y}_0, \tilde{y}_1], ..., (\tilde{y}_{m+1}, 1]\), so \(T(x) \subset \{\tilde{y}_0, ..., \tilde{y}_{m+1}, 1\}\). Furthermore, the sequence \(\{\tilde{x}_k\}\) is constructed such that when the state is \(\tilde{x}_k\) the monopolist is indifferent between selecting \(\tilde{y}_{k-1}\) and \(\tilde{y}_k\), i.e. \(\pi(\tilde{y}_{k-1}; \tilde{x}_k) = \pi(\tilde{y}_k; \tilde{x}_k)\). Let \(h_{m+1}(x) = \pi(1; x) - \pi(\tilde{y}_m; x)\), and for \(k \leq m\) let \(h_k(x) = \pi(\tilde{y}_k; x) - \pi(\tilde{y}_{k-1}; x)\). Then \(h_k(\tilde{x}_k) = 0\) for all \(k\), \(\frac{dh_{m+1}}{dx} = -(\nu - \tilde{p}_{k-1}) > 0\), and \(\frac{dh_k}{dx} = -(\tilde{p}_k - \tilde{p}_{k-1}) > 0\) for \(k \leq m\). Hence for all \(k\) we have \(h_k(x) \leq 0\) as \(x \leq \tilde{x}_k\). We conclude that \(t(x)\) solves (3). It is straightforward to verify that (4) is also satisfied, so \(\{P(\cdot), R(\cdot), t(\cdot)\}\) is a stationary triplet for \(\mu \geq \underline{\mu}\).

Finally, let \(\{P_0(\cdot), R_0(\cdot), t_0(\cdot)\}\) be the triplet associated with some stationary equilibrium having \(\hat{q}\) as a steady state. We will show that \(\{P_0, R_0, t_0\}\) is unique, and hence must equal \(\{P, R, t\}\). Define \(\bar{p}_1 = \sup_{q > \hat{q}} P_0(q)\).

Suppose first that \(\bar{p}_1 = \underline{\nu}\). Then \(P_0(q) = \nu\) for \(q \in (0, \hat{q}]\) and \(P_0(q) = \nu\) for \(q \in (\hat{q}, 1]\), so \(P_0(\cdot)\) is uniquely determined. Now for \(q > \hat{q}\) equation (4) implies \((1 - \rho)\nu + \rho P_0(t_0((1 - \mu)q)) = P_0(q) = \nu\) so we must have \(t_0(x) > \hat{q}\) for \(x > (1 - \mu)\hat{q}\). We conclude that \(t_0(x) = 1\) and \(R_0(x) = \left(\frac{\nu}{1 - \mu} + (1 - x)\right)\nu\) for \(x \in ((1 - \mu)\hat{q}, 1 - \mu]\). Furthermore, since \(\hat{q}\) is a steady state, we also have \(t_0(x) = \hat{q}\) and \(R_0(x) = \left(\frac{\nu}{1 - \mu} + (\hat{q} - \hat{q})\right)\nu\), for \(x \in [0, (1 - \mu)\hat{q}]\). Hence \(\{P_0, R_0, t_0\}\) is uniquely determined. Note that this case is characterized by \(\mu = \underline{\mu}\), for at the state \((1 - \mu)\hat{q}\) the monopolist must be indifferent between selecting \(y = \hat{q}\) and \(y = 1\).

Next, suppose that \(\bar{p}_1 > \underline{\nu}\). Then we claim that \(\bar{p}_1 = \hat{p}_1\). To prove this claim, we will show that there exists \(\varepsilon > 0\) s.t. \(t_0((1 - \mu)q) = \hat{q}\) for all \(q \in (\hat{q}, \hat{q} + \varepsilon]\). Observe first that \(t_0((1 - \mu)q) < q\) for all \(q \in (\hat{q}, 1]\) s.t. \(P_0(q) > \nu\). Indeed, \(t_0((1 - \mu)q) \geq q\) would imply that \(P_0(t_0((1 - \mu)q)) \leq P_0(q)\), and so (4) would yield \(P_0(q) = (1 - \rho)\nu + \rho P_0(t_0((1 - \mu)q)) \leq (1 - \rho)\nu + \rho P_0(q)\), or \(P_0(q) \leq \nu\). Now pick some \(q_0 \in (\hat{q}, 1]\) with \(P_0(q_0) > \nu\) and iteratively define \(q_k = t_0((1 - \mu)q_{k-1})\). If there existed no such \(\varepsilon\), then \(\{q_k\}\) would be a decreasing sequence bounded below by \(\hat{q}\). Then \(P_0(q_0) = (1 - \rho^k)\nu + \rho^k P(q_k)\) would yield \(\lim_{k \to \infty} P_0(q_0) = \nu\), contradicting that \(P_0(q_0) > \nu\).

Now let \(\bar{q}_1 = \max\{x \leq 1 - \mu : t_0(x) = \hat{q}\}\); we will show that \(\bar{q}_1 = \tilde{x}_1\). There will be three cases.

First, suppose that \(\bar{q}_1 = 1 - \mu\). Then \(P_0(q) = \bar{p}_1\) for all \(q \in (\hat{q}, 1]\), \(R_0(x) = \hat{R}(x)\) and \(t_0(x) = \hat{q}\) for all \(x \in [0, 1 - \mu]\), so \(\{P_0, R_0, t_0\}\) is uniquely determined. In this case we have \(\tilde{x}_1 = 1 - \mu\) and \(m = 0\).

