Exercise sheet 5.

April 7th

Due April 14th in class.

Exercise 17. Prove that $\mathrm{Ad}(G)$-conjugacy classes of maximal F-tori of $\mathrm{GL}_n/_F$ are in 1-1 correspondence with isomorphism classes of n-dimensional étale commutative F-algebras (i.e. products of separable field extensions). Here two étale algebras E_1 and E_2 are isomorphic if there is an isomorphism $E_1 \xrightarrow{\sim} E_2$ that restricts to the identity on F.

Exercise 18. (*) For any n-dimensional étale commutative F-algebra E, define $C_{E/F} \subset \{\pm 1\} \subset F^\times$ to be the set

$$\{\det(\sigma) \text{ as an } F\text{-linear map on } E \mid \sigma \in \mathrm{Gal}(E/F)\}.$$

Prove that there exists a bijection between

(1) $\mathrm{SL}_n(F)$-conjugacy classes of rational maximal tori of $\mathrm{SL}_n(F)$.
(2) The set $\{[E], \alpha\}$ where E is an isomorphism class of n-dimensional étale commutative F-algebras as above, and $\alpha \in F^\times / N_{E/F}(E^\times) \cdot C_{E/F}$.

such that a rational maximal torus in $\mathrm{SL}_n(F)$ is compact iff the corresponding étale algebra is a field.