Exercise sheet 2.

March 10th

Due March 17th in class.

Exercise 7. Show that there exist an open compact Lie subgroup $K \subset GL_n(F)$ and a lattice $\mathfrak{K} \subset \mathfrak{gl}_n$ such that

$$\log(h) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}(h - \mathrm{id})^i}{i}$$

defines an isomorphism from K to $\mathfrak{K}$, and that

$$\exp(X) = \sum_{i=0}^{\infty} \frac{X^i}{i!}$$

defines the inverse of log from $\mathfrak{K}$ to K. Both series are computed in the space of $n \times n$ matrices in F. Show also that $\log(h^{-1}) = -\log(h)$ for $h \in K$.

Exercise 8. Let $n \in \mathbb{Z}_{\geq 1}$. Suppose $\mathbb{G} = \mathrm{SL}_n$, $G = \mathrm{SL}_n(F)$ and X is the nilpotent matrix with $X_{ij} = 1$ when $j = i + 1$ and zero elsewhere. Let $\mathbb{O} = \mathrm{Ad}(\mathbb{G})X$. Describe directly (i.e. without using Galois cohomology, but in terms of F) how

$$\mathbb{O}(F) = \{Y \in \mathfrak{sl}_n(F) \mid Y = \mathrm{Ad}(g)X \text{ for some } g \in \mathrm{SL}_n(\bar{F})\}.$$

is a disjoint union of $\mathrm{Ad}(G)$-orbits, and justify your description.