Next, let \(\bar{q}_1 < 1 - \mu\). Define \(\bar{p}_2 = \sup\{P_0(q) | q > \frac{\bar{q}_1}{1 - \mu}\}\), and consider the case where \(\bar{p}_2 = \nu\). Then as above, we have \(P_0(q) = \nu\) for \(q \in \left(\frac{\bar{q}_1}{1 - \mu}, 1\right]\), \(R_0(x) = R_1(x)\) and \(t_0(x) = 1\) for \(x \in (\bar{q}_1, 1 - \mu]\). Since \(R_0(x) = \hat{R}(x)\) for all \(x < \bar{q}_1\), the continuity of \(R_0\) at \(x = \bar{q}_1\) then implies that \(\hat{R}(\bar{q}_1) = R_1(\bar{q}_1)\), i.e. \(\bar{q}_1 = x^* = \tilde{x}_1\).
Finally, let \( \tilde{q}_1 < 1 - \mu \) and \( \tilde{p}_2 > v \). Then we claim that \( \tilde{p}_2 = \tilde{p}_2 \). Indeed, as above we can show that there exists \( \varepsilon > 0 \) s.t. \( t_0(q) = \frac{\tilde{q}_1}{1 - \varepsilon} \) for all \( q \in (\tilde{q}_1, \tilde{q}_1 + \varepsilon] \), establishing the claim. Next, we show that \( \tilde{q}_1 = \tilde{x}_1 \). Define \( V(x) = \pi(\tilde{x}; x) - \frac{\mu p_x}{(1 - \mu)(1 - y)} \). Then since \( t_0(x) = \tilde{q} \) for \( x \leq \tilde{q}_1 \), and \( t_0(q) = \tilde{y}_1 \) for all \( q \in (\tilde{q}_1, \tilde{q}_1 + \varepsilon] \), we must have \( V(\tilde{q}_1) = 0 \). Furthermore, the definition of \( \tilde{x}_1 \) implies \( V(\tilde{x}_1) = 0 \). Since \( V'(x) < 0 \) we must have \( \tilde{q}_1 = \tilde{x}_1 \).

The same argument can now be applied inductively. Given \( \tilde{q}_k = \max\{x : t_0(x) = \tilde{q}_{k-1}\} \), it must be that \( \tilde{q}_k = \tilde{x}_k \). In the case where \( \tilde{q}_k = 1 - \mu \) we have \( \tilde{x}_k = 1 - \mu \) and \( m = k - 1 \). When \( \tilde{q}_k < 1 - \mu \) and \( \tilde{p}_{k+1} = v \) then \( \tilde{q}_k = x^* = \tilde{x}_k \) and \( m = k - 1 \). Finally, when \( \tilde{p}_{k+1} > v \) then \( m > k-1 \) and \( \tilde{q}_{k+1} = \max\{x : t_0(x) = \tilde{q}_k\} \).

We conclude that in each case \( \{P_0, R_0, t_0\} \) coincides with a single instance of (A-5) and (A-6). Since each \( \mu \) uniquely defines the sequence \( \{\tilde{x}_k\}_{k=0}^\infty \), there is at most one triplet \( \{P, R, t\} \), and hence at most one triplet \( \{P_0, R_0, t_0\} \) associated with any \( \mu \geq \underline{\mu} \).

**Proof of Corollary 2:**

By construction we have \( P(\tilde{y}_k) = \tilde{p}_k = (1 - \rho^k)v + \rho^k v \) and \( \tilde{y}_k = \frac{\tilde{p}_k - 1 \tilde{y}_{k-1}}{\rho (1 - \mu)} \). So
\[
\frac{P(\tilde{y}_k) - P(\tilde{y}_{k-1})}{\tilde{y}_k - \tilde{y}_{k-1}} = -\rho^{k-1}(1 - \rho)(v - \tilde{y}) \frac{1}{\tilde{y}_k - 1}(\tilde{p}_k - 1) \frac{1}{\rho (1 - \mu)} \sim \frac{1}{\mu \tilde{y}_k - 1}.
\]

Using \( p_{k-1} - v = \rho^{k-1}(v - \tilde{y}) \), this can be rewritten as
\[
\frac{P(\tilde{y}_k) - P(\tilde{y}_{k-1})}{\tilde{y}_k - \tilde{y}_{k-1}} = -\left(1 - \rho\right)\frac{\tilde{p}_{k-1} - \mu(\tilde{p}_{k-1} - v)}{\mu \tilde{y}_{k-1}}
\]

As \( z \) tends to zero, \( \tilde{y}_k - \tilde{y}_{k-1} \) converges to zero. Hence, the above equation converges to \( P'(y)y = -\frac{\lambda \tilde{y}}{y} P(y) \). Solving this differential equation gives \( P(y) = k_0 y^{1 - \frac{\lambda \tilde{y}}{y}} \). Since \( P(\tilde{q}) = v \), the particular solution has \( k_0 = \frac{v}{1 - \frac{\lambda \tilde{y}}{y}} \). Next, upon dividing both sides by \( z \) the equation \( \dot{\tilde{y}} = \frac{v}{1 - \frac{\lambda \tilde{y}}{y}} \) converges to \( \dot{y} = \lambda y \frac{P(y) - v}{P(y)} = \lambda y \frac{v - v}{y} = \lambda y (1 - \frac{v}{y}) \).

