Depth filtration on affine character sheaves & affine Harish-Chandra bimodules

0. Main goal:

$$F = \mathbb{F}_q((t))$$

induced depth filtration

$$\begin{tikzcd}
\text{char sheaves on } G(F) \arrow[r, "\mathrm{tr}\,\mathrm{Fr}"] & \text{char distributions on } G(F) \\
\text{cat reps } G(F) \arrow[u, "\mathrm{tr}"] \arrow[r, "\mathrm{tr}\,\mathrm{Fr}"] & \text{reps } G(F) \arrow[u, "\mathrm{tr}"]
\end{tikzcd}$$

has depth filtration

has depth filtration

expect: LHS to have a depth filtration

finite model: (Lusztig, 80s)

$$\begin{tikzcd}
\ell\text{-adic char sheaves on } G_{\mathbb{F}_q} \arrow[r, "\mathrm{tr}(\mathrm{Fr})"] & \ell\text{-adic char distributions on } G(\mathbb{F}_q) \\
\ell\text{-adic cat representations of } \mathrm{Sh}(G_{\mathbb{F}} \arrow[u, "\mathrm{tr}(\mathrm{id})"] \arrow[r, "\mathrm{tr}(\mathrm{Fr})"] & \ell\text{-adic representations of } G(\mathbb{F}_q) \arrow[u, "\mathrm{tr}(\mathrm{id})"]
\end{tikzcd}$$

(class functions)

v.s.

1-category

2-category

1-category

"q"

$Sh(X \times Y) \leftrightarrow Sh(X) \otimes Sh(Y)$

today: explain (provable) variant w/ following modifications:

$$Sh(G(\mathbb{F}_q((t)))) \quad \leadsto \quad \text{D-mod}(G(\mathbb{C}((t)))) = \text{D-mod}(LG)$$

$$Sh(G(\mathbb{F}_q)) \quad \leadsto \quad \text{D-mod}(G(\mathbb{C})) = \text{D-mod}(G)$$

our actual object of study:

adjoint-equivariant D-modules on LG ($\&$ G) & their depth filtration

+ new feature$/\mathbb{C}$: interaction w/ Harish-Chandra bimodules

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① idea of construction:

$\leadsto$ 2-categories $\overset{\dim}{\leadsto}$ categories $\overset{\dim}{\leadsto}$ vector $\overset{\dim}{\leadsto}$ numbers spaces

We will start a filtration on the 2-category $D\text{-mod}(LG)\text{-mod}$ (Moy – Prasad filtration – D. Yang) and apply traces.

Cartoon: $W^n \cdots \subset W' \subset W \subset V$

$$\dim W^n \leq \dim W' \leq \dim W \leq \dim V$$

next cartoon :

category

$$\overset{\circ}{e} \rightleftarrows e$$

$$tr(\overset{\circ}{e}) \longrightarrow tr(e)$$

<u>example</u> $Z \overset{i}{\hookrightarrow} X$ closed subvariety

$$\begin{array}{ccc} D\text{-mod}(Z) & \underset{i^!}{\overset{i_*}{\rightleftarrows}} & D\text{-mod}(X) \\ \| & & \| \\ \overset{\circ}{e} & & e \end{array}$$

$tr(D\text{-mod}(y))$ any var y

$$\begin{array}{c} tr(D\text{-mod}(y)) \\ \cong \\ C^{BM}(y) \end{array}$$

$$i_* : \quad C_*^{BM}(Z) \longrightarrow C_*^{BM}(X)$$

lesson: need a bit more:

(all functors are cts)

$$\overset{\circ}{e} \rightleftarrows e$$

what we first think about

what we need to get a map on traces

... & for $tr(\overset{\circ}{e}) \longrightarrow tr(e)$ to be a retract we need that $\leftarrow$ also has a right adjoint.

