Characters of p-adic groups

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Contents

0. Notations throughout the course

Let p be a prime number. Let $\mathbb{Q}_p$ be the p-adic completion of $\mathbb{Q}$. Let F be a finite extension of $\mathbb{Q}_p$. We will denote by $\mathcal{O} = \mathcal{O}_F$ the ring of integers in F, by $\mathfrak{m} = \mathfrak{m}_F \lhd \mathcal{O}_F$ the maximal ideal and by $k = k_F := \mathcal{O}_F/\mathfrak{m}_F$ the residue field. We denote by $q := \#k$. Sometimes we take an algebraic closure $\bar{F}/F$. Let $\mathbb{G}$ be a connected reductive linear algebraic group over F (see §2 for reviews on these). We will denote the identity element(s) by $e \in \mathbb{G}$, and $e \in G$, and in any group. Let $\mathcal{C}$ be any field of characteristic **different** from p. More assumptions on $\mathcal{C}$ will be added in individual sections, and for a quick reading it might be convenient to assume that $\mathcal{C}$ is algebraically closed.

These notations are (hopefully) fixed for the rest of the semester, although in examples we might take specific $\mathbb{G}$, p or $\mathcal{C}$.

Let $G = \mathbb{G}(F)$, in particular a p-adic Lie group. We will often call such G a p-adic reductive group. We write $\mathfrak{g} := T_eG = (T_e\mathbb{G})(F)$ the Lie algebra. It has a Lie bracket map

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

see §2 for a review on this. We will denote by $\mathfrak{g}^* := \mathrm{Hom}_F(\mathfrak{g}, F)$ the linear dual space of $\mathfrak{g}$. There is an adjoint action $\mathrm{Ad} : G \to \mathrm{End}_F(\mathfrak{g})$. We denote by Ad^* the composition $\mathrm{Ad}^* : G \to \mathrm{End}(\mathfrak{g}) \cong \mathrm{End}_F(\mathfrak{g}^*)$.

1. Introduction

To begin with, most of the material in this course originates from [HC99], with partial revisions in [Kot05], and with further adaptations of our own, partly inspired by [VW01]. We thank Stephen DeBacker for extremely helpful discussions on the subject. We thank Gemini and ChatGPT for providing references.

Let us begin with a finite group H. We consider a complex representation $\pi \in \mathrm{Rep}(H)$, i.e. there is an underlying complex vector space V, and $\pi : H \to \mathrm{Aut}_{\mathcal{C}}(V)$ is a group homomorphism. It is known that if π is irreducible then it is finite-dimensional. If it is finite-dimensional, we can define its character

$$\begin{array}{rccc} \Theta_\pi : & H & \to & \mathbb{C} \\ & h & \mapsto & \mathrm{tr}(\pi(h)). \end{array}$$

Characters Θ_π for isomorphism classes of irreducible representations (π, V) form an orthonormal basis of over H. Every finite-dimensional representation (π, V) of H is the direct sum of a finite number of irreducible representations, and therefore the isomorphism class of (π, V) is determined by Θ_π. Similar statement is true if we replace "finite group" by "compact Hausdorff topological group," and "representation" by "continuous representation."

Now we begin with a p-adic reductive group $G = \mathbb{G}(F)$; if you have not seen them, you are encourage to think of the example

$$G = \mathrm{GL}_n(\mathbb{Q}_p) = \{g \text{ is an } n \times n \text{ matrix in } \mathbb{Q}_p \mid \det(g) \neq 0\}.$$

equipped with the topology inherited from $(\mathbb{Q}_p)^{n^2}$. In particular G is (always) a totally disconnected locally compact Hausdorff topological group. We would like a representation theory of G similar to that of compact Hausdorff groups. The colloquial way of saying it is that

p-adic (reductive) groups try very hard to behave like finite (reductive) groups.

In some sense the goal of the course is to address "How?" from the point of view of characters. To begin with, let us ask what kind of continuity is reasonable. Any neighborhood of $\mathrm{id} \in G$ contains an open (pro-p) subgroup of G. Using it we can prove

Exercise 1. *Let H be a p-adic Lie group. Let $\mathcal{C} = \mathbb{C}$ or $\mathcal{C} = \overline{\mathbb{Q}}_\ell$ with its usual topology. Let n be an integer and $\rho : H \to \mathrm{GL}_n(\mathcal{C})$ be a continuous group homomorphism. Show that $\ker(\rho)$ contains an open subgroup.*

For the rest of the introduction, $\mathcal{C}$ is a fixed algebraically closed field of characteristic different from p.

Definition 1.1. *Consider a $\mathcal{C}$-representation (π, V) (i.e. V is a vector space over $\mathcal{C}$, often infinite-dimensional) of a p-adic Lie group H. A vector $v \in V$ is* **smooth** *if its stabilizer $\mathrm{Stab}_H(v) \subset H$ is open. A representation is called* **smooth** *if every vector is smooth.*

From now on, unless otherwise stated, representations will always be smooth $\mathcal{C}$-representations of G. All maps on $\mathcal{C}$-representations are assumed to be $\mathcal{C}$-linear.

Definition 1.2. *A smooth representation (π, V) is called* **admissible** *if $\dim_{\mathcal{C}} \pi^J < +\infty$ for any open subgroup $J \subset G$.*

The next theorem is a big one. We will not attempt to prove it.

Theorem 1.3. *(Jacquet, Vignéras) Every irreducible smooth representation of G is admissible.*

Most of this course will be about irreducible admissible representations. Because of Theorem 1.3, they are the same as irreducible smooth representations, and thus are natural objects of study. Maybe this is a good time to pause, and ask why we care about representations of G. I can think of a few reasons:

(1) A p-adic reductive group G acts on automorphic forms, with coefficient ring $\mathcal{C} = \mathbb{C}$, $\mathbb{Q}_\ell$, $\mathbb{F}_\ell$, $\mathbb{Z}_\ell$, etc.. Such automorphic forms live in representations that can be decomposed into irreducible admissible representations of (various) G.
(2) There is a subject, of fundamental importance to mathematics, called Langlands program. It relates properties about representations of G to structures in number theory, algebraic geometry, and mathematical physics.
(3) Representation theory of G are connected to that of affine Hecke algebras and double affine Hecke algebras, of quantum groups, and to the geometry of affine Grassmannians and various related moduli spaces.
(4) Probably because of the above reasons, representation theory of p-adic reductive groups is itself rich and fun.

Despite so many fancy stuff mentioned, this course will be content to develop a character theory for a p-adic reductive group G, due to Harish-Chandra and several others. This is similar to the basic character theory of finite groups developed in the 1900's by Frobenius and others; a foundation for characters of irreducible representations is provided, but without a method to actually determine the characters. In particular we will see very little (if any) connection to the fancy stuff.

Let's go back to characters. Already in the case of compact groups, e.g. $H = \mathbb{Z}_p$, to say that characters form an orthonormal basis of functions we need to choose a measure on the group.

Definition 1.4. *We denote by $\Omega(G)$ be the set of open compact subgroups of G. We consider M_G the set of functions $\nu : \Omega(G) \to \mathbb{Z}[1/p]$ such that*

$$J, J' \in \Omega(G),\ J \subset J' \implies \nu(J') = [J' : J] \cdot \nu(J).$$

Lemma 1.5. *M_G is a free module of rank 1 over $\mathbb{Z}[1/p]$.*

Exercise 2. *Show that M_G is indeed a free module of rank 1 over $\mathbb{Z}[1/p]$. Show that any open compact subset $X \subset G$ can be written as a finite union of J-cosets*

$X = \bigsqcup_{i=1}^m g_iJ$ *for some* $J \in \Omega(G)$*. Show that the quantity* $\nu(X) := m \cdot \nu(J) \in \mathbb{Z}[1/p]$ *depends only on* X *and* ν *but not on the choice of* g_i *and* J*.*

Let us fix $\nu \in M_G$ so that $M_G = \mathrm{span}_{\mathbb{Z}[1/p]}(\nu)$, thought of our measure on G. It will be clear that what we do below do not depend on the choice. We will discuss some important property of the measures, mostly that $\nu(gJg^{-1}) = \nu(J)$, in later sections. Let X be any topological space. The following definition is most often applied to $X = G$.

Definition 1.6. *Denote by* $\mathcal{C}^\infty(X)$ *the space of locally constant* $\mathcal{C}$*-valued functions on* X*, and by* $\mathcal{C}_c^\infty(X) \subset \mathcal{C}^\infty(X)$ *the space of compactly supported locally constant* $\mathcal{C}$*-valued functions on* X*.*

Definition 1.7. *The space of* **locally constant compactly supported measures***, sometimes just* **test measures***, on* G *is*

$$M_c^\infty(G) := \mathcal{C}_c^\infty(G) \otimes_{\mathbb{Z}[1/p]} M_G = \mathcal{C}_c^\infty(G) \otimes_{\mathcal{C}} (\mathcal{C} \otimes_{\mathbb{Z}[1/p]} M_G).$$

For $\mu = f \otimes \nu$ *where* $f \in \mathcal{C}_c^\infty(G)$*, we will talk about left, right and conjugation action of* G *on* μ *via that on* f*.*

Remark 1.8. As the name suggest, any $\mu = f \otimes \nu \in M_c^\infty(G)$ is sort of a measure; for any $f' \in \mathcal{C}^\infty(G)$, we define

$$\mu(f') := \left(\sum_{g_i \in G/J} (ff')(g_i) \right) \cdot \nu(J) \in \mathcal{C} \tag{1}$$

where $J \in \Omega(G)$ is such that ff' is right J-invariant. The quantity $\mu(f')$ is independent of the choice of J.

Let π be a smooth representation. Whenever $X \subset G$ is open compact, we denote by $1_X \in \mathcal{C}_c^\infty(G)$ the function that takes the value 1 on X and 0 elsewhere. Consider any $\mu \in M_c^\infty(G)$ such that

$$\mu = \left(\sum_{i=1}^m c_i 1_{g_iJ} \right) \otimes \nu$$

for some $J \in \Omega(G)$, $g_i \in G$ and $c_i \in \mathcal{C}$. We define

$$\pi(\mu)_J : \pi^J \to \pi, \ \ \pi(\mu)_J v := \int_G \pi(g)v \cdot \mu(g) := \left(\sum_{i=1}^m c_i \pi(g_i) v \right) \cdot \mu(J)$$

One verifies that when $J \subset J'$, we have $\pi(\mu)_J|_{\pi^{J'}} = \pi(\mu)_{J'}$. Hence the maps can be glued to be $\pi(\mu) : \pi \to \pi$. For $g \in G$, denote by $l_g(f) \in \mathcal{C}_c^\infty(G)$ given by $(l_g(f))(h) := f(g^{-1}h)$, and $l_g(\mu) = l_g(f) \otimes \nu$. Then we have

$$l_g(f) = \sum_{i=1}^m c_i 1_{gg_iJ} \implies \pi(l_g(\mu)) = \pi(g) \circ \pi(\mu) \in \mathrm{End}(\pi)$$

Observe that there exists $J_\mu \in \Omega(G)$ such that $\mu = l_g(\mu)$ for any $g \in J_\mu$. Then $\pi(g)\pi(\mu)v = \pi(\mu)v$ for any $g \in J_\mu$, i.e. $\pi(\mu)v \in \pi^{J_\mu}$. Let us highlight

Lemma 1.9. *Suppose π is admissible. Then $\dim_{\mathcal{C}} \mathrm{im}(\pi(\mu)) < +\infty$ for any $\mu \in M_c^\infty(G)$.*

Definition 1.10. *Suppose π is admissible. Then the $\mathcal{C}$-linear functional*

$$\begin{array}{rccc} \Theta_\pi : & M_c^\infty(G) & \to & \mathcal{C} \\ & \mu & \mapsto & \mathrm{tr}(\pi(\mu)) \end{array}$$

is called the **character** *of π.*

Example 1.11. Suppose $\dim_{\mathcal{C}} \pi < +\infty$. For $\mu = f \otimes \nu$ we have

$$\Theta_\pi(f \otimes \nu) = \int_G \mathrm{tr}(\pi(g))\mu(g) := \left(\sum_{g_i \in G/J} \mathrm{tr}(\pi(g_i)) f(g_i) \right) \cdot \nu(J) = \mu(\mathrm{tr}(\pi)) \tag{2}$$

where $J \in \Omega(G)$ is such that $\pi = \pi^J$ and that μ is left J-invariant and $\mathrm{supp}(\mu) = \bigsqcup_i Jg_i$. In this case we usually say Θ_π is represented by the function $g \mapsto \mathrm{tr}(\pi(g))$.

For $g \in G$ and $\mu = f \otimes \nu \in M_c^\infty(G)$ with $f \in \mathcal{C}_c^\infty(G)$, denote by $c_g(f) \in \mathcal{C}_c^\infty(G)$ the function $(c_g(f))(h) = f(g^{-1}hg)$ and $c_g(\mu) := c_g(f) \otimes \nu$.

Definition 1.12. *A* **generalized function** *on G is a linear functional $\Theta : M_c^\infty(G) \to \mathcal{C}$. An invariant generalized function is a generalized function $\Theta : M_c^\infty(G) \to \mathcal{C}$ such that $\Theta(c_g(\mu)) = \Theta(\mu)$ for any $g \in G$, $\mu \in M_c^\infty(G)$.*

In particular, when π is irreducible admissible, our character Θ_π is an invariant generalized function.

Remark 1.13. For $f \in \mathcal{C}_c^\infty(G)$ and $\mu = f \otimes \nu \in M_c^\infty(G)$, we have that $\mathrm{supp}(f) = \mathrm{supp}(\mu)$ is always open. Suppose $K \subset G$ is open compact, we denote by $f|_K \in \mathcal{C}_c^\infty(K) \subset \mathcal{C}_c^\infty(G)$ the obvious restriction, and also $\mu|_K := (f|_K) \otimes \nu$. When Θ is a generalized function, we denote by $\Theta|_K$ the "restricted" generalized function given by $\Theta|_K(\mu) := \Theta(\mu|_K)$.

Naturally, one expects that the theory for Θ_π for irreducible smooth (thus admissible) representations is important for the study of these representations. It is however difficult to directly operate (invariant) generalized functions. Ideally, we would like a "function" t_π on G such that for any $\mu \in M_c^\infty(G)$ we have

$$\Theta_\pi(\mu) = \int_G t_\pi(g)\mu(g) = \mu(t_\pi)$$

in appropriate sense, generalizing (2). However, recall that for complex representation of finite groups, $\mathrm{tr}(\pi(e)) = \dim_{\mathcal{C}}(\pi)$. Hence if $\dim_{\mathcal{C}}(\pi) = +\infty$, we are supposed to have $t_\pi(e)$ blows up to infinity.

In classical analysis, useful generalized function (e.g. those in $L^2(\mathbb{R})$) are often defined almost everywhere. Similar phenomenon happens here. We will define a non-zero analytic (in fact algebraic) function [CC: reference to a later section to be inserted]

$$D_G : G \to F$$

and

$$G^{\mathrm{rs}} := \{g \in G \mid D_G(g) \neq 0\} \subset G$$

the **regular semisimple locus**, that is open, dense, and such that $G \backslash G^{\mathrm{rs}}$ has measure zero. This gives an injection

$$\mathcal{C}_c^\infty(G^{\mathrm{rs}}) \hookrightarrow \mathcal{C}_c^\infty(G)$$

Our first main goal is

Theorem A. *(Harish-Chandra) Suppose π is irreducible admissible. Then there exists a function $t_\pi \in \mathcal{C}^\infty(G^{\mathrm{rs}})$ such that for any μ with $\mathrm{supp}(\mu) \subset G^{\mathrm{rs}}$ we have*

$$\Theta_\pi(\mu) = \mu(t_\pi)$$

in the sense of (1).