**Proof of Theorem 3:**

Suppose there is a stationary equilibrium with \( y^* < \tilde{q} \) as a steady state. Then by the proof of Lemma 2 we have \( P(y) = v \) for all \( y \leq y^* \), \( P(y) < v \) for \( y \in (y^*, \tilde{q}] \), and \( P(y) = v \) for \( y \in (\tilde{q}, 1] \).

This implies that over the interval \((y^*, 1)\) the stock is strictly increasing, i.e. \( t((1 - \mu)y) > y \) (see the argument at the end of the proof of Theorem 4). It follows that over the interval \((y^*, 1]\) the stationary triplet must coincide with the Coase Conjecture equilibrium triplet.

First, we show that the existence of a reputational equilibrium implies \( \mu \in [\underline{\mu}, \bar{\mu}] \). To establish \( \mu > \underline{\mu} \), observe that since \( P(y) = v \) for all \( y \leq y^* \) we must have \( t((1 - \mu)y) = y^* \) for all \( y \leq y^* \),
and hence $R((1 - \mu)y^*) = \frac{\mu y^* - \pi}{1 - \delta}$. Furthermore, continuity of the function $R$ implies that $y^*$ solves $\zeta(y) = 0$, where $\zeta(y) = \frac{\mu y}{1 - \delta} - R^C((1 - \mu)y)$ and $R^C(\cdot)$ is the Coase Conjecture profit function. Now the definition of $\mu$ implies $\zeta(\hat{q}) < 0$ for all $\mu < \hat{\mu}$. As $\zeta(\cdot)$ is strictly increasing, the existence of $y^* < \hat{q}$ therefore requires $\mu > \hat{\mu}$.

To establish that $\mu \leq \hat{\mu}$, we provide a proof by contradiction. Suppose that for some $\mu > \hat{\mu}$ a reputational equilibrium exists. Define $K = \min\{k : \bar{x}_j < \bar{x}_{j-1} \text{ for } j \leq k\}$. Since the Coase Conjecture equilibrium does not exist for $\mu > \hat{\mu}$, we necessarily have $\bar{x}_K > 0$. Furthermore, since $\bar{x}_{K+1} \geq \bar{x}_K$, when the initial state is $\bar{x}_K$ the monopolist prefers selling to $\bar{y}_K$ to selling to $\bar{y}_{K-1}$, i.e. $\mu \bar{y}_K + \delta R^C(\bar{x}_K) \geq R^C(\bar{x}_K)$ where $R^C(\bar{x}_K)$ is given by (A-4). This implies that $\zeta(\bar{y}_K) = \frac{\mu \bar{y}_K}{1 - \delta} - R^C(\bar{x}_K) > 0$; since $\zeta(\cdot)$ is strictly increasing we would have to have $y^* < \bar{y}_K$.

But this is impossible, as the construction (A-2)-(A-4) cannot be extended to initial states below $\bar{x}_K$.

Second, we show that a reputational equilibrium exists for any $\mu \in [\hat{\mu}, \bar{\mu}]$. For such $\mu$ we have $\zeta(\hat{q}) > 0$. Because $\zeta(0) = -R(0) < 0$, and $\zeta(\cdot)$ is strictly increasing, there exists a unique $y^* \in (0, \hat{q})$ such that $\zeta(y^*) = 0$. Now let $P(y) = \nu$ for $y \in [0, y^*]$, and $R(x) = (y^* - x)\nu + \delta \frac{\mu y}{1 - \delta}$ and $t(x) = y^*$ for $x \in [0, (1 - \mu)y^*]$, and let $\{P, R, t\}$ be given by (A-2)-(A-4) elsewhere. It is then easily checked that $\{P, R, t\}$ is a stationary triplet.

Next, we prove that $\hat{\mu} > \mu$ for all $\delta > 0$. We will establish that if $\mu \leq \hat{\mu}$ then the sequence $\{\bar{x}_k\}$ is strictly decreasing and $m$ is finite. The continuity of $\bar{x}_k$ in $\mu$ then implies that the same property holds in a right neighborhood of $\hat{\mu}$, so $\hat{\mu} > \mu$.

First, we show that $\bar{x}_2 < \bar{x}_1$ holds for any $\mu \leq \hat{\mu}$. Indeed, since $\mu \leq \hat{\mu}$ iff $\frac{\mu y}{1 - \delta} \leq \pi(\bar{y}_0; \bar{x}_1)$, we have $h_2(\bar{x}_1) = \pi(\bar{y}_1; \bar{x}_1) - \pi(\bar{y}_0; \bar{x}_1) = P(\bar{y}_1) \mu \bar{q} + \delta \pi(1; \bar{x}_1) - \delta(1; \bar{x}_1) \leq P(\bar{y}_1) \mu \bar{q} - (1 - \delta) \frac{\mu y}{1 - \delta} = -(\nu - P(\bar{y}_1)) \mu \bar{q} < 0$. Since $h_2(\bar{x}_2) = 0$ and $\frac{dh_2}{dx} < 0$ we must have $\bar{x}_2 < \bar{x}_1$.

