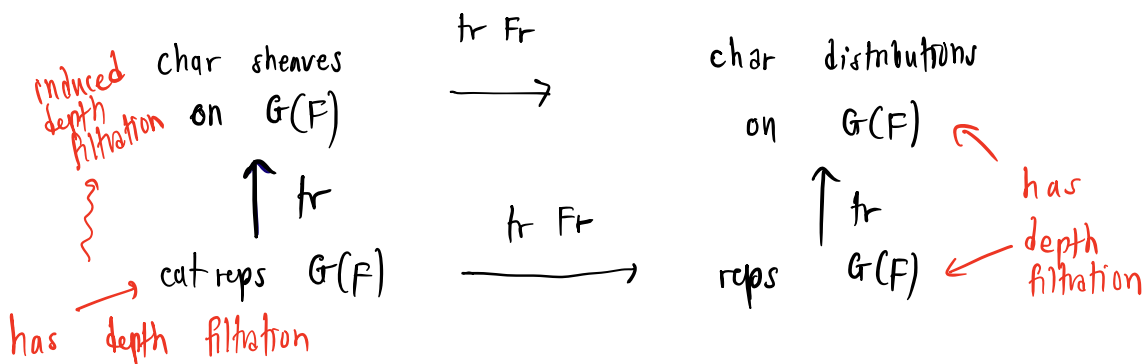


Depth filtration on  
 affine character sheaves &  
 affine Harish-Chandra bimodules

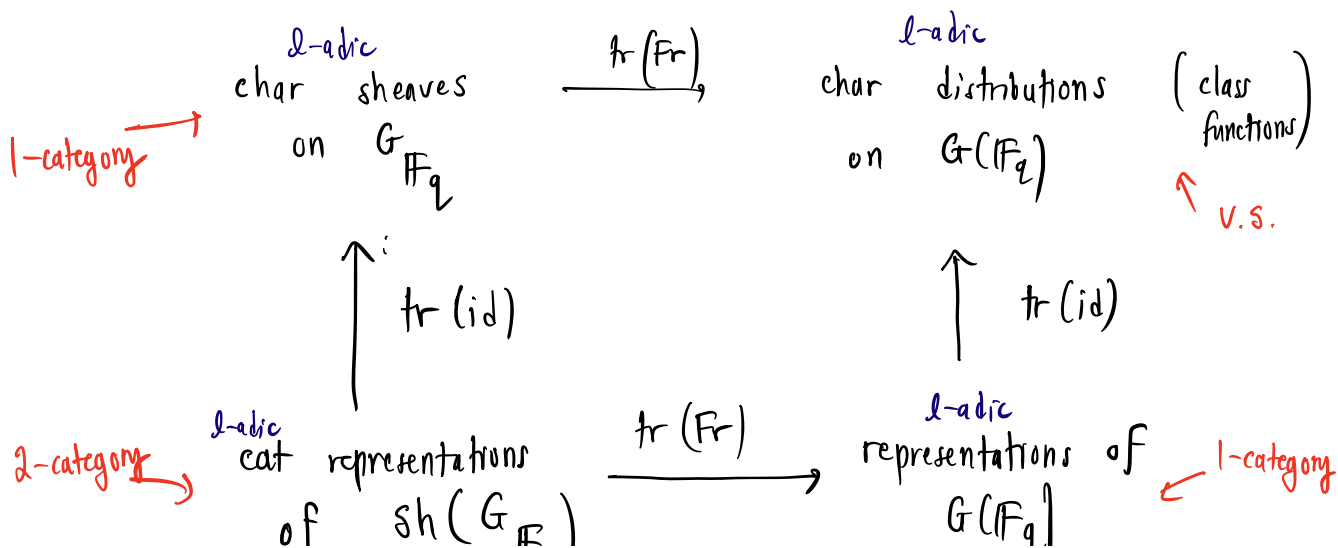
0. Main goal:

$$F = \mathbb{F}_q((t))$$



expect: LHS to have a depth filtration

finite model: (Lusztig, 80s)