As discussed, at least when $\mathcal{C} = \mathbb{C}$ the function t_π is doomed to blow up near $e \in G$. Hence t_π should **not** extend to a locally constant function on G, and consequently $\mu(t_\pi)$ cannot be directly defined if $e \in \mathrm{supp}(\mu)$. However, our second main goal is to show that

Theorem B. *(Harish-Chandra) Suppose $\mathcal{C} = \mathbb{C}$ and π is irreducible admissible. Then*

(1) The function $t_\pi \cdot |D_G|^{1/2}$ is **locally bounded***; for any compact subset $E \subset G$ there exists $C_{\pi,E} \in \mathbb{R}$ such that*

$$g \in G^{\mathrm{rs}} \cap E \implies t_\pi(g) \cdot |D_G(g)|^{1/2} \leq C_{E,\pi}$$

(2) t_π **locally integrable***, meaning that for any $\nu \in M_G$, any open compact set $K \subset G$ and any sequence of increasing open compact subsets K_n with $K \cap G^{\mathrm{rs}} = \lim_{n \to +\infty} K_n$, we have that*

$$\nu\left(t_\pi|_{K_n}\right) \in \mathbb{C}$$

converges absolutely. Consequently, for any $\mu \in M_c^\infty(G)$ we may define

$$\mu(t_\pi) := \lim_{n \to +\infty} \left(\mu|_{K_n}\right)(t_\pi)$$

and it is independent of the choice of K_n.

(3) Moreover, t_π represents Θ_π, meaning that

$$\Theta_\pi(\mu) = \mu(t_\pi)$$

for any $\mu \in M_c^\infty(G)$.

Remark 1.16. Theorem B is not as analytical as it might seen. Modulo certain coefficients in $\mathbb{C}$ - in fact often algebraic numbers - what essentially appears in

$$\lim_{n \to +\infty} \left(\mu|_{K_n}\right)(t_\pi) \to \Theta_\pi(\mu)$$

is a sequence of rational numbers converging to another rational number. It's often of the flavor

$$\frac{p}{p-1} = 1 + p^{-1} + p^{-2} + \ldots$$

In this sense it is possible to make sense of this for any $\mathcal{C}$ with $\mathrm{char}(\mathcal{C}) = 0$.

Exercise 3. *Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, which acts on $\mathbb{P}^1(\mathbb{Q}_p)$ in a natural way. Consider the space $\pi = \mathcal{C}_c^\infty(\mathbb{P}^1(\mathbb{Q}_p))$ on which G also acts. Show that π is generated by a single vector, i.e. there exists $v \in \mathcal{C}_c^\infty(\mathbb{P}^1(\mathbb{Q}_p))$ such that*

$$\mathcal{C}_c^\infty(\mathbb{P}^1(\mathbb{Q}_p)) = \mathrm{span}_{\mathcal{C}}\langle g.v \mid g \in \mathrm{GL}_2(\mathbb{Q}_p)\rangle.$$

Our proof for Theorem A, B and C will actually work with admissible representations generated by finitely many vectors. In the next exercise we motivate part of Theorem B.

Exercise 4. (*) *Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, which actfs on $\mathbb{P}^1(\mathbb{Q}_p)$ in a natural way, which acts on $\mathbb{P}^1(\mathbb{Q}_p)$ in a natural way. Consider the space $\pi = \mathcal{C}_c^\infty(\mathbb{P}^1(\mathbb{Q}_p))$ on which G also acts. For $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ with distinct eigenvalues λ_1, λ_2, we show that*

$$t_\pi(g) = \begin{cases} 0 & \text{when } \lambda_1, \lambda_2 \notin \mathbb{Q}_p \text{ (but in a quadratic extension)}, \\ \frac{|\lambda_1| + |\lambda_2|}{|\lambda_1 - \lambda_2|} & \text{when } \lambda_1 \neq \lambda_2 \in \mathbb{Q}_p. \end{cases}$$

See the exercise sheet for more detail.

To prove Theorem B, we will first prove our third main goal that

Theorem C. *(Howe, Harish-Chandra) Let $g \in G$. There exist a finite-dimensional space F_g of generalized functions on G, such that for any irreducible admissible representation π, there exist $\phi_{\pi,g} \in F_g$ and an open compact subset $U_{\pi,g} \subset G$ containing g such that*

$$\Theta_\pi|_{U_{\pi,g}} = \phi_{\pi,g}|_{U_{\pi,g}}$$

The dimension $\dim_{\mathcal{C}} F_g$ is bounded by a constant that depends only on G but not on g.

Note that we don't need to assume $\mathcal{C} = \mathbb{C}$ nor assume $\mathrm{char}(\mathcal{C}) = 0$ in Theorem C. The elements in F_g need only be defined on some neighborhood of g. On such a neighborhood we will perform **Fourier transform**, and show that we can represent the results as invariant distributions on some **dual nilpotent cone**, which is one point in the special case when $g \in G^{\mathrm{rs}}$, thereby proving Theorem A.

After that we prove Theorem B by proving the corresponding property for these Fourier transforms, which will be another highly nontrivial task.

When $\mathrm{char}(\mathcal{C}) = \ell > 0$ (recall that in this case we assume $\ell \neq p$), the classical theory of $\mathcal{C}$-representations of finite groups was developed particularly by Brauer. More precisely, let $R := W(\mathcal{C})$ be the Witt ring over $\mathcal{C}$. Then a finite-dimensional semisimple $\mathcal{C}$-representation of a finite group is determined by its Brauer character[1], which is an R-valued function on the finite group. For our p-adic reductive group G, we can also define a Brauer character $\tilde{\Theta}_\pi$, an R-linear functional on $R_c^\infty(G) \otimes_{\mathbb{Z}[1/p]} M_G$. We have

Theorem 1.18. *(Vignéras, Dat, Tsai) The same statement of Theorem C holds for $\tilde{\Theta}_\pi$. With any fixed choice of ring embedding $R = W(\mathcal{C}) \hookrightarrow \mathbb{C}$, the statements of Theorem B also holds for $\tilde{\Theta}_\pi$.*

[1] If we have time to talk about this, we will give some review.

When $\mathrm{char}(\mathcal{C}) = 0$ and $g = e \in G$ in either Theorem C, or when $\mathrm{char}(\mathcal{C}) > 0$ and $g = e$ in Theorem 1.18, the generalized function $\phi_{\pi,e}$ controls (some of) the so-called degenerate Whittaker models for π, in particular the Whittaker models for π. This is the important theorem of Rodier [Rod75], Mœglin–Waldspurger [MW87] and Varma [Var14]. We also hope to talk about this if we have time.

2. Reviews on F, $\mathbb{G}$ and $G = \mathbb{G}(F)$

2.1. **$\mathbb{Q}_p$ and its finite extensions.** We recommend the first two chapters in *Local Fields and Their Extensions* by Fesenko–Vostokov, https://ivanfesenko.org/wp-content/uploads/2021/10/vol.pdf, for a reference.

Recall that p is a fixed prime number. The field $\mathbb{Q}$ of rational numbers has a norm (with values in $p^{\mathbb{Z}} \sqcup \{0\}$) given by

$$|0|_p := 0, \ \ |p^v \cdot \frac{a}{b}|_p := p^{-v} \text{ for any } v \in \mathbb{Z} \text{ and } a, b \in \mathbb{Z} \text{ with } \gcd(a,p) = \gcd(b,p) = 1.$$

This norm is called the p-adic norm on $\mathbb{Q}$. We define $\mathbb{Q}_p$, the field of p-adic numbers, to be the completion of $\mathbb{Q}$ under it (as a metric space). This makes $\mathbb{Q}_p$ a locally compact Hausdorff normed field, just like $\mathbb{R}$. We denote by $\mathbb{Z}_p$ the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$.

Let $F/\mathbb{Q}_p$ be any finite extension. We will review properties about F. We have to be minimally very familiar with the case $F = \mathbb{Q}_p$, and in fact in this course all results are equally strong when assuming $F = \mathbb{Q}_p$; see Remark 2.30. Let $\mathcal{O} = \mathcal{O}_F$ be those elements in F that are integral over $\mathbb{Z}_p$. Then

(1) $\mathcal{O}_F$ is a local ring. It has a principal maximal ideal $\mathfrak{m} = \mathfrak{m}_F$, and its residue field $k = k_F := \mathcal{O}_F/\mathfrak{m}_F$ is commonly called the residue field of F.
(2) We have $p \in \mathfrak{m}_F$. The field $k_F \supset \mathbb{Z}/p$ is a finite field. We denote by $q := \#k_F$, a power of p. The quotient ring $\mathcal{O}_F/\mathfrak{m}_F^n$ is a finite ring of order q^n.
(3) For $x \in F$, define $|x|_F := [x\mathcal{O}_F : \mathcal{O}_F]$ to be the relative index. More precisely, we define
   (a) If $x = 0 \in F$, then $|x|_F := 0$.
   (b) If $x \in \mathcal{O}_F$, then $|x|_F := \frac{1}{[\mathcal{O}_F : x\mathcal{O}_F]}$.
   (c) If $x \notin \mathcal{O}_F$, then $x^{-1} \in \mathcal{O}_F$, and $|x|_F := [x\mathcal{O}_F : \mathcal{O}_F] = \frac{1}{[\mathcal{O}_F : x^{-1}\mathcal{O}_F]}$

   Then $|\cdot|_F : F \to q^{\mathbb{Z}} \sqcup \{0\} \subset \mathbb{R}_{\geq 0}$ defines a norm. The restriction of $|\cdot|_F$ to $\mathbb{Q}_p$ agrees with $(|\cdot|_p)^{[F:\mathbb{Q}_p]}$. If E/F is another finite extension, then $(|\cdot|_E)\,|_F = (|\cdot|_F)^{[E:F]}$.

   We will often skip the subscript $_F$ and just denote the norm by $|\cdot|$. When we refer to topology on F, we always mean the topology induced by $|\cdot|$. The natural map

$$\mathcal{O}_F \to \varprojlim_n \mathcal{O}_F/\mathfrak{m}_F^n$$

   is an isomorphism of topological rings. In particular $(\mathcal{O}_F, +)$ is a pro-p-group, and is totally disconnected. Likewise F is totally disconnected.
(4) We may define a valuation $\mathrm{val}_F : F \to \mathbb{Z} \sqcup \infty$ with $\mathrm{val}(x) := -\log_q(|x|)$. This makes F a complete discrete valuation field. In particular $\mathcal{O}_F$ is a PID. Also $\mathrm{val}_F(x, y) \geq \min(\mathrm{val}_F(x), \mathrm{val}_F(y))$ and equivalently $|x + y| \leq \max(|x|, |y|)$.

(5) For any $n \in \mathbb{Z}$, we have

$$\mathfrak{m}_F^n = \{x \in F \mid \operatorname{val}(x) \geq n\} = \{x \in F \mid |x| \leq q^{-n}\}.$$

(6) $\mathcal{O}_F$ is open compact under $|\cdot|$. Hence any $\mathfrak{m}_F^n$ ($n \in \mathbb{Z}$) is also open compact.

Remark 2.1. Unlike $\mathbb{R}$, the field $\mathbb{Q}_p$ has (many) infinite algebraic extensions; any algebraic closure of $\mathbb{Q}_p$ has countable but infinite dimension over $\mathbb{Q}_p$.

Remark 2.2. For any algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, the norms $(|\cdot|_F)^{\frac{1}{[F:\mathbb{Q}_p]}}$ on all finite extensions agree, so that we have a well-defined norm $|\cdot|_p : \overline{\mathbb{Q}}_p \to \mathbb{R}_{\geq 0}$.

2.2. F**-analysis.** We recommend [Ser06, Part II, Chapter I and II] for a reference for most results. We have the following analogues of the concepts in introductory real analysis[2]:

(1) F^n is a normed space, with $|(x_1, ..., x_n)| := \max(|x_1|, ..., |x_n|)$. All norms defined on a finite-dimensional vector space are equivalent in the sense that any two differ by at most a constant, and therefore define the same topology.
(2) A subset in F^n is compact iff it is sequential compact, iff it is closed and bounded.
(3) Chain Rule for analytic functions from F^{n_1} to F^{n_2}; a function is **analytic** if it is locally given by converging power series.
(4) Inverse Function Theorem and Implicit Function Theorem, for F^n or F-analytic manifolds.
(5) Definition of differential forms*, that are F-valued on F-analytic manifolds.
(6) $\mathbb{R}$-valued Lebesgue integrals of (absolute values of) differential forms using $|\cdot|$. (See however Lemma 2.11 for the version we actually use.
(7) Fubini's theorem.

We also give a partial list of results that, to the best of my knowledge, has **no** simple analogue over F:

(1) Connectedness of $\mathbb{R}$ and $\mathbb{R}^n$.
(2) Riemann integrals.
(3) Fundamental theorem of calculus, Stokes' theorem.

2.3. F**-analytic manifolds.** We recommend [Ser06, Part II, Chapter III] for a reference for most results.

Definition 2.3. *An n-dimensional F-analytic manifold is a second countable Hausdorff topological space X together with an open covering $X = \bigcup U_i$ and homeomorphisms $c_i : U_i \xrightarrow{\sim} \mathcal{O}_F^n$ such that $c_j|_{U_i \cap U_j} \circ \left(c_i|_{U_i \cap U_j}\right)^{-1}$ is a converging power series, defined from an open subset of $\mathcal{O}_F^n$ to another.*

There is an obvious notion of F-valued functions (locally given by converging power series) on X, and similarly for analytic maps between F-analytic manifolds. There is an obvious notion of the tangent space T_xX for any $x \in X$, which is a vector space of dimension n over F.

[2]the subject that is sometimes called "advanced calculus."

Remark 2.4. The dimension is unique as part of this data. This can be quickly seen as follows: if we have two different charts with different dimensions, then we have an invertible analytic map from an open subset of $\mathcal{O}_F^n$ to an open subset of $\mathcal{O}_F^m$, for some $n > m$. This is impossible by the implicit function theorem.

Remark 2.5. However, unlike $\mathbb{R}$, the topological space $\mathbb{Q}_p$ is similar to a Cantor set; there exists a homeomorphism $\mathbb{Q}_p \cong \mathbb{Q}_p \times \mathbb{Q}_p$, or $\mathbb{Z}_p \cong \mathbb{Z}_p \times \mathbb{Z}_p$. (Meanwhile, I think non-trivial algebraic topology can be used to show that $\mathbb{R}^m \ncong \mathbb{R}^n$ topologically for any $m < n$.) The thing is that such a homeomorphism cannot be given by an analytic map.

Definition 2.6. *A (closed) sub-manifold $Y \subset X$ of dimension m for some $m \leq n$ is a closed subset that in local coordinates can be given by $\mathcal{O}_F^m \subset \mathcal{O}_F^n$.*

Remark 2.7. Since X, like $\mathcal{O}_F^n$, is totally disconnected, it has many open sub-manifolds of the same dimension. For example $\mathfrak{m}_F^n \subset \mathcal{O}_F^n$ is an open (and also closed) sub-manifold, both of dimension n.

Fix an n-dimensional F-analytic manifold X for the rest of this subsection.