Next, we show that there exists $\Delta > 0$ such that $\bar{x}_3 < \bar{x}_2 - \Delta$. Indeed, $h_3(\bar{x}_2) = \pi(\bar{y}_2; \bar{x}_2) - \pi(\bar{y}_1; \bar{x}_2) = P(\bar{y}_2) \frac{\mu x_2}{1 - \mu} - \delta \pi(\bar{y}_1; \bar{x}_2) - \pi(\bar{y}_0; \bar{x}_2) = P(\bar{y}_2) \frac{\mu x_2}{1 - \mu} - (1 - \delta) \pi(\bar{y}_0; \bar{x}_1) - (1 - \delta)(\bar{x}_1 - \bar{x}_2) \nu \leq P(\bar{y}_2) \frac{\mu x_2}{1 - \mu} - (1 - \delta) \frac{\mu y}{1 - \delta} - (1 - \delta)(\bar{x}_1 - \bar{x}_2) \nu < -(\nu - P(\bar{y}_2)) \frac{\mu x_2}{1 - \mu} - (1 - \delta)(\bar{x}_1 - \bar{x}_2) \nu < 0$. The first inequality follows because $\mu \leq \hat{\mu}$, the second one because $\nu < \bar{x}_1 = (1 - \mu) \hat{q}$. Since $h_3(\bar{x}_2) = -(\nu - P(\bar{y}_2)) + \delta R(1 - \mu)$ is independent of $x$, it follows that $\bar{x}_2 - \bar{x}_3 = \frac{h_3(\bar{x}_2)}{h_3'(\bar{x}_2)} > \frac{(1 - \delta)(\bar{x}_1 - \bar{x}_2) \nu}{P(\bar{y}_2) - P(\bar{y}_1)} = \Delta > 0$.

Finally, we establish that for each $k \geq 3$ we have $\bar{x}_{k-1} - \bar{x}_k \geq \Delta$, implying $m \leq 2 + \frac{1 - \bar{x}_2}{\epsilon}$. We have shown above that the result is true at $k = 3$. Next, we show that if the result for some $k \geq 3$ then it also holds for $k + 1$. Indeed, $\pi(\bar{y}_k - \bar{x}_k) \geq \pi(\bar{y}_0; \bar{x}_k) = (1 - \bar{x}_k) \nu + \delta R(1 - \mu) = \pi(\bar{y}_0; \bar{x}_1) + (\bar{x}_1 - \bar{x}_k) \nu$. Therefore, following the argument in the previous paragraph, we have
\[ h_{k+1}(\bar{x}_k) < -(1 - \delta)(\bar{x}_1 - \bar{x}_k)\psi_k < 0. \] Since \( h'_{k+1}(x) = -P(\bar{y}_k) + P(\bar{y}_{k-1}) \) is independent of \( x \), and since \( h'_{k+1}(\bar{x}_{k+1}) = 0 \) we obtain \( \bar{x}_k - \bar{x}_{k+1} = \frac{h_{k+1}(\bar{x}_k)}{h'_{k+1}} > \frac{(1-\delta)\bar{x}_1 - \bar{x}_k\psi_k}{P(\bar{y}_{k-1}) - P(\bar{y}_k)} > \Delta. \) Q.E.D.

**Proof of Lemma 3:**

Define \( \xi_k = \frac{\bar{x}_k}{\bar{x}_{k-1}}. \) It then follows from (A-1) that

\[
\xi_k = (1 - \mu)^{-1} \left( 1 - (\xi_{k-1} - 1)\delta^{-(k-2)}\psi_k \right), \tag{A-7}
\]

where \( \psi_k = \frac{\psi}{(1-\rho)^{k+2} - (\psi + \delta^{k-2})} > 1. \) Fix \( \mu \leq \bar{\mu}(\delta) \) so that \( \xi_k(\delta) < 1 \) for all \( k = 1, ..., m + 1 \). Next, observe that \( \xi_2 \) is independent of \( \delta \). Thus, (A-7) implies that \( \xi_3 \) is increasing in \( \delta \), i.e. \( \xi_3(\delta) < 1 \) implies \( \xi_4(\delta') < \xi_3(\delta) < 1 \) for all \( \delta' < \delta \). Next, suppose that \( \xi_{k-1}(\delta') < \xi_{k-1}(\delta) < 1 \); it then follows from (A-7) that \( \xi_k(\delta') < \xi_k(\delta) < 1. \) We conclude that if \( \mu \leq \bar{\mu}(\delta) \) then \( \mu < \bar{\mu}(\delta') \) for all \( \delta' < \delta \), so \( \bar{\mu} \) is a decreasing function of \( \delta \).

Next, we argue that \( \bar{\mu}(1) > 0. \) Indeed, when \( \delta = 1 \) and \( \mu = 0 \) the difference equation (A-1) reduces to

\[ \bar{x}_k - \bar{x}_{k-1} = (\bar{x}_{k-1} - \bar{x}_{k-2})\psi, \]

where \( \psi = \frac{\psi}{(1-\rho)^{k+2}} > 1. \) Hence \( \bar{x}_k - \bar{x}_{k-1} = \psi^{k-1}(\bar{x}_2 - \bar{x}_1) < 0. \) Let \( m \) be such that \( \bar{x}_m < 0 \leq \bar{x}_{m-1}. \) Then since \( \bar{x}_k \) is a continuous function of \( \mu \), it follows that there exists \( \mu' > 0 \) such that for all \( \mu \in [0, \mu'] \) we have \( \bar{x}_k - \bar{x}_{k-1} > 0 \) and \( \bar{x}_m < 0 \leq \bar{x}_{m-1}. \) Q.E.D.

