On the Bernstein center of Hecke algebras at deeper level.

(joint with Reda Boumasmoud) :-

<u>Notation</u> :-

F - non-arch local field

val_F.

$\underline{G}$ - conn. reductive gp $/F$.

Let
$$X_F(\underline{G}) = \{ \chi : \underline{G} \longrightarrow \underline{G}_m \mid \chi \text{ is } F\text{-rational} \}$$

$G = \underline{G}(F)$

Define $\quad {}^{\circ}G = \{ g \in G \mid val_F(\chi(g)) = 0 \ \forall \ \chi \in X_F(\underline{G}) \}$

<u>Note</u>! $\quad {}^{\circ}G \trianglelefteq G$ open and contains every compact subgp of G.

Let $\quad X_{un}(G) = Hom\left(G/{}^{\circ}G, \mathbb{C}^{\times} \right)$.

gp of unramified characters of G.

<u>Inertial eq. classes</u> :-

Consider pairs $(\underline{M}, \sigma)$, $\underline{M}$ - F-Levi subgp of $\underline{G}$

σ - is supercuspidal repn of M.

We say $(\underline{M}_1, \sigma_1)$ and $(\underline{M}_2, \sigma_2)$ are inertially eq. if $\exists\ g \in G \ni$ $M_2 = g^{-1} M_1 g$ and $\sigma_1^g = \sigma_2 \otimes \chi$ for some $\chi \in X_{un}(M_2)$

$\sigma_1^g(x) = \sigma_1(g x g^{-1})$.

Write $[M, \sigma]_G$ for the inertial eq. class of the pair (M, σ) and let $\mathcal{B}(G)$ be the set of inertial eq. classes in G.

<u>Fact</u> :- Let (π, V) be an irr. smooth repn of G. Then $\exists$ an $\mathcal{F}$-parabolic $P = MN$ and an irr. s.c repn of $M \ni$

$\pi \hookrightarrow \mathrm{ind}_P^G \sigma$ (normalized parabolic ind)

The inertial eq. class $[M, \sigma]_G$ is called inertial support of π; denoted $\mathcal{I}(\pi)$.

§ Bernstein decomposition :-

Let $\mathcal{R}(G)$ be the category of all complex repns of G.

smooth ... be an

Let $s = [M, \sigma]_G \in \mathcal{B}(G)$ be an inertial eq. class. A repn π of $\mathcal{R}(G)$ lies in $\mathcal{R}^s(G)$ $\iff$ every irr. subquotient of π has inertial support s.

(irr. objects in $\mathcal{R}^s(G)$ are comp factors of $\mathrm{ind}_P^G \sigma\nu$)

The Bernstein decomp says

<u>Thm</u>: $$\mathcal{R}(G) = \prod_{s \in \mathcal{B}(G)} \mathcal{R}^s(G)$$

§ <u>The center of $\mathcal{R}^s(G)$:</u>

Let $s = [M, \sigma]_G$ and $t = [M, \sigma]_M \in \mathcal{B}(M)$

The action of $N_G(M)$ on M induces an action of $N_G(M)$ on $\mathcal{B}(M)$.

Let
$$N^t = \{ n \in N_G(M) \mid {}^n\sigma \cong \sigma\nu \text{ for some } \nu \in X_{un}(M) \}.$$

and let $W^t = N^t / M$.

Let $\mathfrak{Z}^t$ denote the center of $\mathcal{R}^t(M)$

Let $\mathfrak{Z}^s$ denote the center of $\mathcal{R}^s(G)$

Then
$$\mathfrak{Z}^s = \left(\mathfrak{Z}^t\right)^{W^t}.$$

What are we interested in?

Let K be compact open subgp of G. We would like to "describe" the center of the Hecke algebra $\mathcal{H}(G, K)$.

Let $\mathcal{R}_K(G)$ be the subcategory of $\mathcal{R}(G)$ consisting of repns generated by their K-fixed vectors.