Definition 2.8. *Fix $k \in \{0, 1, ..., n\}$. A differential k-form on an open subset $U \subset X$ is an assignment*

$$U \ni u \mapsto \sum_{1 \leq j_1 < ... < j_k \leq n} f_{\vec{j}} dx_{j_1} dx_{j_2} ... dx_{j_k}$$

where each $f_{\vec{j}}$ is an analytic F-valued function on a neighborhood of U, and $(x_1, ..., x_n) : V \xrightarrow{\sim} \mathcal{O}_F^n$ is any coordinate of a chart on an open subset $V \subset X$ covering u.

For different coordinates/charts, differential forms are identified in the obvious way using Jacobians.

Definition 2.9. *Suppose X is an n-dimensional F-analytic manifold, and ω is a nowhere-vanishing differential n-form on it. Consider any chart $c : U \xrightarrow{\sim} \mathcal{O}_F^n$ such that $c_*\omega$ is of the form $f dx_1 dx_2 ... dx_n$ and that $|f|$ is constant, for which we define $|\omega|(U) := q^{n/2} \cdot |f|$. Suppose $E \subset X$ is a disjoint union $E = \bigsqcup_{i=1}^m U_i$ for finitely many such U_i, then $|\omega|(E) := \sum |\omega|(U_i)$.*

[CC: The extra factor $q^{n/2}$ is added on March 17th.]

Remark 2.10. Why the factor $q^{n/2}$? The ultimate mathematical reason is to be compatible with the result in [MW87]. What happens, in my opinion, is that the result in [MW87] shows that there is a canonical choice of measure, under which the preimage of a *Lagrangian*[3] in $(\mathcal{O}_F/\mathfrak{m}_F)^n$ has measure 1.

I am not sure, but I think we will only deal with the case when n is even so that $q^{n/2}$ is always an integer. There is another point of view about this factor that is brought up in [Tsa20], that it is given by a weight 0 pure local system of perverse degree 0.

[3]Meaning a Lagrangian subspace as in `https://en.wikipedia.org/wiki/Symplectic_vector_space#Lagrangian_form`.

Lemma 2.11. *Suppose we have a ring homomorphism $\iota : \mathbb{Z}[q^{-n/2}] \to \mathcal{C}$. Then the above definition is well-defined for any open compact subset $E \subset X$, giving $|\omega|(E) \in \mathbb{Z}[q^{-1/2}]$. Consequently, for any $f \in \mathcal{C}_c^\infty(X)$, we can write $f = \sum_j c_j 1_{E_j}$ and define $\int_X f|\omega| := \sum_j c_j \iota(|\omega|(E_j))$. Again the integral is well-defined.*

2.4. **p-adic Lie groups.** We recommend [Ser06, Part II, Chapter IV] for a reference for most results.

Definition 2.12. *A Lie group over F (of dimension n) is an F-analytic manifold H (of dimension n) together with a multiplication map $m_H : H \times H \to H$ that is analytic. A Lie subgroup is a closed subgroup that is also a sub-manifold.*

Remark 2.13. Lie groups over F are more often called a p-adic Lie group. When $F/\mathbb{Q}_p$ is finite of degree d, a Lie group over F of dimension n can be easily verified to be a Lie group over $\mathbb{Q}_p$ of dimension nd over $\mathbb{Q}_p$, so the notation is not quite an abuse of language.

Lemma 2.14. *Suppose H is a p-adic Lie group. Then there exists an open compact Lie subgroup $J \subset H$ such that*

(1) As a topological group, J is a pro-p-group, meaning that it is isomorphic to an inverse limit of J/J_m for some open compact normal Lie subgroups $J_m \subset J$ with index $[J : J_m]$ being a power of p.

(2) In fact, possibly after shrinking J, we may take

$$J_m = \{h^{p^m} \mid h \in J\}.$$

If H has dimension n, then $[J : J_m] = (\#(\mathcal{O}_F/p\mathcal{O}_F))^{nm}$.

(3) Consequently, any open subset of J contains a J_m-coset for some m. Same for any open subset of H.

Sketch. We claim that with any local coordinate matching $0 \in \mathcal{O}_F^n$ with e, the analytic map m_H is of the form

$$m\left((x_1, ..., x_n), (y_1, ..., y_n)\right) = (x_1 + y_1 + f_1(x, y), ..., x_n + y_n + f_n(x, y))$$

where each $f_i(x, y)$ has no constant nor linear terms Indeed, any analytic function $m : \mathcal{O}_F^n \times \mathcal{O}_F^n \to \mathcal{O}_F^n$ satisfying $m(x, 0) = m(0, x) = x$ has this form. The formula basically says $m(x, y) = x + y + O(\max(|x|, |y|)^2)$. One then verifies that if we take J to be the subset $(\mathfrak{m}_F^N)^n$ for a sufficiently large N depending on f_i, all asserted properties in the lemma hold. ☐

An alternative sketch. Alternatively, any group that we actually work with is a closed subgroups of $\mathrm{GL}_n(F)$, and the result follows from that for $\mathrm{GL}_n(F)$ which can be directly checked. ☐

2.5. **Varieties.** A **variety** (always over F) will be defined as a geometrically reduced separated scheme of finite type over F. In this course, schemes are never needed, and the readers unfamiliar with varieties are advised to refer to various online resources for more friendly introductions to varieties. We will only sketch the concept below.

Consider $\bar{F}/F$ any algebraic closure. A lot of time we work with affine varieties, and $\mathrm{char}(F) = 0$, so an affine variety is the vanishing locus $\mathbb{X} = \mathbb{X}(\bar{F}) = (I = 0)$ in $\mathbb{A}^N := \bar{F}^N$ of some ideal $I \lhd F[x_1, ..., x_N]$ for some $N \in \mathbb{Z}_{\geq 0}$, such that $F[x_1, ..., x_N]/I$

is reduced. As $F[x_1, ..., x_N]$ is Noetherian, I is generated by a finite number of elements, and usually we just say the variety is the vanishing locus of those generators. We write $\mathbb{X}(F) = \mathbb{X}(\bar{F}) \cap F^n$. When $I = 0$, the variety is denoted $\mathbb{A}^N$, for which is corresponding locus is the whole $\bar{F}^n$. A variety is glued from finitely many affine varieties in some way. A **morphism** between varieties over F is a map that is given locally by polynomial equations with coefficients in F.

A closed subvariety (over F) is something that is locally of the form $(J = 0) \subset (I = 0)$, where $J \supseteq I$ is a possibly bigger ideal. A (Zariski) closed F-subset of a variety is $\mathbb{Y}(\bar{F}) \subset \mathbb{X}(\bar{F})$ for some closed subvariety $\mathbb{Y} \subset \mathbb{X}$. A (Zariski) open subset is the complement in $\mathbb{X}(\bar{F})$ of a Zariski closed subset. A variety is said to be **irreducible** if it is not the union of two closed subsets, and **connected** if it contains a proper subset that is both Zariski closed and open.

Remark 2.15. One might be a bit confused about that we only defined closed F-subset, but not closed subset in general. I am mostly being lazy; every subset of a variety that we'll every encounter in this course should be defined over F anyway.

A variety is **smooth** at $x \in \mathbb{X}(\bar{F})$, if Zariski locally (possibly after shrinking) it is of the form $x \in (f_1 = f_2 = ...f_n = 0) \subset \mathbb{A}^N$ such that the matrix $\left[\frac{\partial f_i}{\partial x_j}(x)\right]$ has rank $n \leq N$. In this case it is said to be smooth of dimension $N - n$. A variety is said to be smooth (of dimension n) if it is smooth at every $\bar{F}$-point $x \in \mathbb{X}(\bar{F})$ (of dimension n).

Lemma 2.16. *For any variety $\mathbb{X}$ (over F) that is smooth at every point $x \in X := \mathbb{X}(F)$, the set X has a canonical structure of an F-analytic manifold. For any $x \in X$, we have a canonical isomorphism $T_x(X) \cong (T_x\mathbb{X})(F)$; here $T_x\mathbb{X}$ is the tangent space at x to X as a smooth variety. An algebraic differential k-form on $\mathbb{X}$ gives an (analytic) differential k-form on X.*

Sketch. For any $x \in X = \mathbb{X}(F)$, the variety is locally a closed subvariety of $\mathbb{A}^N_F$ cut out by $N - n$ equations with non-singular differential. By implicit function theorem, this is then an n-dimensional analytic sub-manifold of F^N. The rest are tautology. □

Lemma 2.17. *Let $\mathbb{X}$ be a connected smooth variety over F with $\mathbb{X}(F) \neq \emptyset$. Then $\mathbb{X}(F)$ is (Zariski) dense in $\mathbb{X}$.*

Sketch. Suppose on the contrary the Zariski closure $\mathbb{Y}$ of $\mathbb{X}(F)$ in $\mathbb{X}$ is a proper closed subvariety. Since $\mathbb{X}$ is connected smooth, therefore irreducible, this implies that $\dim \mathbb{Y} \lneq \dim \mathbb{X}$.

We then argue that any closed subvariety with $\dim \mathbb{Y} \lneq \dim \mathbb{X} = n$ cannot contain a non-trivial n-dimensional F-analytic manifold, thus a contradiction. We can either argue that $\mathbb{Y}(F) \subset \mathbb{X}(F)$ has measure zero in Lebesgue sense, or cut $\mathbb{Y}$ into a finite number of smooth sub-varieties of smaller dimensions and argue that their union cannot contain an n-dimensional F-analytic manifold. This is essentially because an F-analytic manifold of dimension $< n$ cannot be an F-analytic manifold of dimension n, as in Remark 2.4. □

Remark 2.18. The same proof works with $F = \mathbb{R}$, and in fact also for $F = \mathbb{F}_q((t))$; see [Poo17, Proposition 3.5.75].

Example 2.19. As non-examples, $\mathbb{X}_1 = (x^2 + y^2 = 3) \subset \mathbb{A}^2$ over $\mathbb{Q}_3$ is smooth but $\mathbb{X}_1(\mathbb{Q}_3) = \emptyset$. Meanwhile $\mathbb{X}_2 = (x^2+y^2 = 0) \subset \mathbb{A}^2$ is not smooth, and $\mathbb{X}_2(F) = \{(0,0)\}$ is just one point and not dense.

2.6. **Linear algebraic groups and reductive groups.** An **algebraic group** over F is a variety $\mathbb{G}$ over F together with morphisms $m_{\mathbb{G}} : \mathbb{G} \times \mathbb{G}$, $i_{\mathbb{G}} : \mathbb{G} \to \mathbb{G}$ and $e \in \mathbb{G}(F)$ that makes $\mathbb{G}(\bar{F})$ a group. A **(closed/algebraic) subgroup** $\mathbb{H} \subset \mathbb{G}$ is a Zariski closed variety $\mathbb{H} \subset \mathbb{G}$ that again satisfies $m_{\mathbb{G}}(\mathbb{H} \times \mathbb{H}) \subset \mathbb{H}$, $i_{\mathbb{G}}(\mathbb{H}) \subset \mathbb{H}$ and $e \in \mathbb{H}(F)$.

An algebraic group is said to be **connected**[4] if it is connected as a variety. Any linear algebraic group over F is automatically smooth. Lemma 2.16 then implies

Lemma 2.20. *Let $\mathbb{G}$ be an algebraic group. Then $G = \mathbb{G}(F)$ is a Lie group over F.*

As a variety, GL_n is the closed subvariety of

$$\mathbb{A}^{n^2+1} = \{(g, x) \mid g \text{ is an } n \times n \text{ matrix with } \det(g) \cdot x = 1\}.$$

It is often more convenient to view it as an open subvariety $(\det(g) \neq 0) \subset \mathbb{A}^{n^2}$. It has the obvious multiplication map $m_{\mathrm{GL}_n}(g, g') = gg'$, an inverse map $i_{\mathrm{GL}_n}(g) = g^{-1}$, and an identity $e = \mathrm{Id}_n \in \mathrm{GL}_n(F)$. That makes it an algebraic group. A **linear algebraic group** is a closed subgroup of GL_n.

Example 2.21. Let $\mathbb{G}_a$ be the algebraic group that as a variety is $\mathbb{A}^1$, and with the group law given by addition. It can be identified

$$\mathbb{G}_a \cong \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} := \{g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2 \mid a = d = 1,\ c = 0\}.$$

Let B_2 be the group

$$B_2 \cong \begin{bmatrix} * & * \\ & * \end{bmatrix} := \{g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2 \mid c = 0\} \subset \mathrm{GL}_2.$$

Note that the $\det \neq 0$ condition forces $a, d \neq 0$. We have that $\mathbb{G}_a \subset B_2 \subset \mathrm{GL}_2$ are closed subgroups.

Let V be an n-dimensional vector space over F. The **general linear group** GL_V is the group identified with GL_n using any basis $V \cong F^n$.

The **special linear group** $\mathrm{SL}_n \subset \mathrm{GL}_n$ (or $\mathrm{SL}_V \subset \mathrm{GL}_V$) is the closed subgroup consisting of matrices of determinant 1. In particular $\mathrm{SL}_n := \mathrm{SL}_{F^n}$.

Suppose V be equipped with a symmetric bilinear form $(\cdot, \cdot)$. The **orthogonal group**[5] is $\mathrm{O}_V \subset \mathrm{GL}_V$ is the closed subgroup consisting of orthogonal operators with $(gv_1, gv_2) = (v_1, v_2)$. The **special orthogonal group** $\mathrm{SO}_V \subset \mathrm{O}_V$ is the closed subgroup consisting of orthogonal operators with determinant 1. If $(\cdot, \cdot)$ **is** non-degenerate, then $\mathrm{SO}_V \subset \mathrm{O}_V$ is a connected component, and has index 2.

Suppose V be equipped with an anti-symmetric bilinear form $\langle \cdot, \cdot \rangle$. The **symplectic group** group $\mathrm{Sp}_V \subset \mathrm{GL}_V$ is the closed subgroup consisting of symplectic operators with $\langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle$.

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[4]When an algebraic group is connected, it is automatically geometrically connected.

[5]Usually, when we say “orthogonal group”, “special orthogonal group” and “symplectic group” we assume that the underlying bilinear form is non-degenerate.

From now on, when we write O_V, SO_V or Sp_V we **assume that the bilinear form is non-degenerate unless otherwise stated**. In such non-degenerate cases, the groups GL_V, O_V and Sp_V are called **classical groups**. It is common in the area to *selectively* include SO_V as a classical group, or exclude GL_V from classical groups, depending on context. In the old days sometimes SL_V is also included as a classical group.

Definition 2.22. *An element $g \in \mathrm{GL}_n(F)$ is called* **semisimple** *if it is diagonalizable after some finite extension. It is* **unipotent** *if all eigenvalues $= 1$.*

Let $\mathbb{G} \subset \mathrm{GL}_n$ be a linear algebraic group. An element is **semisimple** *(resp.* **unipotent***) if it is so in $\mathrm{GL}_n(F)$. A Jordan decomposition of $g \in G = \mathbb{G}(F)$ is $g = su = us$ where $s \in G$ is semisimple and $u \in G$ is unipotent.*

For the following theorem see [Bor91, §4].