**Proof of Theorem 4:**

For all \( i = 1, ..., N - 1 \), analogously to the two-step case, given \( q_i \) as a steady state construct a sequence \( \{\bar{x}^i_k\}_{k=0}^\infty \) to the right of \( q_i \) as follows. Let \( \bar{x}^i_0 = (1 - \mu) q_i \) and let \( \bar{P}^i(q) = (1-\rho) f(q) + \rho f(q_i) \) for \( q \geq q_i. \) Given \( \bar{P}^i(\cdot) \), let \( \bar{x}^i_1 \) be the highest state such that the seller prefers selecting \( y = q_i \) to selecting \( y \in (q_i, \bar{y}^i_1] \). For \( k \geq 2 \) let \( \bar{\ell}^i(q) \) be the seller’s optimal choice over \( (\bar{y}^i_{k-2}, \bar{y}^i_{k-1}] \) and redefine \( \bar{P}^i(q) = (1-\rho) f(q) + \rho f(\bar{\ell}^i(q)) \) for \( q \geq \bar{y}^i_{k-1}. \) Finally, let \( \bar{x}^i_k \) be the highest state such that the seller prefers selecting \( y \in (\bar{y}^i_{k-2}, \bar{y}^i_{k-1}] \) to selecting \( y \in (\bar{y}^i_{k-1}, \bar{y}^i_k]. \) Let \( \bar{R}^i(q) \) be the seller’s payoff function associated with \( \bar{P}^i(q). \)

Also analogously to the two-step case, given \( q_i \) as a steady state construct a sequence \( \{\bar{x}^i_k\}_{k=0}^\infty \) to the left of \( q_i \) as follows. Let \( \bar{x}^i_0 = (1 - \mu) q_i \) and let \( \bar{P}^i(q) = (1-\rho) f(q) + \rho f(q_i) \) for \( q \leq q_i. \) Given \( \bar{P}^i(\cdot) \), let \( \bar{x}^i_1 \) be the smallest state such that the seller prefers selecting \( y = q_i \) to selecting \( y \in (\bar{y}^i_1, q_{i-1} \). For \( k \geq 2 \) let \( \bar{\ell}^i(q) \) be the seller’s optimal choice over \( (\bar{y}^i_{k-1}, \bar{y}^i_k] \) and redefine \( \bar{P}^i(q) = (1-\rho) f(q) + \rho f(\bar{\ell}^i(q)) \) for \( q < \bar{y}^i_{k-1}. \) Finally, let \( \bar{x}^i_k \) be the smallest state such that the seller prefers selecting \( y \in (\bar{y}^i_{k-1}, \bar{y}^i_k] \) to selecting \( y \in (\bar{y}^i_{k}, \bar{y}^i_{k-1}] \). Let \( m^i \) be the largest value of \( k \).
such that $0 \leq \bar{x}_j < \bar{x}_{j-1}$ for all $j \leq k$, and let $\bar{R}^q$ be the seller's payoff function associated with $\bar{P}^q$.

Let $A_0 = \{i | \lim_{k \to \infty} \bar{x}_k^i \geq 1 - \mu \}$. If $A_0 = \emptyset$, let $q_{i_0} = 1$. Otherwise, let $i_0 = \min A_0$. Then define $P(q) = \bar{P}^{i_0}$ for $q > q_{i_0}$. Also let $R(q)$ and $t(q)$ be the equilibrium value function and policy function associated with $P(q)$.

(P) Now we will construct $P(q)$ for $q \leq q_{i_0}$. Define $i_1$ such that $\lim_{k \to \infty} \bar{x}_k^i \geq \lim_{k \to \infty} \bar{x}_k^i$ for all $i < i_0$. Note that $\lim_{k \to \infty} \bar{x}_k^i > (1 - \mu) q_{i_0-1}$ because $\lim_{k \to \infty} \bar{x}_k^i > (1 - \mu) q_{i_0-1}$. If $\bar{R}^{i_0} ((1 - \mu) q_{i_0-1}) < \bar{R}^{i_1} ((1 - \mu) q_{i_0-1})$, let $q'$ be such that $\bar{R}^{i_0} ((1 - \mu) q') = \bar{R}^{i_1} ((1 - \mu) q')$, and set $P(q) = \bar{P}^{i_0} (q)$ for $q \in (q', q_{i_0}]$ and $P(q) = \bar{P}^{i_1} (q)$ for $q \in (q_{i_0}, q')$. Now return to (P) with $i_1$ taking the role of $i_0$. (P') If $\bar{R}^{i_0} ((1 - \mu) q_{i_0-1}) \geq \bar{R}^{i_1} ((1 - \mu) q_{i_0-1})$ a proof similar to the one showing $\mu < \bar{\mu}$ in the two-step case yields $\bar{x}_{m_0} \leq q_{i_0-2}$. Hence, $\bar{P}^{i_0} (q)$ exists for $q \in (q_{i_0-2}, q_{i_0-1})$. Define $i_2$ such that $\lim_{k \to \infty} \bar{x}_k^i \geq \lim_{k \to \infty} \bar{x}_k^i$ for all $i < i_0 - 1$. If $\bar{R}^{i_0-1} ((1 - \mu) q_{i_0-2}) < \bar{R}^{i_2} ((1 - \mu) q_{i_0-2})$, let $q'$ be such that $\bar{R}^{i_0-1} ((1 - \mu) q') = \bar{R}^{i_2} ((1 - \mu) q')$, and set $P(q) = \bar{P}^{i_0-1} (q)$ for $q \in (q', q_{i_0-1}]$ and $P(q) = \bar{P}^{i_2} (q)$ for $q \in (q_{i_0}, q')$. Now return to (P) with $i_2$ taking the role of $i_0$. If $\bar{R}^{i_0-1} ((1 - \mu) q_{i_0-2}) \geq \bar{R}^{i_2} ((1 - \mu) q_{i_0-2})$ then return to (P') with $i_2$ taking the role of $i_1$ and $q_{i_0-2}$ taking the role of $q_{i_0-1}$. Q.E.D.

