Exercise sheet 3.

March 17th

Due March 24th in class.

Exercise 9. Let $O^* \in \mathfrak{g}^*/G$. Show that $\mathcal{C}_c^\infty(\overline{O^*}) \to \mathcal{C}_c^\infty(\partial O^*)$ is surjective.

Exercise 10. The **wave-front set** of $O \in \mathfrak{g}/G$ is defined to be

$$\{Y \in \mathfrak{g} \mid \exists t_i \in (F^\times)^2 \text{ and } X_i \in O \text{ such that } \lim t_i \to 0 \text{ and } \lim t_i X_i = Y\}.$$

The determination of wave-front sets, for example for SO_V and Sp_V, is an open problem. (The answer is known, but rather non-trivial, when $F = \mathbb{R}$ and $(F^\times)^2 = \mathbb{R}_{>0}$.)

Suppose $\mathbb{G} = \mathrm{GL}_n$. Show that the wave-front set of any $O \in \mathfrak{g}/G$ contains the closure of the nilpotent orbit $\mathrm{Ad}(G)N \in \mathfrak{g}^{\mathrm{nil}}/G$ that appears in its rational canonical form of O. That is, N is the strictly lower triangular part of the rational canonical form, as in https://en.wikipedia.org/wiki/Frobenius_normal_form#Example.

Exercise 11. (*) Show that in Exercise 10, the wave-front set actually **is** the closure of the indicated orbit.

Exercise 12. (*) Let p be any odd prime, $\mathcal{C} = \mathbb{Q}$ and $\mathbb{G} = \mathrm{SL}_2$ over F. The Lie algebra $\mathfrak{sl}_2$ can be identified with its dual using $\langle A, B\rangle = \mathrm{tr}(AB)$. Let $\mathfrak{g}(\mathcal{O}_F) = \mathfrak{sl}_2(\mathcal{O}_F) \subset \mathfrak{sl}_2(F)$ be those traceless 2×2 matrices in $\mathcal{O}_F$. Let $X^* := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in (\mathfrak{g}^*)^{\mathrm{nil}}$ and $O^* := \mathrm{Ad}^*(G)X^*$. Show that

$$I_{O^*}(1_{\mathfrak{g}(\mathcal{O}_F)}) = \frac{q}{2}.$$

[CC: Keep in mind Remark 2.10.]