Theorem 2.23. *Let $\mathbb{G}$ be a linear algebraic group. The notion of being semisimple and unipotent does not depend on the embedding $\mathbb{G} \hookrightarrow \mathrm{GL}_n$. Every element $g \in G = \mathbb{G}(F)$ has a unique Jordan decomposition in G, and that also does not depend on the embedding $\mathbb{G} \hookrightarrow \mathrm{GL}_n$.*

Definition 2.24. *An algebraic group $\mathbb{U}$ is* **unipotent** *if it is a linear algebraic group and every element $u \in \mathbb{U}(\bar{F})$ is unipotent.*

Definition 2.25. *An algebraic group $\mathbb{G}$ is* **reductive** *if it is a linear algebraic group, and that every unipotent connected normal closed subgroup is trivial.*

Remark 2.26. A previous version of the note skipped the "connected" in "unipotent connected normal closed subgroup." We thank Hao-An for pointing this out. The previous definition turns out to be equivalent due to Remark 2.29, but is not a good idea over a general field.

Remark 2.27. In old books it is common to require in definition that a reductive group is connected. It seems more common nowadays to not assume that.

Remark 2.28. In Example 2.21, the groups $\mathbb{G}_a$ and B_2 are not reductive. (They both have $\mathbb{G}_a$ as an unipotent normal subgroup.) The groups GL_V and SL_V are reductive. Without assuming that the bilinear form is non-degenerate, the groups O_V, SO_V and Sp_V are reductive if and only if the underlying symmetric / anti-symmetric form is non-degenerate or trivial. Probably the easiest way to check reductivity in these cases is to do some linear algebra and argue that the conjugates of any non-trivial unipotent subgroup generates some non-unipotent element.

Remark 2.29. We have the following facts:

(1) Every unipotent subgroup is automatically connected. (Thanks to that $\mathrm{char}(F) = 0$. Roughly, over a field of characteristic 0, any closed subgroup that contains all powers of u must connect u to the identity e.)
(2) If $\mathbb{H}$ is a linear algebraic group for which every $h \in \mathbb{H}(F)$ is unipotent, then $\mathbb{H}$ is unipotent. (Thanks to Lemma 2.17.)
(3) A linear algebraic group is reductive if and only if it does not contain a normal closed subgroup isomorphic to $\mathbb{G}_a$.

(4) A linear algebraic group $\mathbb{G}$ is reductive if and only if $\mathbb{G}(F)$ does not contain a normal closed (topological) subgroup isomorphic to $F = \mathbb{G}_a(F)$, or just $\mathbb{Q}_p$.

However, we remark that in our opinion the important thing about reductive groups is not their definition, but many consequential properties that we will gradually see (or you might have seen in related subjects).

Remark 2.30. Suppose $\mathbb{G}$ is an algebraic group over F. Then there exists an algebraic group $\mathrm{Res}^F_{\mathbb{Q}_p}\mathbb{G}$ over $\mathbb{Q}_p$ that gives a canonical isomorphism $\mathbb{G}(F) \cong (\mathrm{Res}^F_{\mathbb{Q}_p}\mathbb{G})(\mathbb{Q}_p)$ as p-adic Lie groups. (This is called the **Weil restriction.** The group $\mathrm{Res}^F_{\mathbb{Q}_p}\mathbb{G}$ is linear/connected/reductive if and only if $\mathbb{G}$ is (this uses that thankfully $\mathbb{Q}_p$ is perfect). Therefore, in some sense we can always reduce to the case when $F = \mathbb{Q}_p$, though in my opinion that's generally not a good idea.

Exercise 5. *Let E/F be any quadratic extension. For $x \in E$ denote by $\bar{x}$ the (unique) Galois conjugate of x over F. Let V be an n-dimensional vector space, on which we are given a hermitian form $\langle\cdot,\cdot\rangle : V \times V \to E$, i.e. that is bi-additive and such that*

$$\langle c_1 v_1, c_2 v_2\rangle = c_1\bar{c}_2\langle v_1, v_2\rangle,\ \forall c_1, c_2 \in E,\ v_1, v_2 \in V.$$

Construct an algebraic group $\mathbb{G}$ over F, such that as Lie groups over F we have

$$\mathbb{G}(F) = \text{ group of unitary operators on } (V, \langle\cdot,\cdot\rangle).$$

Show that your $\mathbb{G}$ is reductive only if $\langle\cdot,\cdot\rangle$ is non-degenerate or identically zero.

Exercise 6. *As a follow-up exercise, suppose that $\langle\cdot,\cdot\rangle$ is non-degenerate in Exercise 5. Compute $\mathbb{G}(E)$. Show that your $\mathbb{G}$ is reductive.*

2.7. Lie algebra and exponential map.

Definition 2.31. *Let H be a p-adic Lie group. Then* $\mathrm{Lie}(H) := T_eH$ *is the tangent space at the identity. Likewise, when $\mathbb{H}$ is an algebraic group, we put* $\mathrm{Lie}(\mathbb{H}) := T_e\mathbb{H}$.

Example 2.32. The Lie algebra $\mathrm{Lie}(\mathrm{GL}_V(F))$ (resp. $\mathrm{Lie}(\mathrm{GL}_V)$) is canonically identified with the space of linear operators on V. In particular $\mathrm{Lie}(\mathrm{GL}_n(F))$ (resp. $\mathrm{Lie}(\mathrm{GL}_n)$) is the space of $n \times n$ matrices in F. We write $\mathfrak{gl}_n := \mathrm{Lie}(\mathrm{GL}_n(F))$ and We write $\mathfrak{gl}_V := \mathrm{Lie}(\mathrm{GL}_V(F))$.

Suppose $H \subset H'$ is a Lie subgroup. Then obviously $\mathrm{Lie}(H) \subset \mathrm{Lie}(H')$ is a subspace. For example, $\mathfrak{sl}_V \subset \mathfrak{gl}_V$ is the subspace of traceless $n \times n$ matrices, $\mathfrak{so}_V \subset \mathfrak{gl}_V$ is the space of anti-self-adjoint operators, and likewise $\mathfrak{sp}_V \subset \mathfrak{gl}_V = \{X \in \mathfrak{gl}_V \mid \langle Xv, w\rangle = -\langle v, Xw\rangle\}$.

Definition 2.33. *Let H be a p-adic Lie group and $g \in H$. We denote by*

$$\begin{array}{rccc} c_g: & H & \to & H \\ & h & \mapsto & ghg^{-1} \end{array}$$

and $\mathrm{Ad}(g) := dc_g|_e : \mathrm{Lie}(H) \to \mathrm{Lie}(H)$ *its differential. We will sometimes also write* $\mathrm{Ad}(g)$ *in terms of c_g.*

Definition 2.34. *Write* $\mathfrak{h} := \mathrm{Lie}(H)$. *We also denote by*

$$\begin{array}{rccc} \mathrm{Ad}: & H \times \mathfrak{h} & \to & \mathfrak{h} \\ & (h, X) & \mapsto & \mathrm{Ad}(h)X \end{array}$$

It is easy to see that this map is analytic. We also write it as $\mathrm{Ad} : H \to \mathrm{GL}_{\mathfrak{h}}$*, which is also analytic. They are called the* **adjoint action** *and the* **adjoint representation**.

Definition 2.35. *We define* $\mathrm{ad} : \mathfrak{h} \to \mathfrak{gl}_{\mathfrak{h}}$ *as* $\mathrm{ad} = d(\mathrm{Ad})|_e$*, the differential of* $\mathrm{Ad} : H \to \mathrm{GL}_{\mathfrak{h}}$ *at the identity. We also write* $[Y, X] := \mathrm{ad}_Y(X) := \mathrm{ad}(Y, X)$*, called the* **Lie bracket** *of* Y *and* X.

Lemma 2.36. *We have the following facts:*

(1) $c_g(h_1 h_2) = c_g(h_1)c_g(h_2)$
(2) $\mathrm{Ad}_g([h_1, h_2]) = [\mathrm{Ad}_g(h_1), \mathrm{Ad}_g(h_2)])$
(3) *(Jacobi identity)* $\mathrm{ad}_Y([X_1, X_2]) = [\mathrm{ad}_Y(X_1), X_2] + [X_1, \mathrm{ad}_Y(X_2)]$.

Proof. The first one is obvious. Each next one is a differential of the previous one. □

Remark 2.37. It is obvious from the definition that the Lie bracket is F-bilinear. We will later see that it is alternating.

We remark that the definitions of $\mathrm{Lie}(H)$, Ad and ad can be done in the same way for algebraic groups. When $H = \mathbb{H}(F)$, the two definitions agree. In what follows we will do something non-algebraic. Suppose $U \subset H$ is an open subset with $e \in U$, and $c : U \to T_eU = \mathrm{Lie}(H)$ is an analytic map such that $dc|_e = \mathrm{id}_{\mathrm{Lie}(H)}$. We consider

$$\log_{H,c}(h) := \lim_{i \to +\infty} p^{-i} c(h^{p^i}) \tag{3}$$

provided that the limit exists in T_eH.

Definition 2.38. *We say* $h \in H$ *is* **topologically unipotent** *if* $\lim_{i \to +\infty} h^{p^i} = e$.

Lemma 2.39. *Suppose* h *is topologically unipotent. The limit* (3) *exists for any* U *and* c *as above, and is independent of the choice of* U *and* c*. If* $f : H \to H'$ *is any Lie group homomorphism, then* $(df)(\log(h)) = \log_{H'}(f(h))$ *for any topologically unipotent* $h \in H$*. In particular* $\mathrm{Ad}_g(\log(h)) = \log(\mathrm{Ad}_g(h))$.

We will only sketch Lemma 2.39. However, Exercise 7 provides a more informative proof (in my opinion) in the situations that we need.

Sketch of Lemma 2.39. We work in a local chart that matches e with $0 \in \mathcal{O}_n^F$. After shrinking U, we write $U = (\mathfrak{m}_F^N)^n \subset H$ and $\mathrm{Lie}(H) = T_eU = F^n$. We have $c : (\mathfrak{m}_F^N)^n \to F^n$ is a power series with $dc|_0 = \mathrm{id}_{F^n}$. This implies $c(x) = x + O(|x|)^2$. We have seen that multiplication map on $\mathcal{O}_F^n$ has the first order approximation $m(x, y) = x + y + O(\max(|x|, |y|)^2)$. In particular, $x^p = m(x, m(x, m(x, ...)) = px + O(|x|)^2$. Therefore, in our local coordinate we have

$$|h^{p^{i+1}}| = |p| \cdot |h^{p^i}|$$

for $i \gg 0$ with $|h^{p^i}|$ sufficiently small, and

$$p^{-(i+1)}c(h^{p^{i+1}}) = p^{-(i+1)}\left(p \cdot c(h^{p^i}) + O(|h^{p^i}|)^2\right) = p^{-i}c(h^{p^i}) + p^{-(i+1)} \cdot O(|h^{p^i}|)^2.$$

Hence $p^{-i}c(h^{p^i})$ is a Cauchy sequence that converges. For a different choice of c (and U), they again differ by $x + O(|x|)^2$ after shrinking, and similar techniques shows that they give the same limit. □

Definition 2.40. *We define* $\log(h) := \log_{H,c}(h) \in T_eH$ *thanks to Lemma 2.39.*

Lemma 2.41. *There exists an open subgroup* $K \subset H$ *such that every element in* K *is topologically unipotent, and such that* $\log|_K : K \to \log(K) \subset \mathrm{Lie}(H)$ *is an isomorphism of analytic manifolds.*

The proof is similar to that of Lemma 2.39. We skip it.

Definition 2.42. *Let* K *be as in Lemma 2.41. We define* $\exp : \log(K) \xrightarrow{\sim} K$ *to be the inverse of* $\log|_K$*.*

In fact, what is closer to our use is

Exercise 7. *Show that there exist an open compact Lie subgroup* $K \subset GL_n(F)$ *and a lattice* $\mathfrak{K} \subset \mathfrak{gl}_n$ *such that*

$$\log(h) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}(h - \mathrm{id})^i}{i}$$

defines an isomorphism from K *to* $\mathfrak{K}$*, and that*

$$\exp(X) = \sum_{i=0}^{\infty} \frac{X^i}{i!}$$

defines the inverse of $\log$ *from* $\mathfrak{K}$ *to* K*. Both series are computed in the space of* $n \times n$ *matrices in* F*. Show also that* $\log(h^{-1}) = -\log(h)$ *for* $h \in K$*.*

The properties in Exercise 7 also holds for any conjugate of K. Since (3) already characterizes log uniquely, we have

Corollary 2.43. *Let* $\mathbb{H}$ *be a linear algebraic group over* F*, and* $H = \mathbb{H}(F)$*. Then there exists an* H*-conjugation invariant neighborhood* $e \in U \subset H$ *and* $0 \in \mathfrak{u} \subset \mathfrak{h}$ *such that* $\log$ *maps* U *isomorphically to* $\mathfrak{u}$*, with inverse* $\exp$*, given by any embedding* $\mathbb{H} \hookrightarrow \mathrm{GL}_n$*.*

Proof. Apply Lemma 2.39 and Exercise 7 to (any) $\mathbb{H} \hookrightarrow \mathrm{GL}_n$. □

Example 2.44. Certainly, Corollary 2.43 holds whenever H is a closed Lie subgroup of some $\mathrm{GL}_n(F)$. One might ask if there exists a p-adic Lie group that cannot be embedded into $\mathrm{GL}_n(F)$ for any n. Such groups do exist. Examples can be found in `https://mathoverflow.net/questions/91789/non-linear-lie-group` by replacing the "$\mathbb{R}$" in any answer by a p-adic field in appropriate ways. For example, consider

$$H = \{\begin{bmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{bmatrix} \mid a, b, c \in \mathbb{Q}_p\} \supset \Gamma = \{\begin{bmatrix} 1 & & b \\ & 1 & \\ & & 1 \end{bmatrix} \mid b \in \mathbb{Z}_p\}.$$

Then H/Γ is a 2-dimensional p-adic Lie group, in fact with an abelian Lie algebra. We can verify that every element in H/Γ is topologically unipotent, but Corollary 2.43 does not hold for H/Γ.

Remark 2.45. If we replace F by $\mathbb{R}$, for real Lie groups we have

$$\exp(X) := \lim_{n \to +\infty} c(\frac{1}{n}X)^n \in G$$

for $X \in \mathrm{Lie}(H)$. This generalizes the well-known fact for that for $G = \mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times$ we have $\exp(X) = \lim(1 + \frac{X}{n})^n$ for $X \in \mathfrak{g} = \mathbb{R}$. One can then define log (for real Lie group) to be the inverse of exp.

Lemma 2.46. *When $G = \mathrm{GL}_n$, we have $[X, Y] = XY - YX$.*

Corollary 2.47. *Let $\mathbb{H}$ be a linear algebraic group over F, and $H = \mathbb{H}(F)$. Let $X, Y \in \mathrm{Lie}(H)$. Then the* **Lie bracket** *of X and Y satisfies*

$$[X, Y] = \lim_{\substack{a,b \in F \\ a,b \to 0}} \frac{\log(\exp(aX)\exp(bY)\exp(-aX)\exp(-bY))}{ab}$$

In particular, we have $[X, Y] = [Y, X]$. If $H \subset H'$ is a Lie subgroup, then for $X, Y \in \mathrm{Lie}(H)$ we have that $[X, Y]$ is the same in $\mathrm{Lie}(H')$ as in $\mathrm{Lie}(H)$.