**Proof of Theorem 5:**

(i) Let $\pi^* = \max_{q \in [0, 1]} q f(q)$ be the monopoly profit, and suppose that $\pi^* > q_N f(q_N) = v_N$. Define $\mu$ as the solution to the equation $f(\mu) = 0$, where $f(\mu) = \frac{\mu \pi^*}{1 - \mu} - (1 - (1 - \mu) q_{N-1}) v_N - \frac{\delta q_N}{1 - \delta}$. Then since $f(1) = \frac{\pi^* - q_N}{1 - \delta} > 0$ and since $f(0) = -(1 - q_{N-1}) v_N < 0$, we have $0 < \mu < 1$.

Furthermore, note that $y \in [0, q_{N-1}]$ and $\mu < \mu$ imply $\frac{\mu f(y)}{1 - \delta} < (1 - (1 - \mu) y) v_N + \frac{\delta q_N}{1 - \delta}$. Suppose that $\mu < \mu$ and that there existed a stationary equilibrium with steady state $y_s \in [0, q_{N-1}]$. Let $\hat{y}$ be the largest such steady state. Then when the state is $q = (1 - \mu) \hat{y}$ the monopolist would earn $\frac{\mu f(\hat{y})}{1 - \delta}$. If instead the monopolist were to select $y = 1$ and remain there forever after, she would earn $(1 - (1 - \mu) \hat{y}) v_N + \frac{\delta q_N}{1 - \delta}$, which exceeds $\frac{\mu f(\hat{y})}{1 - \delta}$ for $\mu < \mu$. We conclude that $y = 1$ is the unique steady state when $\mu < \mu$. Thus the Coase Conjecture equilibrium is the unique equilibrium.

(ii) Given a stationary triplet, we say that a steady state $y_s$ is reachable from $(1 - \mu) q^*$ if $\lim_{n \to \infty} q^n ((1 - \mu) q^*) = y_s$, where $g(x) = (1 - \mu) t(x)$. Note that a reachable steady state from $(1 - \mu) q^*$ is the smallest steady state above $q^*$. Define $q^*_1 = \min \arg \max_{q > q^*} q^* f(q_i)$.

First we claim that there exists $\bar{\mu}_1 < 1$, such that $q^*_1$ cannot be a reachable steady state from $(1 - \mu) q^*$ for any $\mu > \bar{\mu}$. Suppose not; let $P(q)$ be the acceptance function associated with some
stationary equilibrium having \( q_1^* \) as a reachable steady state. Then for \( x \in \left[ (1 - \mu) q^*, (1 - \mu) q_1^* \right] \), it must be the case that \( t(x) \notin (q^*, q_1^*) \). Indeed, let \( q^* = q_{i-K} < \ldots < q_i = q_1^* \), and suppose that \( K > 1 \). Analogously to the two-step case construct a sequence \( \{\tilde{x}_k\}_{k=0}^{\infty} \) starting at the steady state \( q_1^* \), i.e. \( \tilde{x}_0 = (1 - \mu) q_1^* \). First we show that \( t(x) \neq q_{i-1} \). If \( t(x) = q_{i-1} \) then by the definition of \( q_1^* = q_i \), we would have \( \tilde{x}_1 = (1 - \mu) q_{i-1} \) and \( \tilde{x}_2 = \frac{q_{i-1} - q_i}{v_{i-1}} \leq 0 \), yielding a contradiction. Let \( t(x) = q_{i-1} \). Next, for \( K > 2 \) we show that \( t(x) \neq q_{i-2} \). Since \( q_1^* \) is a reachable steady state, we must have \( t((1 - \mu) q) > q \) for \( q < q_1^* \). Thus \( t((1 - \mu) q_{i-2}) > q_{i-2} \). Since \( t((1 - \mu) q_{i-2}) \neq q_{i-1} \), we have \( t((1 - \mu) q_{i-2}) = q_i \) and \( P ((1 - \mu) q_{i-2}) = (1 - \rho) v_{i-2} + \rho v_i \). If \( t(x) = q_{i-2} \) then we would have \( \tilde{x}_1 = (1 - \mu) q_{i-2} \) and \( \tilde{x}_2 = \frac{q_{i-2} - q_i}{v_{i-2} - v_i} \leq 0 \), a contradiction. Using a similar argument we conclude that \( t(x) \notin \{q_{i-K+1}, \ldots, q_{i-1}\} \). Hence, \( P ((1 - \mu) q^*) = (1 - \rho) v_{i-K} + \rho v_i \). Define \( \tilde{\mu}_1 \) as the solution to \( \tilde{x}_2 = \frac{q_{i-K} v_{i-K} - q_i v_i}{v_{i-K} - v_i} = (1 - \tilde{\mu}_1) q_{i-K} \). Note that \( \tilde{\mu}_1 = \frac{q_i - q^* f(q^*)}{q^* f(q^*) - f(q_i^*)} > 1 \). Then for \( \mu > \tilde{\mu}_1 \), \( (1 - \mu) q_{i-K} < \tilde{x}_2 \) and \( t((1 - \mu) q_{i-K}) \leq t(\tilde{x}_2) \leq q_{i-K} \), contradicting that \( q_1^* \) is a reachable steady state from \( (1 - \mu) q^* \).