(A1) Assumption :- The category $\mathcal{R}_K(G)$ is closed under taking subquotients

Bushnell- Kutzko, this is

(In Bushn... e_K a
the same as saying
special idempotent)

Then (Bernstein)

<u>Thm</u>:- $\exists$ a finite set $\mathcal{G}_K \subseteq \mathcal{B}(G)$
such that

$$\mathcal{R}_K(G) = \prod_{s \in \mathcal{G}_K} \mathcal{R}^s(G)$$

The functor

$$\mathcal{R}_K(G) \longrightarrow \boxed{\boxed{\mathcal{H}(G, K)}\text{-mod}}$$

$$V \longrightarrow V^K$$

is an eq. of categories with

$$\mathcal{H}(G, K)\text{-mod} \longrightarrow \mathcal{R}_K(G)$$

$$V^K \longrightarrow \mathcal{H}(G) * e_K \otimes_{\mathcal{H}(G,K)} V^K$$

In particular,

$$\mathfrak{Z}(\mathcal{H}(G, K)) \cong \mathfrak{Z}(\mathcal{R}_K(G))$$

$$\cong \prod_{s \in \mathcal{G}_K} \mathfrak{Z}(\mathcal{R}^s(G)).$$

So, under assumption (A1), we
... center

..., want to describe the ...
$\mathfrak{Z}^s$, $s \in \mathcal{B}_K$, and also $\mathfrak{Z}(\mathcal{H}(G, K))$
via the theory of types.

Theory of ~~types~~ types :- Consider a pair (J, ρ) where

J - compact open subgp of G

ρ - an irr. repn of J

Let $s = [M, \sigma]_G \in \mathcal{B}(G)$. We say (J, ρ) is an s-type if $\forall$ irr. repns of G, $(\pi, V) \in \mathcal{R}^s(G)$

ie $\mathfrak{g}(\pi) = s \iff \mathrm{Hom}_J(\rho, \pi) \neq 0$

Fact :- If (J, ρ) is a s-type, then the

$$\mathcal{R}^s(G) \longrightarrow \mathcal{H}(G, \rho)\text{-mod}$$
$$\pi \longrightarrow \mathrm{Hom}_J(\rho, \pi)$$

is an eq. of categories.

Here $\mathcal{H}(G, \rho) = \mathrm{End}_G(c\text{-ind}_J^G \rho)$

... (ρ, π)

acts naturally on $\mathrm{Hom}_J(\rho, \pi)$
$\quad\quad \shortparallel$
$\mathrm{Hom}_G(\text{c-ind}_J^G \rho, \pi)$.

In particular, $\underline{\mathfrak{Z}(\mathcal{R}^S(G))} \cong \underline{\underline{\mathfrak{Z}(\mathcal{H}(G,\rho))}}$
when (J,ρ) is an S-type.

<u>On existence of types:</u>

$(A2)_1$: Assume G splits over a tamely ramified extn of F.
Then J.K. Yu gave a construction of s.c repns of G.

Fintzen: If $\boxed{p \nmid |W|}$ $(A2)_2$, then this construction exhausts all supercuspidal repns of G.

Let $(A2)$: G splits over a tamely ramified extn & $p \nmid |W|$

then Kim & Yu gave a construction of types for each Bernstein block,

Fintzen: This Kim-Yu construction holds under $(A2)$

... assume

For our purposes: We will ass[illegible] every Bernstein block has a type (J, ρ) as above.

Want: Describe the center of $\mathcal{H}(G, \rho)$.

Supercuspidal types:-

Let π be an irr. supercuspidal repn of G and let $s = [G, \pi]_G$.

Assume

(1) $\pi = \text{c-ind}_{\tilde{J}}^{G} \tilde{\rho}$,

$\tilde{J}$ - open, compact mod center subgp of G.

$\tilde{\rho}$ - irr. repn of $\tilde{J}$.