We also need

Definition 2.48. *An element $X \in \mathfrak{gl}_n(F)$ is called* **semisimple** *if it is diagonalizable after some finite extension. It is* **nilpotent** *if all eigenvalues $= 0$.*

Let $\mathbb{G} \subset \mathrm{GL}_n$ be a linear algebraic group. An element $X \in \mathfrak{g}$ is **semisimple** *(resp.* **nilpotent***) if it is so in $\mathfrak{gl}_n(F)$. A Jordan decomposition of $X \in \mathfrak{g}$ is $X = X_s + X_n$ such that $[X_s + X_n]$ where $X_s \in \mathfrak{g}$ is semisimple and $X_n \in \mathfrak{g}$ is nilpotent.*

We have the following analogue of Theorem 2.23; see also [Bor91, §4].

Theorem 2.49. *The notion of being semisimple or nilpotent does not depend on the embedding $\mathbb{G} \hookrightarrow GL_n$. Every element $\mathfrak{g}$ has a unique Jordan decomposition which also does not depend on the embedding.*

3. Reductive $\Longrightarrow$ unimodular

[CC: Currently, we plan to skip this section and not do it in class. It is unclear to me if contents in this section are logically needed for later sections.]

In this section we see the first place (among the several) where reductivity of $\mathbb{G}$ is used. Let $\mathbb{H}$ be any linear algebraic group, $H = \mathbb{H}(F)$ and $\mathfrak{h} := \mathrm{Lie}\, H$.

Traditionally, Lie brackets are defined for vector fields. Indeed, the tangent space $\mathrm{Lie}\,\mathbb{H} = T_e\mathbb{H}$ can be identified via left translation ($l_h : g \mapsto hg$) to $T_g\mathbb{H}$ by any $h \in \mathbb{H}(\bar{F})$. Hence $\mathrm{Lie}\,\mathbb{H}$ can be identified with the space of left-invariant algebraic vector fields on $\mathbb{H}$. Likewise $\mathfrak{h}$ can be identified with the space of left-invariant analytic vector fields on H. In both the algebraic and analytic case, the Lie bracket agree with the commutator in the sense of derivations[6].

Define $\mathfrak{h}^*$ as the dual vector space, but as an affine space over F. Then $\mathfrak{h}^*$ can be identified with the space of left-invariant analytic 1-forms on H. Consequently, $\bigwedge^{\mathrm{top}}(\mathfrak{h}^*)$ is the space of left-invariant analytic top forms on H. Note that $\bigwedge^{\mathrm{top}}(\mathfrak{h}^*)$ is the dual space of $\bigwedge^{\mathrm{top}}(\mathfrak{h})$.

Lemma 2.11 allows us to integrate any such top form and get $|\omega|$ that assigns every open compact subset of G a positive value ("measure") in $\mathbb{Z}[1/p]$. By construction $|\omega|$ is left-invariant, meaning that $|\omega|(gE) = |\omega|(E)$ for any $g \in G$. We are then wondering if $|\omega|$ is also right-invariant, or equivalently if $|\omega|(gEg^{-1}) = |\omega|(E)$ for any $g \in G$. This leads to

[6] See https://chatgpt.com/s/t_69a95e881e608191a335067caf887d2c for more detail.

Definition 3.1. *The* **modulus character** *is* $\Delta_H = \det(\mathrm{Ad}) : H \to F^\times$ *is the action of* H *on* $\bigwedge^{\mathrm{top}}(\mathfrak{h}^*)$ *through* $\mathrm{Ad} : H \to \mathrm{Aut}_F(\mathfrak{h}) = \mathrm{Aut}_F(\mathfrak{h}^*)$.

By definition Δ_H measures whether left-invariant top forms are right-invariant. The goal of this section is to prove that

Theorem 3.2. *Let* $\mathbb{G}$ *be a reductive group over* F *and* $G = \mathbb{G}(F)$. *Then* $|\Delta_G| : G \to q^{\mathbb{Z}}$ *is trivial. In this case we say* G *is* **unimodular**. *If* $\mathbb{G}$ *is also connected then in fact* Δ_G *is trivial.*

Example 3.3. It may takes more than one minute, but not too hard to check that Theorem 3.2 is true for $\mathbb{G} = \mathrm{GL}_V$, SL_V, O_V, SO_V and Sp_V. Therefore, although we will have two pages of algebraic groups to prove Theorem 3.2, it is not needed if one is restricting to such subgroups.

We review some definitions.

Definition 3.4. *Let* $\mathbb{H}$ *be a linear algebraic group. The* **identity component** *(also* **neutral component***)* $\mathbb{H}^\circ$ *is the connected component of the variety* $\mathbb{H}$ *that contains* $e \in \mathbb{H}(F)$. *The* **component group** *is* $\mathbb{G}/\mathbb{G}^\circ$ *which is a linear algebraic group of dimension* 0.

Definition 3.5. *The* **center** *of* $\mathbb{H}$ *is given by* $Z(\mathbb{H})(\bar{F}) = \{g \in \mathbb{H}(\bar{F}) \mid gh = hg,\ \forall h \in \mathbb{H}(\bar{F})\}$. *The* **connected center** *is* $Z(\mathbb{H})^\circ$, *the identity component of* $Z(\mathbb{H})$.

Corollary 3.6. *Suppose that* $\mathbb{H}$ *is connected. Let* $H = \mathbb{H}(F)$. *We have* $Z(\mathbb{H})(\bar{F}) = \{g \in \mathbb{H}(\bar{F}) \mid gh = hg,\ \forall h \in H\}$. *Thus* $Z(\mathbb{H})$ *is a closed subgroup defined over* F. *We have* $Z(\mathbb{H})(F) = Z(H)$.

Proof. Suppose $g \in Z(\mathbb{H})(\bar{F})$ is such that $c_g(h) = h$ for every $h \in H$. We have to prove that c_g is trivial on $\mathbb{H}$. This follows immediately from Lemma 2.17. Therefore, $Z(\mathbb{H})$ is defined by algebraic equations defined over F (by Noetherian property, actually a finite number of such). The last sentence then follows from definition. □

Remark 3.7. We warn that Definition 3.5 would be problematic (and "incorrect" from the modern point of view), if $\mathrm{char}(F)$ was positive; the center of the abstract group $\mathrm{SL}_2(\bar{\mathbb{F}}_2)$ is trivial, but $Z(\mathrm{SL}_2/_{\mathbb{F}_2}) = \mu_2/_{\mathbb{F}_2}$ is not supposed to be (geometrically) reduced; as a scheme, we have $\mu_2(\mathbb{F}_2[\epsilon]/\epsilon^2) = \{x \in \mathbb{F}_2[\epsilon]/\epsilon^2 \mid x^2 = 1\} = \{1, 1+\epsilon\}$ which is non-trivial.

Definition 3.8. *The* **derived subgroup** $\mathbb{H}^{\mathrm{der}} \subset \mathbb{H}$ *is the Zariski closure of the subgroup generated by* $(\mathbb{H}(\bar{F}), \mathbb{H}(\bar{F}))$.

Example 3.9. Write $H = \mathbb{H}(F)$. Denote by $Z(H)$ the usual center, and H^{der} the closure of the commutator (H, H); it can be proved that H^{der} is always a closed Lie subgroup, though we won't need it.

We have the following table for those in Example 2.21:

$\mathbb{H}$	$Z(\mathbb{H})$	$Z(\mathbb{H})^\circ$	$\mathbb{H}^{\text{der}}$	H	$Z(H)$	H^{der}
GL_n	GL_1	GL_1	SL_n	$\mathrm{GL}_n(F)$	$F^\times$	$\mathrm{SL}_n(F)$
SL_n	μ_n	$\{e\}$	SL_n	$\mathrm{SL}_n(F)$	$\mu_n(F)$	$\mathrm{SL}_n(F)$
B_2	GL_1	$\{e\}$	$\mathbb{G}_a$	$B_2(F)$	$F^\times$	$\mathbb{G}_a(F)$
$\mathbb{G}_a$	$\mathbb{G}_a$	$\mathbb{G}_a$	$\{e\}$	$\mathbb{G}_a(F) = F$	F	$\{e\}$
O_V	μ_2	$\{e\}$	SO_V	$O_{V(F)}$	$\mu_2(F) = \{\pm 1\}$	$*$
SO_V	$\mu_{\gcd(2,\dim V)}$	$\{e\}$	SO_V	$\mathrm{SO}_{V(F)}$	$\mu_{\gcd(2,\dim V)}(F)$	$*$
Sp_V	μ_2	$\{e\}$	Sp_V	$\mathrm{Sp}_{V(F)}$	$\{\pm 1\}$	$\mathrm{Sp}_{V(F)}$

In the case of SO_V, it is assumed that $\dim V \neq 2$ (otherwise SO_V is a torus and thus abelian). Also

$$\mu_n := \{x \in \mathrm{GL}_1(\bar{F}) \mid x^n = 1\}$$

is a closed subgroup of GL_1 of dimension 0, and $\mu_n(F) = \{x \in F^\times \mid x^n = 1\}$. Lastly,

$$* = \mathrm{im}(\mathrm{Spin}_V(F) \to \mathrm{SO}_V(F)) = \ker(\mathrm{SO}_V(F) \xrightarrow{\text{spinor norm}} F^\times/(F^\times)^2 = H^1(F, \mu_2))$$

(The proof Cheng-Chiang knows uses Bruhat-Tits theory.) In particular $H^{der} \subset \mathbb{H}^{\text{der}}(F)$ could be proper.

Proposition 3.10. *Suppose $\mathbb{G}$ be a connected reductive group. Then we have $\mathbb{G} = Z(\mathbb{G})^\circ \cdot \mathbb{G}^{\text{der}}$. This means, by definition, that any element in $g \in G = \mathbb{G}(\bar{F})$ can be written as $g = zg'$ for $z \in Z(\mathbb{G})^\circ(\bar{F})$ and $g' \in \mathbb{G}^{\text{der}}(\bar{F})$.*

[CC: Add some sketch.]

Example 3.11. The analogous statement $G = Z(G) \cdot G^{\text{der}}$ for F-points is **not** true. We have $\mathrm{GL}_2(F) \supsetneq F^\times \cdot \mathrm{SL}_2(F)$. (Why?)

Corollary 3.12. *Suppose $\mathbb{G}$ is connected reductive. Then $\mathrm{Ad} : \mathbb{G} \to \mathrm{GL}_{\mathrm{Lie}\,\mathbb{G}}$ has image in $\mathrm{SL}_{\mathrm{Lie}\,\mathbb{G}}$.*

Proof. By definition $\mathrm{Ad}(Z(\mathbb{G})) = \{e\}$. Also $\mathrm{Ad}(\mathbb{G}^{\text{der}}(\bar{F})) \subset (\mathrm{GL}_{\mathrm{Lie}(\mathbb{G})})^{\text{der}}(\bar{F}) = \mathrm{SL}_{\mathrm{Lie}(\mathbb{G})}(\bar{F})$. Hence the corollary follows from Proposition 3.10. $\square$

Remark 3.13. We remark that $\mathbb{G}$ is geometrically reduced, so the (scheme-theoretic) image is geometrically reduced. In this case it is appropriate to study the image at the variety level, i.e. using $\mathrm{Ad}(\mathbb{G}(\bar{F})) \subset \mathrm{GL}_{\mathbb{G}}(\bar{F})$.

Remark 3.14. As a non-example, when $\mathbb{H} = B_2$ as in Example 2.21, the action of $\mathrm{Ad}(\begin{bmatrix} a & 1 \\ 0 & 1 \end{bmatrix})$ has determinant a.

Proof of Theorem 3.2. Firstly, when $\mathbb{G}$ is connected reductive, Corollary 3.12 says $\det(\mathrm{Ad}) : \mathbb{G} \to \mathrm{GL}_1 = \mathrm{GL}_{\bigwedge^{\text{top}}(\mathrm{Lie}^*\mathbb{G})}$ is trivial. This is the algebraic version of Δ_G; passing to F-points, we see that $\Delta_G : G \to F^\times$ is trivial.

Now suppose $\mathbb{G}$ is only reductive. Write[7] $G^\circ := \mathbb{G}^\circ(F)$. Then $|\Delta_G| : G \to q^{\mathbb{Z}}$ is trivial on G° because $\mathrm{Lie}\, G = (\mathrm{Lie}\,\mathbb{G})(F) = (\mathrm{Lie}\,\mathbb{G}^\circ)(F) = \mathrm{Lie}\, G^\circ$. But $G/G^\circ \hookrightarrow$

[7]Warning: this is an abuse of language; G° is not a connected component of G. In fact it is **not** determined by G as a p-adic Lie group. The definition really makes use of the algebraic geometry.

$\pi_0(G)(F)$ and the latter is a finite group. Therefore, $\Delta_G : G/G^\circ \to q^{\mathbb{Z}}$ factors through a finite group, and is trivial as $q^{\mathbb{Z}}$ is torsion free. □

In the rest of this section let $\mathbb{G}$ be reductive, but not necessarily connected. Theorem 3.2 implies that, in fact, every top form in $\bigwedge^{\mathrm{top}}(\mathfrak{g}^*)$ is **also** right-invariant. And therefore any of them gives an **bi-invariant** measure $|\omega|$; this is often called a **Haar measure**.

Note that $|\omega|$ for non-zero ω can be restricted to a non-trivial element $\nu_\omega \in M_G$; indeed, to every open compact subgroup $J \in \Omega(G)$ it assigns a value $\nu(J) = |\omega|(J) \in \mathbb{Z}[1/p]$, and it is not hard to check the property in Definition 1.4. We have the following result that a priori has nothing to do with measures.

Corollary 3.15. *Let $J \in \Omega(G)$ and $g \in G$. Then*

$$[J : J \cap gJg^{-1}] = [gJg^{-1} : J \cap gJg^{-1}]$$

Proof. Choose any $\omega \neq 0$ as above. We have

$$[J : J \cap gJg^{-1}] = \frac{\nu_\omega(J)}{\nu_\omega(J \cap gJg^{-1})} = \frac{\nu_\omega(gJg^{-1})}{\nu_\omega(J \cap gJg^{-1})} = [gJg^{-1} : J \cap gJg^{-1}]$$

This proves the corollary. □

Reversing the logic, one can see that Corollary 3.15 is equivalent to Theorem 3.2. We end this section by using the "measure" to integrate functions as in Lemma 2.11. Recall that $\mathcal{C}$ is an algebraically closed field with $\mathrm{char}(\mathcal{C}) \neq p$.

Definition 3.16. *A* **distribution** *on G is a $\mathcal{C}$-linear functional $\delta : \mathcal{C}_c^\infty(G) \to \mathcal{C}$. It is* **left-invariant** *(resp.* **right-invariant**, *resp.* **invariant***) if $\delta(l_g(f)) = \delta(f)$ (resp. $\delta(r_g(f)) = \delta(f)$, resp. $\delta(c_g(f)) = \delta(f)$) for any $f \in TF$ and $g \in G$. Here $(l_g(f))(h) := f(g^{-1}h)$, $(r_g(f))(h) := f(hg)$ and $(c_g(f))(h) := f(g^{-1}hg)$. It is* **bi-invariant** *if it is both left-invariant and right-invariant.*

We have seen that $|\omega|$ gives a distribution. An equivalent form of Theorem 3.2 is that $|\omega|$ is not only left-invariant, but bi-invariant. It is not hard to see that

Lemma 3.17. *The space of left-invariant distributions on G is 1-dimensional, and can be identified with M_G.*

On the other hand, $|\omega|$ is bi-invariant, and therefore

Corollary 3.18. *Every left-invariant distribution on G is automatically bi-invariant.*

4. Orbital integrals

This section partly follows [Kot05, §3, §6 and §17], with the major difference that Kottwitz only works with $\mathcal{C} = \mathbb{C}$, and we don't even assume $\mathrm{char}(\mathcal{C}) = 0$. Every statement in [Kot05] can be made true with $\mathrm{char}(\mathcal{C}) = 0$. **Not** every statement in [Kot05] is true when $\mathrm{char}(\mathcal{C}) > 0$, so we need to be careful. A detailed study of what really happens when $\mathrm{char}(\mathcal{C}) > 0$ in done in [VW01]; we will not dive into that.