Secondly, given \( q_1^*, \ldots, q_{i-1}^* \), define \( q_k^* = \min \{ q_i f(q_i) : q_i > q^* \text{ and } q_i \notin \{q^*, q_1^*, \ldots, q_{i-1}^*\} \} \) and let \( q' = \max \{ q_i < q_k^* : q_i f(q_i) > q_k^* f(q_k^*) \} \). Mimicking the argument above, we can show for any \( x \in \left[ (1 - \mu) q', (1 - \mu) q_k^* \right] \), it must be the case that \( t(x) \notin \{q', q_k^* \} \). Letting \( \tilde{\mu}_k = \frac{(q' - q_k^*) f(q')}{q^* f(q^*) - f(q_k^*)} < 1 \), it follows that for any \( \mu > \tilde{\mu}_k \), \( q_k^* \) is not a reachable steady state from \( (1 - \mu) q^* \).

Since the inductive step stops after finitely many iterations, we conclude that no \( q_i > q^* \) can be a reachable steady state from \( (1 - \mu) q^* \) when \( \mu > \tilde{\mu} = \max \{ \tilde{\mu}_k \} \). Therefore \( t((1 - \mu) q^*) \leq q^* \), implying \( P (q^*) \geq f(q^*) \) and \( R(0) \geq q^* f(q^*) + \frac{\delta q^* f(q^*)}{1 - \delta} \). Since the outcome path of any stationary equilibrium can always be duplicated in a rental equilibrium by selecting an appropriating sequence of rentals, and since the rental profit is uniquely maximized by charging \( f(q^*) \), we also must have \( R(0) \leq q^* f(q^*) + \frac{\delta q^* f(q^*)}{1 - \delta} \). We conclude that \( R(0) = q^* f(q^*) + \frac{\delta q^* f(q^*)}{1 - \delta} \) and that \( t(q) = q^* \) for all \( q \leq (1 - \mu) q^* \). Thus in any stationary equilibrium \( q^* \) is the unique steady state reachable from 0.

(iii) We start by constructing \( \mu_L \in (\mu, \tilde{\mu}) \) such that at \( \mu_L \) there exists a reputational equilibrium that results in the monopoly outcome (from \( q = 0 \)). For this purpose, define the fictitious demand curve \( \tilde{f}(q) = \min \{ f(q), f(q_{k+1}) \} \), where \( q_k = q^* \) for some \( k < N \). By Theorem 4 there exists a stationary equilibrium for the demand curve \( \tilde{f} \). Let \( g(\mu) \) denote the profit in this equilibrium when the initial state equals \((1 - \mu)q^* \). Note that since the equilibrium constructed in Theorem 4 is the most profitable one, \( g(\mu) \) can only jump up as \( \mu \) increases, and hence is upper semicontinuous and continuous from the right. Additionally, let \( h(\mu) = \frac{\mu q^* f(q^*)}{1 - \delta} \) be the profit from serving replacement.

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demand when monopoly is a steady state. Note that \( h(\cdot) \) is continuous in \( \mu \), and that \( h(0) < g(0) \) and \( h(1) > g(1) \), so \( \text{Im}(g) \subset \text{Im}(h) \). Define \( r(\mu) = h^{-1}(g(\mu)) \). Then \( r : [0, 1] \to [0, 1] \), and so by the Lemma in Roberts and Sonnenschein (1976) has a fixed point \( \mu_L \). Furthermore, it follows from right continuity that \( g(\mu) < h(\mu) \) in a right neighborhood of \( \mu_L \).

We now claim that when the demand curve is \( f(\cdot) \) and \( \mu = \mu_L \) there exists a stationary equilibrium in which the monopolist earns \( \frac{\mu_L q^* f(q^*)}{1-\delta} \) when the state is \( (1-\mu)q^* \). Indeed, let \( P(q) = (1-\rho)f(q) + \rho f(q^*) \) for \( q \leq q^* \), and for \( q > q^* \) let \( P(q) \) be given by the one computed in Theorem 4 for the demand curve \( \tilde{f}(\cdot) \). Then at \( q = (1-\mu)q^* \) the monopolist is indifferent between staying at \( q^* \) forever and moving beyond \( q^* \), thereby earning the continuation profit \( g(\mu_L) = h(\mu_L) = \frac{\mu_L q^* f(q^*)}{1-\delta} \). Meanwhile, for \( q > (1-\mu)q^* \) the profits from going to \( q^* \) and remaining there ever after fall below the profits from going forward (see the argument following (A-4)) in the proof of Theorem 1, so the equilibrium is a reputational one.

