Exercise sheet 1.

March 3th
Due March 10th in class.

Exercise 1. Let H be a p-adic Lie group. Let $\mathcal{C} = \mathbb{C}$ or $\mathcal{C} = \overline{\mathbb{Q}}_\ell$ with its usual topology. Let n be an integer and $\rho : H \to \mathrm{GL}_n(\mathcal{C})$ be a continuous group homomorphism. Show that $\ker(\rho)$ contains an open subgroup.

Exercise 2. Show that M_G is indeed a free module of rank 1 over $\mathbb{Z}[1/p]$. Show that any open compact subset $X \subset G$ can be written as a finite union of J-cosets $X = \bigsqcup_{i=1}^m g_i J$ for some $J \in \Omega(G)$. Show that the quantity $\nu(X) := m \cdot \nu(J) \in \mathbb{Z}[1/p]$ depends only on X and ν but not on the choice of g_i and J.

Exercise 3. Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, which acts on $\mathbb{P}^1(\mathbb{Q}_p)$ in a natural way. Consider the space $\pi = \mathcal{C}_c^\infty(\mathbb{P}^1(\mathbb{Q}_p))$ on which G also acts. Show that π is generated by a single vector, i.e. there exists $v \in \mathcal{C}_c^\infty(\mathbb{P}^1(\mathbb{Q}_p))$ such that

$$\mathcal{C}_c^\infty(\mathbb{P}^1(\mathbb{Q}_p)) = \mathrm{span}_\mathcal{C}\langle g.v \mid g \in \mathrm{GL}_2(\mathbb{Q}_p)\rangle.$$

Remark 1. In the last exercise that, π has at most length 2 when $\mathrm{char}(\mathcal{C}) = 0$, and at most length 3 in general.

Our proof for Theorem A, B and C will actually work with admissible representations generated by finitely many vectors. In the next exercise we motivate part of Theorem B.

Exercise 4. $(^*)^1$ Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and $\pi = \mathcal{C}_c^\infty(\mathbb{P}^1(\mathbb{Q}_p))$ be as in the last exercise.
   (a) Show that this is an admissible representation.
   (b) Assume that Theorem A holds for π, where G^{rs} is the subset consisting of $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ with distinct eigenvalues λ_1, λ_2. Show that

$$t_\pi(g) = \begin{cases} 0 & \text{when } \lambda_1, \lambda_2 \notin \mathbb{Q}_p \text{ (but in a quadratic extension)}, \\ \frac{|\lambda_1|+|\lambda_2|}{|\lambda_1-\lambda_2|} & \text{when } \lambda_1 \neq \lambda_2 \in \mathbb{Q}_p. \end{cases}$$

Here, for general field $\mathcal{C}$, the quantity $\frac{|\lambda_1|+|\lambda_2|}{|\lambda_1-\lambda_2|} \in \mathbb{Z}[1/p]$ is realized in $\mathcal{C}$ via the unique map $\mathbb{Z}[1/p] \to \mathcal{C}$.

Remark 2. We see in the last exercise that, when $\mathcal{C} = \mathbb{C}$, the function $t_\pi \cdot |D_G|^{1/2}$ is locally bounded; here $D_G(g) = (\lambda_1 - \lambda_2)^2$. It is not hard, though seemingly cumbersome, to show that t_π is locally integrable. When $\mathrm{char}(\mathcal{C}) = 2$ and p is odd, we have $p = 1$ in $\mathcal{C}$ and consequently $t_\pi \equiv 0$.

Exercise 5. Let E/F be any quadratic extension. For $x \in E$ denote by $\bar{x}$ the (unique) Galois conjugate of x over F. Let V be an n-dimensional vector space, on which we are given a hermitian form $\langle \cdot, \cdot \rangle : V \times V \to E$, i.e. that is bi-additive and such that

$$\langle c_1 v_1, c_2 v_2 \rangle = c_1 \bar{c}_2 \langle v_1, v_2 \rangle, \ \forall c_1, c_2 \in E, \ v_1, v_2 \in V.$$

Construct an algebraic group $\mathbb{G}$ over F, such that as Lie groups over F we have

$$\mathbb{G}(F) = \text{ group of unitary operators on } (V, \langle \cdot, \cdot \rangle).$$

Show that your $\mathbb{G}$ is reductive only if $\langle \cdot, \cdot \rangle$ is non-degenerate or identically zero.

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[1]Indicating that this problem is more difficult.

Exercise 6. As a follow-up exercise, suppose that $\langle \cdot, \cdot \rangle$ is non-degenerate in Exercise 5. Compute $\mathbb{G}(E)$. Show that your $\mathbb{G}$ is reductive.