4.1. **Some motivation.** Let $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. For an abelian locally compact Hausdorff topological group H, it has a **Pontryagin dual** $\widehat{H} := \mathrm{Hom}_{\mathrm{cont}}(H, S^1)$, the group of continuous group homomorphisms equipped with the compact-open topology. The most important examples are in your calculus textbook:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & S^1 & \longrightarrow & 1 \\ & & \wr\| & & \wr\| & & \wr\| & & \\ 1 & \longrightarrow & \mathrm{Hom}_{\mathrm{cont}}(S^1, S^1) & \longrightarrow & \mathrm{Hom}_{\mathrm{cont}}(\mathbb{R}, S^1) & \longrightarrow & \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}, S^1) & \longrightarrow & 1 \end{array}$$

and in your algebraic number theory textbook:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{A}_{\mathbb{Q}} & \longrightarrow & \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} & \longrightarrow & 1 \\ & & \wr\| & & \wr\| & & \wr\| & & \\ 1 & \longrightarrow & \widehat{\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}} & \longrightarrow & \widehat{\mathbb{A}_{\mathbb{Q}}} & \longrightarrow & \widehat{\mathbb{Q}} & \longrightarrow & 1 \end{array}$$

and it has a p-adic counter-part:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}_p & \longrightarrow & \mathbb{Q}_p & \longrightarrow & \mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow & 1 \\ & & \wr\| & & \wr\| & & \wr\| & & \\ 1 & \longrightarrow & \widehat{\mathbb{Q}_p/\mathbb{Z}_p} & \longrightarrow & \widehat{\mathbb{Q}_p} & \longrightarrow & \widehat{\mathbb{Z}_p} & \longrightarrow & 1 \end{array}$$

Consider $\psi : \mathbb{Z}[1/p] \to S^1$ by $n \mapsto e^{2\pi i n}$. This is continuous with respect to $|\cdot|_p$, and therefore extends to a continuous group homomorphism which we denote by $\psi_p : \mathbb{Q}_p \to S^1$. We define $\psi_F : F \xrightarrow{\mathrm{tr}_{F/\mathbb{Q}_p}} \mathbb{Q}_p \xrightarrow{\psi} S^1$. In particular, F is an abelian locally compact Hausdorff topological group and

$$\begin{array}{ccc} F & \cong & \widehat{F} := \mathrm{Hom}_{\mathrm{cont}}(F, S^1) \\ x & \mapsto & (y \mapsto \psi_F(xy)) \end{array}$$

This is generalized to any vector space, e.g. $\mathfrak{g}$ as well as $\mathfrak{g}^* := \mathrm{Hom}_F(\mathfrak{g}, F)$. We have

$$\begin{array}{ccc} \mathfrak{g}^* & \cong & \widehat{\mathfrak{g}^*} := \mathrm{Hom}_{\mathrm{cont}}(\mathfrak{g}, S^1) \\ X^* & \mapsto & (Y \mapsto \psi_F(\langle X^*, Y\rangle)) \end{array}$$

By definition $\mathrm{Hom}_{\mathrm{cont}}(\mathfrak{g}, S^1)$ is the set of (continuous) characters of $\mathfrak{g}$ with values in S^1. Now, let us try to pretend that $\mathfrak{g}$ and G are the same; at least near the identity they are isomorphic F-analytic manifolds via log and exp. Then, we are hoping that characters of (representations of) G, maybe locally, look like elements in $\mathfrak{g}^*$.

Things cannot be this simple for an obvious reason: G is not abelian - not even locally, e.g. when $G = \mathrm{SL}_2(F)$. In particular characters on G should **not** correspond to a single element in $\mathfrak{g}^*$. Another hint comes already from the theory of characters of finite groups, where characters are compared with *class functions*. Therefore, we would like to study not one element $X^* \in \mathfrak{g}^*$, but rather a **coadjoint orbit** $O^* = \mathrm{Ad}^*(G)(X^*) \subset \mathfrak{g}^*$.

4.2. **Coadjoint orbits.** For the rest of this section, we work over $\mathcal{C}$, any field of characteristic different from p. We are also going to work with $\mathfrak{g}^* := \mathrm{Hom}_F(\mathfrak{g}, F)$. The action $\mathrm{Ad} : G \to \mathrm{Aut}_F(\mathfrak{g})$ and $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}_F(\mathfrak{g})$ gives $\mathrm{Ad}^* : G \to \mathrm{Aut}_F(\mathfrak{g}^*)$ and $\mathrm{ad}^* : \mathfrak{g} \to \mathrm{End}_F(\mathfrak{g}^*)$, where $\mathrm{Ad}^*(g) = (g^t)^{-1}$ and $\mathrm{ad}^*(X) = -X^t$. They are called the **coadjoint action** and **coadjoint representation**. We however have

Theorem 4.1. *There exists a non-degenerate symmetric bilinear pairing $\mathfrak{g} \times \mathfrak{g} \to F$ that is $\mathrm{Ad}(G)$-invariant. Equivalently, there exists a self-adjoint isomorphism $\iota : \mathfrak{g}^* \cong \mathfrak{g}$ that is equivariant under G (for Ad^* and Ad).*

Moreover, Jordan decompositions do not depend on the choice of ι. That is, for any two G-equivariant isomorphisms $\iota_1, \iota_2 : \mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g}$, if $X_s + X_n$ is a Jordan decomposition, then so is $(\iota_2)^{-1}(\iota_1(X_s)) + (\iota_2)^{-1}(\iota_1(X_n))$.

This can be deduced e.g. from Theorem 4.13 in https://www.jmilne.org/math/CourseNotes/LAG.pdf. As a consequence, it is ok if you fix ι and pretend $\mathfrak{g}^* = \mathfrak{g}$ the whole time. Anyhow, fix $X^* \in \mathfrak{g}^* = (\mathrm{Lie}^* \mathbb{G})(F)$. Write $O^* = \mathrm{Ad}^*(G)(X^*)$ and also $\mathbb{O}^* = \mathrm{Ad}^*(\mathbb{G})(X^*)$. Let us put

Definition 4.2. *The centralizers of X^* are*

$$Z_{\mathbb{G}}(X^*) = \{g \in \mathbb{G} \mid \mathrm{Ad}^*(g)X^* = X^*\}.$$
$$Z_G(X^*) := \{g \in G \mid \mathrm{Ad}^*(g)X^* = X^*\}.$$
$$Z_{\mathfrak{g}}(X^*) := \{Y \in \mathfrak{g} \mid \mathrm{ad}^*(Y)X^* = X^*\}.$$

Since F is perfect, by [Bor91, Proposition 6.7] we see that $\mathbb{O}^*$ is a locally closed[8] smooth subvariety of $\mathrm{Lie}^* \mathbb{G}$, and that the action map induces $\mathrm{Lie}\,\mathbb{G}/\mathrm{Lie}\,Z_{\mathbb{G}}(X^*) \cong T_{X^*}\mathbb{O}^*$. Passing to F-points, this says

$$(Y \mapsto \mathrm{ad}^*(Y)X^*]) : \mathfrak{g} \twoheadrightarrow T_{X^*}(\mathbb{O}^*(F)) \text{ with kernel } Z_{\mathfrak{g}}(X^*). \tag{4}$$

Note that O^* is a subset of $\mathbb{O}^*(F)$. The surjectivity (4) however shows that $(g \mapsto \mathrm{Ad}^*(g)X^*) : G \mapsto \mathbb{O}^*(F)$ has surjective differential and therefore its image, i.e. O^* is an open sub-manifold of $\mathbb{O}^*(F)$. The complement is a union of such orbits, and therefore open. That is $O^* \subset \mathbb{O}^*(F)$ is also closed. We remark that the implicit function theorem for $G \to O^*$ also shows

Lemma 4.3. *We have $\mathrm{Lie}\, Z_G(X^*) = Z_{\mathfrak{g}}(X^*)$.*

Example 4.4. Let $\mathbb{G} = \mathrm{SL}_2$ and $X^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then

$$\mathbb{O}^*(F) = \bigsqcup_{a \in F^\times/(F^\times)^2} \mathrm{Ad}^*(G) \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}.$$

Exercise 8. *Let $n \in \mathbb{Z}_{\geq 1}$. Suppose $\mathbb{G} = \mathrm{SL}_n$, $G = \mathrm{SL}_n(F)$ and X is the nilpotent matrix with $X_{ij} = 1$ when $j = i + 1$ and zero elsewhere. Let $\mathbb{O} = \mathrm{Ad}(\mathbb{G})X$. Describe directly (i.e. without using Galois cohomology, but in terms of F) how $\mathbb{O}(F)$ is a disjoint union of $\mathrm{Ad}(G)$-orbits, and justify your description.*

---

[8]Curiously, "locally closed" means an intersection of a Zariski closed and a Zariski open subset, not "closed locally in the ambient space."

Remark 4.5. In general, there is a short exact sequence of *pointed sets* that looks like

$$1 \to Z_G(X^*) \to G \to \mathbb{O}^*(F) \to H^1(F, Z_{\mathbb{G}}(X^*)) \to H^1(F, \mathbb{G})$$

It can be proved that the **Galois cohomology** $H^1(F, \mathbb{G})$ for any linear algebraic group is finite. This implies that $\mathbb{O}^*(F)$ is always a finite disjoint union of $G/Z_G(X^*) = O^*$ for a finite number of $X^* \in \mathbb{O}^*(F)$, or $O^* \subset \mathbb{O}^*(F)$.

Remark 4.6. The "correct" definition of $Z_{\mathbb{G}}(X^*)$ should at least[9] be the scheme-theoretic centralizer $\mathbb{G} \times_{\mathbb{O}^*} \{X^*\}$. The Lie algebra of the scheme-theoretic centralizer is by definition $Z_{\mathrm{Lie}\,\mathbb{G}}(X^*)$. Since $\mathrm{Lie}(Z_{\mathbb{G}}(X^*)) = Z_{\mathrm{Lie}\,\mathbb{G}}(X^*)$, we see that the scheme-theoretic centralizer $Z_{\mathbb{G}}(X^*)$ is smooth, and therefore geometrically reduced, and therefore equal to our variety-theoretic definition.

We have seen that $T_{X^*}O^* = \mathfrak{g}/Z_{\mathfrak{g}}(X^*)$. Observe that

$$\begin{array}{rccc} \omega_{X^*}: & \mathfrak{g}/Z_{\mathfrak{g}}(X^*) \times \mathfrak{g}/Z_{\mathfrak{g}}(X^*) & \to & F \\ & (Y, Z) & \mapsto & \langle [Y, Z], X^* \rangle \end{array}$$

is an anti-symmetric non-degenerate bilinear form; to see it is non-degenerate, we use that $\langle [Y, Z], X^* \rangle = \langle \mathrm{ad}(Y)Z, X^* \rangle = -\langle Z, \mathrm{ad}^*(Y)X^* \rangle$. If the last term is 0 for any $Z \in \mathfrak{g}$, then $\mathrm{ad}^*(Y)X^* = 0$, i.e. $Y \in Z_{\mathfrak{g}}(X^*)$. The very existence of ω_{X^*} shows that

Lemma 4.7. $\dim \mathbb{O}^* = \dim O^* = \dim \mathfrak{g}/Z_{\mathfrak{g}}(X^*)$ *is even.*

It is easy to see that ω_{X^*} for all $X^* \in O^*$ define an analytic 2-form ω_{O^*} on O^*.

Remark 4.8. The 2-form is evidently also algebraic. They are also G-invariant, meaning that they are preserved under the $\mathrm{Ad}^*(G)$-action (resp. $\mathrm{Ad}^*(\mathbb{G})$-action) on O^*. Lastly, they are closed, meaning that $d\omega_{O^*} = 0$, making O^* a symplectic manifold; see [CG10, Claim 1.1.6].

We won't directly use that O^* is symplectic, and Cheng-Chiang is not aware if anything we will use implicitly share the same conceptual origin with this fact, but it is important in geometric representation theory.

Let us consider the top exterior power $\bigwedge^{\frac{1}{2}\dim O^*} \omega_{O^*}$, which is a nowhere-vanishing top form. Lemma 2.11 then gives us a $\mathbb{Z}[1/p]$-valued measure on O^*. This allows us to integrate function, i.e. we have a $\mathcal{C}$-linear functional

$$I_{O^*} : \mathcal{C}_c^\infty(O^*) \to \mathcal{C} \tag{5}$$

This feels like the best approximation to a class function; after all, there is no non-zero locally constant function on $\mathfrak{g}$ supported on O^* because $O^* \subset \mathfrak{g}^*$ is not an open subset. However it has a serious issue: The restriction of a function in $\mathcal{C}_c^\infty(\mathfrak{g}^*)$ to O^* does **not** always have compact support.

Example 4.9. Let $\mathbb{G} = \mathrm{GL}_2$, $G = \mathrm{GL}_2(F)$ and $X^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \in O^*$ for any $a \in F^\times$. In particular the zero matrix is in the closure of O^*, but not in O^*. Hence for any open subset $K \subset \mathfrak{g}^*$ with $0 \in K$, we see that $K \cap O^*$ is not closed in $\mathfrak{g}^*$, and therefore not compact.

[9] Cheng-Chiang knows nothing about derived algebraic geometry.

An important result of Ranga Rao [RR72], also independently by Deligne, proves that

Theorem 4.10. *Suppose $\mathcal{C} = \mathbb{R}$ or $\mathbb{C}$. Then any $I_{O^*}(f)$ converges absolutely. That is, although $\mathrm{supp}(f|_{O^*})$ might not be compact, the integral $I_{O^*}(f|_{O^*})$ always converges absolutely.*

We will nevertheless talk about Theorem 4.10 from a point of view that involves no analysis on $\mathcal{C}$.

4.3. **Basic geometry of $\mathrm{Ad}(G)$-orbits.** To understand what happens around Theorem 4.10, we need to understand the geometry of O^*. In Theorem 4.1, we have seen a G-equivariant isomorphism $\iota : \mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g}$, and thus we can work on $\mathfrak{g}$. Let us first study the semisimple and nilpotent situation separately:

Proposition 4.11. *Let $X_s \in \mathfrak{g}$ be semisimple. Then $Z_{\mathbb{G}}(X_s)$ is a connected reductive group. After a finite base change, it is a Levi subgroup.*

Proof. Let $n \in \mathbb{Z}_{>0}$ be large enough so that $\exp(cX_s)$ for any $c \in \mathfrak{m}_F^n$ is defined. Since X_s is semisimple (diagonalizable over some finite extension in some $\mathfrak{gl}_n(F)$), all $\exp(cX_s)$ are also simultaneously diagonalizable over some finite extension. Let $\mathbb{T}$ be the identity component of the Zariski closure of $\{\exp(cX_s) \mid c \in \mathfrak{m}_F^n\}$. We claim that $\mathbb{T}$ is connected. Observe that $\mathbb{T}^\circ(F)$ is Lie subgroup of G such that $X_s = \frac{d\exp(cX_s)}{dc} \in \mathrm{Lie}(\mathbb{T}^\circ(F))$. Hence $\exp(cX_s) \in \mathbb{T}^\circ$ and thus $\mathbb{T}$ is the Zariski closure of $\mathbb{T}^\circ$ which means $\mathbb{T} = \mathbb{T}^\circ$.