Let $J = \tilde{J} \cap {}^{\circ}G$ and let

$$\rho \underset{\text{irr}}{\hookrightarrow} \tilde{\rho}|_J$$

Then (J, ρ) is a s-type.

(2) Let $I_G(\rho) = \{ g \in G \mid \text{Hom}_{J \cap J^g}(\rho, \rho^g) \neq 0 \}$

$I_G(\rho) \subseteq \tilde{J}$.

Assume $-G$ ...

<u>Remark</u>:- Yu's supercuspidal repns are of the form (1) & they satisfy (2).

To now describe $\mathfrak{Z}(\mathcal{H}(G,\rho))$:

Let π_0 be an irr summand of $\pi|_{{}^\circ G}$. Then, Clifford theory, we have

$$\pi|_{{}^\circ G} = m_{{}^\circ G}(\pi) \cdot \Big(\underbrace{\bigoplus \pi_0^g}_{} \Big)$$

distinct conj of π_0 and all occur with the same mult $m_{{}^\circ G}(\pi)$.

Similarly for $\tilde{\rho}$ repn of $\tilde{J}$:

$$\tilde{\rho}|_J = m_J(\tilde{\rho}) \cdot \Big(\bigoplus \rho^g \Big)$$

distinct conj of ρ

and occur with
mult $m_J(\tilde{\rho})$

<u>Prop (B-G)</u>:- $\boxed{m_{{}^\circ G}(\pi) = m_J(\tilde{\rho})}$

(Assuming (1) & (2))

<u>Pf</u>:- The assumption $I_G(\rho) \subseteq \tilde{J}$ implies that $c\text{-}\mathrm{ind}_J^{{}^\circ G}\rho$ is irreducible, giving a candidate for π_0

Let $\pi_0 = c\text{-}\mathrm{ind}_J^{{}^\circ G}\rho$, play with Frobenius reciprocity & Mackey theory

<u>Remark</u>:- If $\pi = c\text{-}\mathrm{ind}_{\tilde{J}}^{G}\tilde{\rho}$ and $I_G(\rho) \subseteq \tilde{J}$, then results known for the s.c block $\mathcal{R}^S(G)$, where $S = [G, \pi]_G$, can be recast into a statement via types.

(Loosely: $(G, {}^\circ G) \longleftrightarrow (\tilde{J}, J)$)

A concrete statement:
A ... statement is part

The following [illegible] of the work of Bernstein (Roche's notes)

Let $X_G(\pi) = \{ \nu \in X_{un}(G) \mid \pi\nu \cong \pi \}$

Let $T = \bigcap_{\nu \in X_G(\pi)} \mathrm{Ker}(\nu)$

Then <u>Thm</u>:- $\mathfrak{Z}^S = \mathcal{H}(T, \pi_0)$

$\pi_0 \hookrightarrow \pi|_{G^\circ}$

Now, let

$X_J(\tilde{\rho}) = \{ \nu \in \mathrm{Hom}(\tilde{J}/J, \mathbb{C}^\times) \mid \tilde{\rho}\nu \cong \tilde{\rho} \}$

and let $J^\dagger = \bigcap_{\nu \in X_J(\tilde{\rho})} \mathrm{Ker}(\nu)$.

Then

<u>Prop</u> (B-G):- $\mathfrak{Z}(\mathcal{H}(G, \rho)) \cong \mathcal{H}(J^\dagger, \rho)$
$\cong \mathbb{C}[J^\dagger/J]$

<u>Pf</u>:- We show

$I_G(\rho) \cap T = J^\dagger$, and

the rest is easy.

For non-supercuspidal blocks :-

Let $\mathfrak{s} = [M, \sigma]_G$ and $\mathfrak{t} = [M, \sigma]_M$.

where

(1) $\sigma = c\text{-ind}_{\tilde{J}_M}^{M} \tilde{\rho}_M$

$\tilde{J}_M$ - open, compact mod center in M

$\tilde{\rho}_M$ - irr. repn of $\tilde{J}_M$.