Next, we show that there exists \( \mu_H > \mu_L \) such that for every \( \mu \in (\mu_L, \mu_H) \) there exists a reputational equilibrium whose smallest steady state satisfies \( y^* < q^* \). Let \( \{P_0(\cdot), R_0(\cdot), t_0(\cdot)\} \) denote the stationary equilibrium constructed in the proof of Theorem 4 when the demand function is \( \tilde{f}(\cdot) \). For all \( q \in [0, 1-\mu] \) define \( R_1(q) = \max_{y > q^*} \{P_0(y)(y-q) + \delta R_0((1-\mu)y)\} \), the maximal profit attainable when the demand curve is \( f(\cdot) \), the state is \( q \), the monopolist is constrained to select a state \( y > q^* \), and the equilibrium \( \{P_0(\cdot), R_0(\cdot), t_0(\cdot)\} \) is played thereafter. Let \( t_0(q) \) denote the minimum element of the corresponding argmax, and let \( P_1(q) = (1-\rho)f(q) + \rho P_0(t_1((1-\mu)q) \). Let \( R_2(q) = \max_{y \leq q^*} \{P_1(y)(y-q) + \delta R_1((1-\mu)y)\} \) the maximal profit when the acceptance function is \( P_1 \), and monopolist is constrained to select \( y \in [0, q^*] \), and let \( t_2(q) \) be the corresponding policy function. Finally, let \( \bar{y} = \max\{y \in [0, q^*] : R_2(q) \geq R_1(q)\} \). Then provided \( \bar{y} < q^* \), the triple \( \{P_1(\cdot), R_1(\cdot), t_1(\cdot)\} \) defines a stationary equilibrium for \( q > \bar{y} \) (see Deneckere and Liang, 2006).

We now claim that there exists \( \mu_H > \mu_L \) and \( \bar{y} \in (q_{k-1}, q^*) \) such that \( \bar{y}(\mu) \leq \bar{y} \) for all \( \mu \in (\mu_L, \mu_H) \). Let \( p^* = (1-\rho)f(q^*) + \rho f(q_{k+1}) \); note that \( p^* \) is an upper bound to \( P_1(q) \) for all \( q > q_{k-1} \). Consequently, \( H(q) = (q^* - (1-\mu)q)p^* + \delta R_1((1-\mu)q^*) \geq R_2(q) \). Observe now that at \( \mu = \mu_L \) we have \( R_1((1-\mu)L)q^* = R_0((1-\mu)L)q^* = g(\mu_L) = h(\mu_L) = \frac{\mu_L q^* f(q^*)}{1-\delta} = H(q^* + \mu_L q^* f(q^*) > H(q^*) \). Since \( R_1 \) and \( H \) are continuous in \( \mu \), there exists \( \mu_H > \mu_L \) and \( \bar{y} \in (q_{k-1}, q^*) \) such that \( R_1(\bar{y}) \geq H(\bar{y}) \geq R_2(\bar{y}) \), and hence \( \bar{y} \leq \bar{y} \) for all \( \mu \in (\mu_L, \mu_H) \).

Finally, we establish that there exists \( \mu_H \in (\mu_L, \mu_H) \) such that \( \frac{\mu_H f(\bar{y})}{1-\delta} < R_1((1-\mu)\bar{y}) \) for all \( \mu \in (\mu_L, \mu_H) \). Indeed, since \( R_1((1-\mu)L)q^* = \frac{\mu_L q^* f(q^*)}{1-\delta} \), and since \( R_1(q) \) and \( \frac{\mu_L f(q^*)}{1-\delta} \) are decreasing and increasing functions of \( q \), respectively, we have \( \frac{\mu_H f(\bar{y})}{1-\delta} < R_1((1-\mu)\bar{y}) \). The existence of \( \mu_H \)
then follows from the continuity of each of these functions in \( \mu \).

We conclude that for every \( \mu \in (\mu_L, \mu_H) \) there exists a stationary equilibrium \( \{ P_1(\cdot), R_1(\cdot), t_1(\cdot) \} \) for states above \( \bar{y} \) such that \( R_1((1-\mu)\bar{y}) > \frac{\mu \bar{y} f(\bar{y})}{1-\delta} \) and \( R_1((1-\mu)q^*) < \frac{\mu q^* f(q^*)}{1-\delta} \), so there exists \( y^* \in (\bar{y}, q^*) \) for which \( R_1((1-\mu)y^*) = \frac{\mu y^* f(y^*)}{1-\delta} \). The acceptance function \( P(q) = (1-\rho)f(q) + \rho f(y^*) \) for \( q \leq y^* \) and \( P(q) = P_1(q) \) for \( q > y^* \) then defines a reputational equilibrium in which \( t(q) = y^* \) for all \( q \leq (1-\mu)y^* \) and \( t(q) = t_1(q) \) for \( q > (1-\mu)y^* \).

(iv) Consider any stationary equilibrium with profit function \( R \). Since \( q^* \) cannot be a steady state, we must have \( \frac{\mu q^* f(q^*)}{1-\delta} < R ((1-\mu)q^*) \). For \( q < q^* \), we have \( \frac{\mu q^* f(q^*)}{1-\delta} < \frac{\mu^* f(q^*)}{1-\delta} < R ((1-\mu)q^*) < R ((1-\mu)q) \). Therefore no \( q < q^* \) can be a steady state. \( Q.E.D. \)