This also gives $Z_{\mathbb{G}}(X_s) = Z_{\mathbb{G}}(\mathbb{T})$. It is therefore connected by [Bor91, Corollary 11.12]. That $Z_{\mathbb{G}}(X_s)$ is reductive is in [Bor91, §13.17, Corollary 2]. It is a Levi subgroup after some base change by [Bor91, Proposition 20.4]. □

Remark 4.12. Proposition 4.11 implies, and gives a proof[10], of the algebraic group version that $Z_{\mathbb{G}}(X_s)$ is connected for any field of characteristic 0. This is not true for semisimple $s \in G$ in the group. A counter-example is $\mathbb{G} = \mathrm{SO}_V$ for $V = F^3$ with $(\vec{x}, \vec{y}) = x_1y_1 + x_2y_2 + x_3y_3$ and $s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $Z_{\mathbb{G}}(s) \cong \mathrm{O}_2$. The analogue of Proposition 4.11 is also not true over a field of (small) positive characteristic, e.g. for $\mathbb{G} = \mathrm{SO}_3/_{\mathbb{F}_2}$ and $X_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Theorem 4.13. *Let X_s be semisimple. Then $\mathbb{O} = \mathrm{Ad}(\mathbb{G})X_s$ is Zariski closed, and $O = \mathrm{Ad}(G)X_s \subset \mathfrak{g}$ is a closed sub-manifold.*

Proof. The first assertion is [Bor91, Theorem 9.2]. The second assertion follows. □

Next, we look at nilpotent elements.

Theorem 4.14. *Let $X_n \in \mathfrak{g}$ be nilpotent. Then there exists an algebraic group homomorphism $\lambda : \mathrm{GL}_1 \to \mathbb{G}$ such that*

$$\mathrm{Ad}(\lambda(t))X_n = t^2 X_n$$

[10]The other simple proof Cheng-Chiang knows uses real/complex Lie groups and is very similar.

for any $t \in F^\times$.

Proof. We sketch the construction in [Car93, §5.5]. Elements $H \in \mathfrak{g}$ and $Y \in \mathfrak{g}$ are constructed so that

$$[X_n, Y] = H,\ [H, X_n] = 2X_n,\ [H, Y] = -2Y.$$

Then one construct

$$\lambda(t) = \exp(-t^{-1}Y)\exp(tX_n)\exp(-t^{-1}Y)\exp(Y)\exp(-X_n)\exp(Y)$$

for any $t \in \bar{F}^\times$. This gives an algebraic map defined over F. □

Remark 4.15. The construction may look a bit weird, but is in fact very natural and comes from the base case $G = \mathrm{SL}_2$ and $X_n = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. In this case $\lambda(t) = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$.

Corollary 4.16. *Let $X_n \in \mathfrak{g}$ be nilpotent. Then for any $t \in F^\times$, we have $t^2X_n \in \mathrm{Ad}(G)X_n$.*

Remark 4.17. We can compare with the linear algebra fact, that a matrix A is conjugate to tA for some $t \in F^\times$ that is not a root of unity, or every $t \in F^\times$, if and only if A is nilpotent.

Corollary 4.18. *Let $X_n \in \mathfrak{g}$ be nilpotent. Then 0 is contained in the closure of $\mathrm{Ad}(G)X_n$. More generally, suppose $X = X_s + X_n$ is a Jordan decomposition. Then $\mathrm{Ad}(G)X_s$ is contained the closure of $\mathrm{Ad}(G)X$.*

In particular, for general $X \in \mathfrak{g}$, the orbit $\mathrm{Ad}(g)X$ is closed if and only if X is semisimple. Same results are true for the algebraic group versions.

Proof. For X_n nilpotent, $\mathrm{Ad}(G)X_n$ contains t^2X_n which converges to 0 as $|t| \to 0$. For general $X = X_s + X_n$, since $Z_{\mathbb{G}}(X_s)$ is reductive, we have that

$$t^2X_n \in \mathrm{Ad}(Z_G(X_s))X_n \implies X_s + t^2X_n \in \mathrm{Ad}(G)(X_s + X_n).$$

And thus X_s, or consequently $\mathrm{Ad}(G)(X_s)$, is in the closure of $\mathrm{Ad}(G)(X_s + X_n)$. The corresponding algebraic group version follows from the Lie group version directly. □

It is also good to know that

Lemma 4.19. *We have $Z_G(X) \subset Z_G(X_s)$, $Z_G(X) \subset Z_G(X_n)$, $Z_{\mathfrak{g}}(X) \subset Z_{\mathfrak{g}}(X_s)$ and $Z_{\mathfrak{g}}(X) \subset Z_{\mathfrak{g}}(X_n)$.*

Proof. Suppose $\mathrm{Ad}(g)X = X$, then $X = \mathrm{Ad}(g)(X_s + X_n) = \mathrm{Ad}(g)X_s + \mathrm{Ad}(g)X_n$ and therefore $\mathrm{Ad}(g)X_s = X_s$ and $\mathrm{Ad}(g)X_n = X_n$. The result for the Lie algebra follows from Lemma 4.3 for X, X_s and X_n. □

Definition 4.20. *(i) Denote by $\mathfrak{g}/G$ (resp. $\mathfrak{g}^*/G$) the* **set** *of $\mathrm{Ad}(G)$-orbits in $\mathfrak{g}$ (resp. $\mathrm{Ad}^*(G)$-orbits in $\mathfrak{g}^*$).*

(ii) Denote by $\mathfrak{g}^{\mathrm{nil}} \subset \mathfrak{g}$ the subset of nilpotent elements. We call $\mathfrak{g}^{\mathrm{nil}}$ the **nilpotent cone***. We have $\mathfrak{g}^{\mathrm{nil}}/G \subset \mathfrak{g}/G$.*

Remark 4.21. $\mathfrak{g}^{\mathrm{nil}}$ is closed, and is a cone, i.e. X_n nilpotent $\implies$ tX_n nilpotent for $t \in F$; this is obvious with any embedding $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n(F)$.

Theorem 4.22. *The set $\mathfrak{g}^{\mathrm{nil}}/G \subset \mathfrak{g}/G$ of nilpotent orbits is finite. As a consequence, for any $O \in \mathfrak{g}/G$ we have $\{O' \in \mathfrak{g}/G \mid O' \leq O\}$ finite.*

Sketch. That the second sentence follows from the first sentence is Proposition 4.39 (where it will be obvious that the proof has nothing that could depends on this theorem or anything after).

For the first sentence, there are many proofs. Most of the proof Cheng-Chiang knows use Galois cohomology. One first proves that the set of $\mathbb{G}$-orbits in $(\mathrm{Lie}\,\mathbb{G})^{\mathrm{nil}}$ is finite. This can be proved in many ways, including (a) using classification of nilpotent orbits [CM93, §3] similar to the proof of Theorem 4.14, (b) using the reductivity of $\mathfrak{g}$ and reduce to $\mathfrak{gl}_n$, or (c) by comparing with finite groups of Lie type. Once we know that the set of $\mathbb{G}$-orbits in $(\mathrm{Lie}\,\mathbb{G})^{\mathrm{nil}}$ is finite, the rest follows from Remark 4.5.

There is another proof [DeB02] that only works with some additional assumption on p. It does not use the Galois cohomology but uses Bruhat-Tits theory instead, and gives an important description of $\mathfrak{g}^{nil}/G$. □

4.4. **Coinvariants.** We work with $\mathfrak{g}^*$ once again, and transport all results from §4.3 using Theorem 4.1. We however begin with the remark that Corollary 4.18 implies the part of Theorem 4.1 that the notion of semisimple and nilpotent elements in $\mathfrak{g}^*$ does not depend on $\iota : \mathfrak{g}^* \cong \mathfrak{g}$.

Let $X^* \in \mathfrak{g}^*$ and $O^* := \mathrm{Ad}^*(G)X^*$ be as before. We are ready to look at $I_{O^*} : \mathcal{C}_c^\infty(O^*) \to \mathcal{C}$.

Definition 4.23. *The* **coinvariant** $\mathcal{C}_c^\infty(O^*)_G$ *is*

$$\mathcal{C}_c^\infty(O^*)/\langle \mathrm{Ad}^*(g)_* f - f \mid f \in \mathcal{C}_c^\infty(O^*),\ g \in G\rangle$$

Lemma 4.24. *I_{O^*} factors through $\mathcal{C}_c^\infty(O^*)_G$, and $\dim_{\mathcal{C}}(\mathcal{C}_c^\infty(O^*)_G) = 1$.*

[CC: The previous proof was wrong, and a correct (brand new) proof was updated on March 16th.]

Proof. The first assertion is obvious. To prove the second assertion, let us understand what O^* looks like locally. The map $(X^* \mapsto \mathrm{Ad}^*(g)X^*) : G \twoheadrightarrow O^*$ is surjective on tangent spaces and induces a bijection $G/Z_G(X^*) \cong O^*$ (of sets). Since the group structure of G locally near e looks like $m(x, y) = x + y + O(|xy|)$, we may choose open compact subgroups $J_a \subset G$ with $J'_a = J_a \cap Z_G(X^*)$ as in (the proof of) Lemma 2.14, for sufficiently large a, such that locally $J_a \cong (\mathfrak{m}_F^a)^{\dim G} \supset (\mathfrak{m}_F^a)^{\dim Z_G(X^*)} \cong J'_a$, and such that $(\mathfrak{m}_F^a)^{\dim G - \dim Z_G(X^*)} \cong \mathrm{Ad}^*(J_a)X^*$. (Implicit function theorem is being used again and again.)

Every function $f \in \mathcal{C}_c^\infty(\mathrm{Ad}^*(J_a)X^*)$ is locally constant by some $\mathrm{Ad}^*(J_b)$ for some big enough $b > a$, and thus is a function on $(\mathfrak{m}_F^a/\mathfrak{m}_F^b)^{\dim G - \dim Z_G(X^*)}$. The group $J_a/J_b \cong (\mathfrak{m}_F^a/\mathfrak{m}_F^b)^{\dim G}$ acts transitively on such functions. Taking the union for all $b > a$, this shows that $\dim_{\mathcal{C}} \mathcal{C}_c^\infty(\mathrm{Ad}^*(J_a)X^*)_{J_a} = 1$. Since G acts transitively on O^*, and every $f \in \mathcal{C}_c^\infty(O^*)$ can be written as a linear combination of functions on small enough neighborhoods, this shows that $\dim \mathcal{C}_c^\infty(O^*)_G \leq 1$. That $\dim_{\mathcal{C}} \mathcal{C}_c^\infty(O^*)_G = 1$ then follows from that I_{O^*} is non-trivial. □

To proceed, we need a partial order on $\mathfrak{g}/G$ and $\mathfrak{g}^*/G$:

Definition 4.25. *(1) For $O_1, O_2 \in \mathfrak{g}/G$, we say $O_1 \geq O_2$ (and $O_2 \leq O_1$) if O_2 is contained in the closure of O_1. We say $O_1 > O_2$ (and $O_2 < O_1$) if $O_1 \geq O_2$ and $O_1 \neq O_2$. Same definitions are given for $\mathfrak{g}^*/G$.*

Remark 4.26. It is easy to see that $\geq$ defines a partial order on $\mathfrak{g}/G$. It also follows from definition that, for any $O \in \mathfrak{g}/G$ we have

$$\overline{O} := \text{closure of } O = \bigcup_{O' \leq O} O'.$$

Lemma 4.27. *Suppose $O_1 > O_2$. Then $\dim O_1 > \dim O_2$.*

Proof. Write $O_1 = \mathrm{Ad}(G)X_1$ and $\mathbb{O}_1 = \mathrm{Ad}(\mathbb{G})X_1$. Also $O_2 = \mathrm{Ad}(G)X_2$ and $\mathbb{O}_2 = \mathrm{Ad}(\mathbb{G})X_2$. We have seen that O_1 is Zariski dense in $\mathbb{O}_1$. That O_2 is contained in the closure of O_1 implies that $\mathbb{O}_2$ is in the (Zariski) closure of $\mathbb{O}_1$. Hence either $\mathbb{O}_2 = \mathbb{O}_1$, or $\dim O_2 = \dim \mathbb{O}_2 < \dim \mathbb{O}_1 = \dim O_1$. We show that $\mathbb{O}_2 = \mathbb{O}_1$ is impossible. Suppose otherwise, then we have seen that O_1 and O_2 are both closed in $\mathbb{O}_1(F) = \mathbb{O}_2(F)$, which is a contradiction. □

Fix any O^* and consider

$$\overline{O^*} = \bigcup_{(O')^* \leq O^*} (O')^*, \quad \partial O^* = \bigcup_{(O')^* < O^*} (O')^*$$

Note that $O^* \subset \overline{O^*}$ is open as we have seen O^* is locally closed. We have a short exact sequence

$$0 \to \mathcal{C}_c^\infty(O^*) \to \mathcal{C}_c^\infty(\overline{O^*}) \to \mathcal{C}_c^\infty(\partial O^*) \to 0.$$

Here the first map is extension by zero using that $O^* \subset \overline{O^*}$ is open, and the second map is restriction, using that $\partial O^* \subset \overline{O^*}$ is closed.

Exercise 9. *Show that $\mathcal{C}_c^\infty(\overline{O^*}) \to \mathcal{C}_c^\infty(\partial O^*)$ is surjective.*

The short exact sequence induces a right exact sequence

$$\mathcal{C}_c^\infty(O^*)_G \to \mathcal{C}_c^\infty(\overline{O^*})_G \to \mathcal{C}_c^\infty(\partial O^*)_G \to 0 \tag{6}$$

Corollary 4.28. *When $\mathrm{char}(\mathcal{C}) = 0$, the sequence (6) is furthermore exact on the left.*

Proof. When $\mathcal{C} = \mathbb{C}$, this follows from the important Theorem 4.10, where absolute convergence is used to extend the functional I_{O^*} from $\mathcal{C}_c^\infty(O^*)_G$ to $\mathcal{C}_c^\infty(\overline{O^*})_G$ in a non-trivial way. The existence of such a functional implies that $\mathcal{C}_c^\infty(O^*)_G \to \mathcal{C}_c^\infty(\overline{O^*})_G$ is non-trivial, therefore injective, since $\dim_{\mathcal{C}}(\mathcal{C}_c^\infty(O^*)_G) = 1$.

That the sequence is exact for $\mathcal{C} = \mathbb{C}$ implies that it is exact for $\mathcal{C} = \mathbb{Q}$, and therefore exact whenever $\mathrm{char}(\mathcal{C}) = 0$. □

Fact 4.29. [VW01, §B] *For any field $\mathcal{C}$ (of characteristic different from p), the sequence (6) is also exact on the left.*

We see that, although the original statement and proof of Theorem 4.10 involves (real) analysis, Corollary 4.28 is completely algebraic. The curious situation is that Theorem 4.10 and Corollary 4.28 are *not* needed for our theory at this stage, as we will see in the next subsection.