Given now a chain:

$$e^0 \hookrightarrow e^1 \hookrightarrow e^2 \hookrightarrow e^3 \hookrightarrow \cdots \varinjlim_i e^i = e$$

+ enough right adjoints

$$tr(e^0) \rightleftarrows tr(e^1) \rightleftarrows tr(e^2) \cdots \rightleftarrows tr(e)$$

same will apply w/ 2-categories $\xrightarrow{tr}$ 1-categories.

$$\overset{\circ}{\mathbf{C}} \rightleftarrows \mathbf{C}$$

2-category

if you have both right adjoints

$$\mathrm{tr}(\mathbf{C}^{\circ}) \rightleftarrows \mathrm{tr}(\mathbf{C})$$

and similarly for chains.

(2) toy model: character sheaves on $G_{\mathbb{C}}$:

motivation:

$$\mathrm{Fun}(G(\mathbb{F}_q))\text{-mod} \xrightarrow{\mathrm{tr}} \mathrm{Fun}\left(G(\mathbb{F}_q)/^{Ad}G(\mathbb{F}_q)\right)$$

$\uparrow$ group algebra

claim:

$$D\text{-mod}(G_{\mathbb{C}})\text{-mod} \xrightarrow{\mathrm{tr}} D\text{-mod}\left(G_{\mathbb{C}}/^{Ad}G_{\mathbb{C}}\right)$$

quick refresher on this:

objects: $\mathcal{C}$ cocomplete dg-category w/ an action of $D\text{-mod}(G)$

$$\begin{array}{ccccc} D\text{-mod}(G) & \otimes & \mathcal{C} & \longrightarrow & \mathcal{C} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ \mathcal{M} & \boxtimes & c & \longmapsto & \mathcal{M} * c \end{array}$$

+ coherent homotopy for assoc

$$\mathrm{Fun}(G(\mathbb{F}_q)) \otimes V \longrightarrow V$$

$$f \otimes v \longmapsto \sum f(g)\, g \cdot v$$

morphisms: $G \circlearrowright \mathcal{C}$ $\quad G \circlearrowright \mathcal{D}$

$$\rightsquigarrow \quad \mathrm{Hom}_{D\text{-mod}(G)\text{-mod}}(\mathcal{C}, \mathcal{D})$$

1-category:

$$\varprojlim \left(\mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathcal{C}, \mathcal{D}) \rightrightarrows \mathrm{Hom}(D(G) \otimes \mathcal{C}, \mathcal{D}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{Hom}(D(G)^{\otimes 2} \otimes \mathcal{C}, \mathcal{D}) \cdots \right)$$

all cocomplete dg-categories

$$\simeq \mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathcal{C}, \mathcal{D})^G$$

$$\mathrm{Hom}_{G(\mathbb{F}_q)}(V, W) \simeq \mathrm{Hom}_{\mathrm{Vect}}(V, W)^{G(\mathbb{F}_q)}$$

<u>examples</u>: $G \circlearrowright X$ complex variety $\qquad G \times X \xrightarrow{a} X$

$$\rightsquigarrow \quad D\text{-mod}(G) \circlearrowright D\text{-mod}(X)$$

w/ action map:

$$\text{D-mod}(G) \otimes \text{D-mod}(X) \overset{\text{künneth formula}}{\simeq} \text{D-mod}(G \times X) \xrightarrow{a_*} \text{D-mod}(X)$$

$$\mathcal{M} \boxtimes \mathcal{N} \longmapsto \pi_G^! \mathcal{M} \overset{!}{\otimes} \pi_X^! \mathcal{N} \longmapsto a_*(\pi^! \mathcal{M} \overset{!}{\otimes} \pi^! \mathcal{N})$$

$G \circlearrowright X$ & Y

$$\text{Hom}_{\text{D-mod}(G)\text{-mod}}(\text{D-mod}(X), \text{D-mod}(Y)) \simeq \text{Hom}_{\text{DGCat}_{\text{cont}}}(\text{D-mod}(X), \text{D-mod}(Y))^G$$

$$\text{Hom}_{\text{DGCat}_{\text{cont}}}(\text{D-mod}(X), \text{D-mod}(Y)) \simeq \text{D-mod}(X \times Y)^G$$

given any $\mathcal{K} \in \text{D-mod}(X \times Y)$, get functor:
(integral transforms)

$$\text{D-mod}(X) \xrightarrow{\pi_X^!} \text{D-mod}(X \times Y) \xrightarrow{- \overset{!}{\otimes} \mathcal{K}} \text{D-mod}(X \times Y) \xrightarrow{\pi_{Y,*}} \text{D-mod}(Y)$$

and all functors arise in this way.