$J_M = \tilde{J}_M \cap {}^{\circ}M$

$\rho_M \xhookrightarrow[\text{irr}]{} \tilde{\rho}_M|_J$. , $I_M(\rho_M) \subseteq \tilde{J}_M$.

Then by the previous result, we know

$\mathfrak{z}(\mathcal{H}(M, \rho_M)) \cong \mathbb{C}[J_M^{\dagger}/J_M]$ as before.

Let (J, ρ) be a G-cover of (J_M, ρ_M)

Then (J, ρ) is a $\mathfrak{s}$-type

$\mathfrak{s} = [M, \sigma]_G$.

Let $\quad \ldots \; N_G(M) \Big/ (J_M^n, \rho_M^n) \cong (J_M, \rho_M)$

$N_G(\rho_M) = \{ x \in N_G \ldots$

$\text{c-ind}_{J_M^n}^{M} \rho_M^n \cong \text{c-ind}_{J_M}^{M} \rho_M \ldots$

and $W(\rho_M) = N_G(\rho_M)/M$,

<u>Prop</u> (B–G) : $\mathfrak{Z}^s = \mathfrak{Z}(\mathcal{H}(G, \rho))$

$$= \mathbb{C}\left[J_M^{\dagger}/J_M\right]^{W(\rho_M)}.$$

§ On the center of Hecke algebras at deeper level :

<u>Assume</u> <u>(A1)</u>: $\mathcal{R}_K(G)$ is closed under taking subquotients.

<u>Corollary</u> :- (B–G) : Then $\mathfrak{Z}(\mathcal{R}_K(G))$

$$\cong \prod_{\substack{s \in \mathcal{G}_K \\ \| \\ [M, \sigma]_G}} \mathbb{C}\left[J_M^{\dagger}/J_M\right]^{W(\rho_M)}.$$

Which K's are known to satisfy (A1)?

Sanin :- For $K = G_{x,r}$,

Bestvina-

$x \in \mathcal{B}(G, F)$, $r > 0$

the category $\mathcal{R}_K(G)$ is closed under taking subquotients.

The put a criteria Δ on K such that $\mathcal{R}_K(G)$ is closed under taking subquotients, and then proved that $G_{x,r}$, $r > 0$ satisfies $\boxed{\Delta}$

Prior to this:

Bernstein-Deligne: $\heartsuit$

They put some other criteria on K for which $\mathcal{R}_K(G)$ is closed under subquotients,

Further one can show: If K satisfies $\heartsuit$, then

Lemma:
$$\boxed{[M, \sigma]_G \in \mathfrak{G}_K \iff \sigma^{K_M} \neq 0}$$
where $K_M = K \cap P / K \cap N$

Remark :- This lemma does ....

Prop :- (1) Let $\mathcal{A}$ be an alcove in $\mathcal{A}(S, F) \subseteq \mathcal{B}(G, F)$, and let $x \in \mathcal{A}$ and let $r > 0$. Then $G_{x,r}$ satisfies $\heartsuit$.

(2) An example of $G_{x,r} \subseteq GL_3$ with x a non-special pt in the boundary of an alcove that does not satisfy $\heartsuit$ (~~and~~ but in particular satisfies Δ).

In any case, $\boxed{\text{for } K = G_{x,r},\ x \in \mathcal{A},\ r > 0, \text{ the}}$ result reads

$$\mathfrak{Z}(\mathcal{H}(G, K)) = \prod_{[M]} \prod_{t \in (G_{K \cap M})_{sc}} \mathbb{C}\left[J_M^{\dagger} / J_M\right]^{W(f_M)}$$

[M] – G-conj. classes of

where { ... } F-Levi's subgps

and

$$(G_{K \cap M})_{sc} = \{ [M, \sigma]_M \in G_{K \cap M} \mid \sigma \text{ is a s.c repn } \cap M \}$$