Exercise 10. *The* **wave-front set** *of $O \in \mathfrak{g}/G$ is defined to be*

$$\{Y \in \mathfrak{g} \mid \exists t_i \in (F^\times)^2 \text{ and } X_i \in O \text{ such that } \lim t_i \to 0 \text{ and } \lim t_i X_i = Y\}.$$

The determination of wave-front sets, for example for SO_V and Sp_V, is an open problem. (The answer is known, but rather non-trivial, when $F = \mathbb{R}$ and $(F^\times)^2 = \mathbb{R}_{>0}$.)

Suppose $\mathbb{G} = \mathrm{GL}_n$. Show that the wave-front set of any $O \in \mathfrak{g}/G$ contains the closure of the nilpotent orbit $\mathrm{Ad}(G)N \in \mathfrak{g}^{\mathrm{nil}}/G$ that appears in its rational canonical form of O. That is, N is the strictly lower triangular part of the rational canonical form, as in https://en.wikipedia.org/wiki/Frobenius_normal_form#Example.

Exercise 11. *(*) Show that in Exercise 10, the wave-front set actually* **is** *the closure of the indicated orbit.*

4.5. **Nilpotent orbital integrals, whenever they exist?** Let us begin with a definition.

Definition 4.30. *Let $E \subset \mathfrak{g}^*$ be any $\mathrm{Ad}^*(G)$-invariant subset. We denote by $D(E) := \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}_c^\infty(E), \mathcal{C})$ the space of $\mathcal{C}$-linear functionals on $\mathcal{C}_c^\infty(E)$ (with no continuity condition). Suppose E is stabilized by $\mathrm{Ad}^*(G)$. We denote by $D(E)^G$ the subspace of those functionals invariant under $\mathrm{Ad}^*(g)$ for any $g \in G$. Equivalently*

$$D(E)^G := \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}_c^\infty(E)_G, \mathcal{C}).$$

They will be called **invariant distributions** *on E.*

We fix an element $O^* \in (\mathfrak{g}^*)^{\mathrm{nil}}$. The sequence (6) can be dualized to

$$0 \to D(\partial O^*)^G \to D(\overline{O^*})^G \to D(O^*)^G \tag{7}$$

We have already defined $I_{O^*} \in D(O^*)^G$. For any function $f \in \mathcal{C}_c^\infty(O^*)$, and any $t \in F$, we denote by $f_{t^2} \in \mathcal{C}_c^\infty(O^*)$ given by $f_{t^2}(X) := f(t^2 X)$. We have

Proposition 4.31. *$I_{O^*}(f_{t^2}) = |t|^{-\dim O^*} I_{O^*}(f)$ for any $f \in \mathcal{C}_c^\infty(O^*)$ and $t \in F^\times$.*

Proof. We have seen in Theorem 4.14 that $(X \mapsto t^2 X)$ stabilizes O^*. The distribution I_{O^*} is defined by integrating the form $\bigwedge^{\frac{\dim O^*}{2}} \omega_{O^*}$. At $X^* \in O^*$, the 2-form ω_{X^*} is given by $\omega_{X^*}(Y, Z) = \langle [Y, Z], X^* \rangle$. In particular, (the push-forward of) the map $(X \mapsto t^2 X)$ sends ω_{O^*} to $t^{-2}\omega_{O^*}$, and therefore $(X \mapsto t^2 X)$ sends $\bigwedge^{\frac{\dim O^*}{2}} \omega_{O^*}$ to $t^{-\dim O^*} \cdot \bigwedge^{\frac{\dim O^*}{2}} \omega_{O^*}$

The function f_{t^2} is the pullback of f under $(X \mapsto t^2 X)$. Therefore

$$\int_{O^*} f_{t^2} \cdot |\omega| = \int_{O^*} f \cdot (X \mapsto t^2 X)_* |\omega| = \int_{O^*} f \cdot |(X \mapsto t^2 X)_* \omega| = |t|^{-\dim O^*} \int_{O^*} f |\omega|$$

which proves the Proposition. ∎

Definition 4.32. *Let $E \subset \mathfrak{g}^*$ be a $\mathrm{Ad}^*(G)$-invariant subset that is furthermore invariant under $(X \mapsto t^2 X)$ for any $t \in F^\times$. For $k \in \mathbb{Z}$, we say $\delta \in D(E)^G$* **is of weight**[11] *k if*

$$\delta(f_{t^2}) = |t|^k \cdot \delta(f)$$

[11] I just invented this notion; it is not used in the community.

for any $f \in \mathcal{C}_c^\infty(E)$ *and* $t \in F^\times$. *Here* $|t|^k \in q^{\mathbb{Z}}$ *is mapped into* $\mathcal{C}$ *in the obvious way. We say* $\delta \in D(E)^G$ **is of weights** W *for a set* W *of integers if* $\delta = \sum \delta_{k'}$ *is a finite sum of elements of weight* k' *for some* $k' \in W$. *(Not all elements in* W *have to appear.)*

We say a subspace of $D(E)^G$ *is of some weight(s) if every element in that subspace is of those weights.*

Remark 4.33. A non-zero element of weight k is of weight $k+a$ if and only if $q^a = 1$ in $\mathcal{C}$.

Proposition 4.34. *Let* $O^* \in (\mathfrak{g}^*)^{\mathrm{nil}}/G$. *The space* $D(\overline{O^*})^G$ *is of weights* $\{k \in 2\mathbb{Z} \mid -\dim O^* \leq k \leq 0\}$. *We also have* $\dim_{\mathcal{C}} D(\overline{O^*})^G \leq \#\{O' \in (\mathfrak{g}^*)^{\mathrm{nil}}/G \mid O' \leq O\}$.

Proof. We prove by induction on O^*, but before that we need to state a slightly different induction hypothesis. Consider any subset of $(\mathfrak{g}^*)^{\mathrm{nil}}/G$ that is a lower set, i.e. $L \subset (\mathfrak{g}^*)^{\mathrm{nil}}/G$ such that if $O^* \in L$ and $(O^*)' < O^*$, then $(O^*)' \in L$. Denote by

$$c(L) := \bigsqcup_{(O^*)' \in L} (O^*)' \subset \mathfrak{g}^*.$$

The proposition then follows from Lemma 4.35 below, applied to the lower set $L = \{(O^*)' \mid (O^*)' \leq O^*\}$. ☐

Lemma 4.35. *For any lower set* $L \subset (\mathfrak{g}^*)^{\mathrm{nil}}/G$, *the space* $D(c(L))^G$ *is of weights* $\{\dim(O^*) \mid O^* \in L\}$ *and* $\dim_{\mathcal{C}} D(c(L))^G \leq \#L$.

Proof. We use induction on $\#L$. Suppose $L' \subset L$ is any subset that is also a lower set. Then $c(L \backslash L') \subset c(L)$ is closed, and we denote by $c'(L') := c(L) \backslash c(L \backslash L')$ the open subset of $c(L)$. In fact $c(L \backslash L') = \partial c'(L')$. we have the following left exact sequence generalizing (7)

$$0 \to D(c(L \backslash L'))^G \to D(c(L))^G \to D(c'(L'))^G. \tag{8}$$

When $L = \emptyset$ the assertion is trivial. Otherwise, let $O^* \in L$ be maximal and $L' := L \backslash \{O^*\}$. By induction the lemma holds for L'. The result then follows from (8), Lemma 4.24 and Proposition 4.31. ☐

Corollary 4.36. *Let* $O^* \in (\mathfrak{g}^*)^{\mathrm{nil}}/G$. *Suppose* $q^a \neq 1$ *in* $\mathcal{C}$ *for any* $a \in \{k \in 2\mathbb{Z} \mid 2 \leq k \leq \dim O^*\}$. *Then* $D(\overline{O^*})$ *has at most one line of weight* $\dim O^*$. *In this case, such a line exists if and only if* (7) *is furthermore right exact.*

Proof. This follows from (7) in which all the maps respect weights, and Lemma 4.35 applied to $L := \{(O')^* \in (\mathfrak{g}^*)^{\mathrm{nil}}/G \mid (O')^* < O^*\}$. ☐

Definition 4.37. *Suppose* $q^a \neq 1$ *in* $\mathcal{C}$ *for any* $a \in \{k \in 2\mathbb{Z} \mid 2 \leq k \leq \dim O^*\}$. *Then by abuse of language we denote by* $I_{O^*} \in D(\overline{O^*})^G \subset D((\mathfrak{g}^*)^{\mathrm{nil}})^G$ *the unique element in* $D(\overline{O^*})$ *of weight* $\dim O^*$ *that maps to* I_{O^*} *in* (5), *if it exists.*

The next corollary follows from induction and Corollary 4.36.

Corollary 4.38. *Suppose* $q^a \neq 1$ *in* $\mathcal{C}$ *for any even integer* a *with* $0 < a \leq \dim O^*$ *for some* $O^* \in (\mathfrak{g}^*)^{\mathrm{nil}}/G$. *Then for any lower set* $L \subset (\mathfrak{g}^*)^{\mathrm{nil}}/G$, *the space* $D(c(L))^G$ *has a basis given by* $\{I_{O^*}\}$ *in Definition 4.37 for those* $O^* \in L$ *for which* I_{O^*} *exists. In particular,* $D((\mathfrak{g}^*)^{\mathrm{nil}})^G$ *has a basis given by* $\{I_{O^*}\}$ *for those* $O^* \in (\mathfrak{g}^*)^{\mathrm{nil}}/G$ *for which* I_{O^*} *exists.*

In the end, by Corollary 4.28 and Fact 4.29 the element $I_{O^*} \in D(\overline{O^*})^G$ always exist. In fact, the cited proof of Fact 4.29 relies on Corollary 4.28, as some kind of lifting to characteristic 0, and both of them rely on the theorem of Ranga Rao and Deligne (Theorem 4.10). However, Proposition 4.34 and Corollary 4.38 will be what we really need for the character theory of p-adic reductive groups.

Exercise 12. (*) *Let p be any odd prime, $\mathcal{C} = \mathbb{Q}$ and $\mathbb{G} = \mathrm{SL}_2$ over F. The Lie algebra $\mathfrak{sl}_2$ can be identified with its dual using $\langle A, B\rangle = \mathrm{tr}(AB)$. Let $\mathfrak{g}(\mathcal{O}_F) = \mathfrak{sl}_2(\mathcal{O}_F) \subset \mathfrak{sl}_2(F)$ be those traceless 2×2 matrices in $\mathcal{O}_F$. Let $X^* := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in (\mathfrak{g}^*)^{\mathrm{nil}}$ and $O^* := \mathrm{Ad}^*(G)X^*$. Show that*

$$I_{O^*}(1_{\mathfrak{g}(\mathcal{O}_F)}) = \frac{q}{2}.$$

The previous exercise shows that nilpotent orbital integrals as in Definition 4.37 sort of does not exist when $G = \mathrm{SL}_2$ and $\mathrm{char}(\mathcal{C}) = 2$. I thank Gemini Deep Think from Google Deepmind who solely found this example.

4.6. **General orbital integrals.** Let us work with $\mathfrak{g}$ for a moment, and discuss about reduction of a general element to its semisimple and nilpotent parts.

Proposition 4.39. *Let $X, Y \in \mathfrak{g}$. The following are equivalent:*

(i) The element Y is contained in the closure of $\mathrm{Ad}(G)X$, i.e.

$$\mathrm{Ad}(G)Y \leq_{\mathfrak{g}/G} \mathrm{Ad}(G)X.$$

(ii) There exists $g \in G$ such that $\mathrm{Ad}(g)Y_s = X_s$, and by putting $\mathbb{H} := Z_{\mathbb{G}}(X_s)$, $H := \mathbb{H}(F) = Z_G(X_s)$ and $\mathfrak{h} := \mathrm{Lie}\, H$, we have that $\mathrm{Ad}(g)Y_n$ is in the closure of $\mathrm{Ad}(H)X_n$, i.e.

$$\mathrm{Ad}(H)(\mathrm{Ad}(g)Y_n) \leq_{\mathfrak{g}/H} \mathrm{Ad}(H)X_n.$$

Proof. Suppose (ii) holds. Then there exists $h_i \in H$ such that $\lim \mathrm{Ad}(h_i)X_n = \mathrm{Ad}(g)Y_n$. We have

$$\lim \mathrm{Ad}(g^{-1}h_i)X = \mathrm{Ad}(g^{-1})X_s + \mathrm{Ad}(g^{-1})\,\mathrm{Ad}(g)Y_n = Y_s + Y_n = Y.$$

Conversely, let $g_i \in G$ be such that $\lim \mathrm{Ad}(g_i)X = Y$. Choose an embedding $\mathbb{G} \hookrightarrow \mathrm{GL}_n$. The element Y and every $\mathrm{Ad}(g_i)X$, viewed as $n \times n$ matrices, have the same characteristic polynomial P, and we have an equality of $n\times n$ matrices

$$Y_s = Q_P(Y), \quad \mathrm{Ad}(g_i)X_s = Q_P(\mathrm{Ad}(g_i)X)$$

where Q_P is an F-coefficient polynomial that depends only on P. Hence we have

$$\lim \mathrm{Ad}(g_i)X_s = \lim g_i Q_P(X) g_i^{-1} = \lim Q_P(g_i X g_i^{-1}) = \lim Q_P(Y) = Y_s.$$

Since $\mathrm{Ad}(G)X_s$ is closed, we have $Y_s \in \mathrm{Ad}(G)X_s$, i.e. $\mathrm{Ad}(g)Y_s = X_s$ for some g and

$$\lim \mathrm{Ad}(gg_i)X_s = X_s.$$

Since $G \twoheadrightarrow \mathrm{Ad}(G)X_s$ is surjective on the tangent space, we may choose $g_i' \in G$ with $\lim g_i' = e$ such that $\mathrm{Ad}(g_i')X_s = \mathrm{Ad}(gg_i)X_s$, i.e. $(g_i')^{-1}gg_i \in H$. We have

$$\lim \mathrm{Ad}((g_i')^{-1}gg_i)X = \lim \mathrm{Ad}(gg_i)X = \mathrm{Ad}(g)Y$$

$$\begin{aligned} \implies \quad & \lim \mathrm{Ad}((g_i')^{-1} g g_i) X_n = \lim \mathrm{Ad}((g_i')^{-1} g g_i)(X - X_s) \\ = \quad & \mathrm{Ad}(g)Y - \mathrm{Ad}(g)Y_s = \mathrm{Ad}(g)Y_n \end{aligned}$$

which implies (ii). □

We go back to $\mathfrak{g}^*$ again. Fix $X_s^* \in \mathfrak{g}^*$ semisimple. Let $\mathbb{H} := Z_{\mathbb{G}}(X_s^*)$ and $H := \mathbb{H}(F) = Z_G(X_s^*)$. Let $L \subset (\mathfrak{h}^*)^{\mathrm{nil}}/H$ be a lower set. Let $c(L) := \bigsqcup_{O^* \in L} O^* \subset (\mathfrak{h}^*)^{\mathrm{nil}}$ be a closed subset as before. Let us also write

$$c(X_s^*, L) := \bigsqcup_{O^* \in L} \mathrm{Ad}^*(G)(X_s^* + O^*) \subset \mathfrak{g}^*/G.$$

Proposition 4.40. *We have a canonical isomorphism*

$$D(c(X_s^*, L))^G \cong D(c(L))^M.$$

Proof. Let us first construct the map dually, namely we would like a map

$$\mathcal{C}_c^\infty(c(L))_M \to \mathcal{C}_c^\infty(c(X_s^*, L))_G$$

□

[CC: Ahhh, maybe we need to do semisimple descent first ...] [CC: to continue]

5. Fourier transforms

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