so: $\text{Hom}_{\text{D-mod}(G)\text{-mod}}(\text{D-mod}(X), \text{D-mod}(Y)) \simeq \left\{ \begin{array}{l} \Delta G\text{-equivariant D-modules} \\ \text{on } X \times Y \end{array} \right\}$

$\text{D-mod}\left(X \times Y / \Delta G\right).$

<u>example</u>: $\mathfrak{g} = \mathrm{Lie}(G)$ Lie algebra

$\mathfrak{g}\text{-mod}$ cat of representations

then $D(G) \circlearrowright \mathfrak{g}\text{-mod}$.

(given $x \in G(\mathbb{C})$, $\delta_x * - : \mathfrak{g}\text{-mod} \xrightarrow{\sim} \mathfrak{g}\text{-mod}$)

sends a rep $\mathfrak{g} \xrightarrow{\rho} \mathrm{End}(V)$

to "twist"

$\mathfrak{g} \xrightarrow{Ad_x^{-1}} \mathfrak{g} \xrightarrow{\rho} \mathrm{End}(V)$

<u>prop</u>: $\mathrm{tr}\left(\text{D-mod}(G)\text{-mod}\right) \simeq \text{D-mod}\left(G /^{Ad} G\right)$

<u>proof</u>: Warmup: A is a $\mathbb{C}$-algebra, then:

$$\mathrm{tr}(A\text{-mod}) \simeq HH_*(A) := A \overset{L}{\underset{A \otimes A^{op}}{\otimes}} A$$

chain level version

Similarly, if $\mathcal{M}$ is a monoidal dg-category$/\mathbb{C}$, then:

$$\mathrm{tr}(\mathcal{M}\text{-mod}) \simeq HH_*(\mathcal{M}) := \mathcal{M} \underset{\mathcal{M} \otimes \mathcal{M}^{op}}{\otimes} \mathcal{M}$$

Applying this to $\mathcal{M} \simeq \text{D-mod}(G)$, we get:

$$\mathrm{tr}\big(\text{D-mod}(G)\text{-mod}\big) \simeq \text{D-mod}(G) \underset{\text{D-mod}(G)\otimes\text{D-mod}(G)^{op}}{\otimes} \text{D-mod}(G)$$

but $\text{D-mod}(G)^{op} \overset{inv_*}{\simeq} \text{D-mod}(G)$, + Künneth formula

$$\simeq \text{D-mod}(G) \underset{\text{D-mod}(G\times G)}{\otimes} \text{D-mod}(G)$$

but we have:

$$\mathcal{C} \underset{\text{D-mod}(H)}{\otimes} \mathcal{D} \simeq (\mathcal{C}\otimes\mathcal{D})_{\Delta H} \simeq (\mathcal{C}\otimes\mathcal{D})^{\Delta H}$$

$$\simeq \text{D-mod}(G\times G)^{\text{D-mod}(G\times G)}$$

$$\simeq \text{D-mod}\left(G\times G \big/ \underline{\underline{G\times G}}\right)$$

$$\simeq \text{D-mod}\left(G/^{Ad}G\right)$$

this all completely parallels a computation for

$$\mathrm{tr}\big(\mathrm{Fun}(G(\mathbb{F}_q))\text{-mod}\big) \simeq \mathrm{Fun}\left(G(\mathbb{F}_q)/^{Ad}G(\mathbb{F}_q)\right)$$

③ baby version of depth filtration:

$B \subset G$ Borel $\quad 1 - N - B - T - 1$

$$\text{D-mod}(G) \curvearrowright \text{D-mod}(G/B) \circlearrowleft \text{End}_{\text{D-mod}(G)\text{-mod}}\big(\text{D-mod}(G/B), \text{D-mod}(G/B)\big)$$

$$\simeq \text{D-mod}(G/B \times G/B)^{\Delta G}$$

$$\simeq \text{D-mod}(B \backslash G/B) := \mathcal{H}$$

we get functors:

$$\mathcal{N} \longmapsto \text{D-mod}(G/B) \underset{\mathcal{H}}{\otimes} \mathcal{N}$$

$$\mathcal{H}\text{-mod} \rightleftarrows \text{D-mod}(G)\text{-mod}$$

$$\text{Hom}_{\text{D-mod}(G)\text{-mod}}\big(\text{D-mod}(G/B), \mathcal{M}\big) \longleftarrow\!\shortmid \mathcal{M}$$

$$\cong$$

$$\text{Hom}_{\text{D-mod}(B)\text{-mod}}\big(\text{D-mod}(B/B), \text{Res}(\mathcal{M})\big)$$

$$\cong$$

$$\mathcal{M}^B.$$

next time: see this gives a fully faithful embedding on traces...

$$\text{tr}(\mathcal{H}\text{-mod}) \hookrightarrow \text{D-mod}(G/_{\text{Ad}}\, G)$$

(remainder from post-talk discussion)

$$D\text{-mod}\left(G/_{G}^{Ad}\right)^{\text{nilp s.s}}_{rh} \simeq \bigoplus_{\substack{\text{inf}\\ \text{char}\\ \lambda}} \left(\_\_ \right)_{\_}$$

$$\simeq \bigoplus \mathrm{tr}\; D\text{-mod}_{\lambda}(G/B)$$

$$\bigoplus_{x\in \mathbb{A}^1(\mathbb{C})} QCoh(\mathbb{A}^1)_{\hat{x}} \hookrightarrow QCoh(\mathbb{A}^1)$$

$$\begin{tikzcd}
D\text{-mod}(N\backslash G/N)\text{-mod} \arrow[r, phantom, "\simeq"] \arrow[d] & D\text{-mod}(G)\text{-mod} \arrow[d, "\text{Ben-Zvi -- Gunningham -- Orem}"] \\
Z\left(D(N\backslash G/N)\right) \arrow[r, phantom, "\simeq"] & D\text{-mod}\left(G/_{G}^{Ad}\right)
\end{tikzcd}$$

$$\begin{tikzcd}
D\text{-mod}(G)^{\Delta G} \arrow[r, "\Gamma"] & \left(\mathfrak{g}\text{-mod} \otimes \mathfrak{g}\text{-mod}\right)^{\Delta G}
\end{tikzcd}$$

$$\text{D-mod}_k(LG)^{\Delta LG} \xrightarrow{\Gamma} \hat{\mathfrak{g}}_k\text{-mod} \otimes \hat{\mathfrak{g}}_{-k-2h^\vee}\text{-mod}^{\Delta LG}$$

$$\text{HCh-bimodules}_{k,\,-k-2h^\vee}$$

$$D_G \simeq \mathcal{U}(\mathfrak{g}) \otimes \mathcal{O}_G$$

$$CDO_G = \text{Dist}(L^+G \subset LG)$$

$$\text{ind}_{\mathfrak{g}[[z]]}^{\hat{\mathfrak{g}}_k} \text{Fun}(L^+G) \approx \mathbb{V}_{\mathfrak{g}} \otimes \mathcal{O}_{L^+G}$$

$$\underline{\hat{\mathfrak{g}}_k}\text{-mod} \xleftrightarrow{\text{local Langlands}} D_{-k^L}\text{-mod}\left(\underline{{}^LG^L/LN^L_{,\psi}}\right)$$

$$\text{D-mod}\left({}^LG/LN_{,\psi}\right)^{\text{gen}} \subset \text{D-mod}(LG)\text{-mo}$$

$$\mathrm{QCoh}(\mathrm{LocSys}_{G^L}) \subset \mathrm{Q\text{-}IndCoh}_{nilp}(\mathrm{LocSys}_{G^L})$$