"q)"

$$sh(X \times Y) \leftrightarrow sh(X) \otimes sh(Y)$$

today: explain (provable) variant w/  
following modifications:

$$sh(G(\mathbb{F}_q((t))))$$

$$sh(G(\mathbb{F}_q))$$



$$D\text{-mod}(G(\mathbb{C}((t))))$$

LG

$$D\text{-mod}(G(\mathbb{C}))$$

G

our actual object of study:

adjoint - equivariant D-modules on LG (G)  
their depth filtration

+ new feature: interaction w/  
Hanish-Chandra bimodules

---

① idea of construction:

dim dim dim  
→ 2-categories → categories → vector → numbers

spaces

We will start a filtration on the  
 2-category  $D\text{-mod}(LG)\text{-mod}$   
 (Moy-Prasad filtration - D. Yang)  
 and apply traces.

Cartoon:  $W^n \subset W' \subset W \subset V$

$$\dim W^n \leq \dim W' \leq \dim W \leq \dim V$$

next cartoon:

$$e^{\circ} \rightleftarrows e \quad \swarrow \text{category}$$

$$\text{tr}(e^{\circ}) \longrightarrow \text{tr}(e)$$

example

$$Z \xrightarrow{i} X \quad \text{closed subvariety}$$

$$\begin{array}{ccc} D\text{-mod}(Z) & \xrightleftharpoons[i!]{i^*} & D\text{-mod}(X) \\ \parallel & & \parallel \\ e^{\circ} & & e \end{array}$$

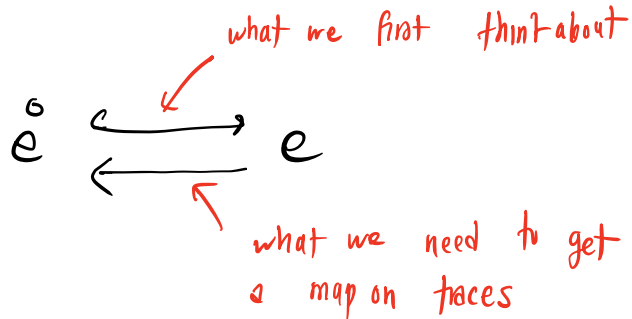
$$\begin{array}{c} \text{tr}(D\text{-mod}(y)) \quad \text{any var } y \\ \parallel \\ \mathbb{C}^{\text{BM}}(y) \end{array}$$

- \* . /

$$i_* : C_*^{BM}(Z) \rightarrow C_*^{BM}(X)$$

lesson: need a bit more:

(all functors are cts)



... & for  $\text{tr}(e^0) \rightarrow \text{tr}(e)$  to be a retract we need that  $\leftarrow$  also has a right adjoint.

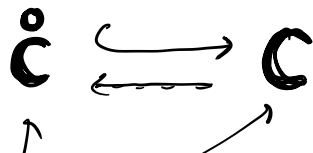
Given now a chain:

$$e^0 \hookrightarrow e^1 \hookrightarrow e^2 \hookrightarrow e^3 \hookrightarrow \dots \xrightarrow{\text{inj}} \varinjlim e^i$$

+ enough right adjoints

$$\text{tr}(e^0) \hookrightarrow \text{tr}(e^1) \hookrightarrow \text{tr}(e^2) \dots \hookrightarrow \text{tr}(e)$$

same will apply w/ 2-categories  $\xrightarrow{\text{tr}}$  1-categories.



2-category

if you have both right adjoints

$$\mathrm{tr}(\mathbf{C}^\circ) \rightleftarrows \mathrm{tr}(\mathbf{C})$$

and similarly for chains.

② toy model: character sheaves on  $G_{\mathbb{C}}$ :

motivation:

$$\begin{array}{ccc} \mathrm{Fun}(G(\mathbb{F}_q))\text{-mod} & \xrightarrow{\mathrm{tr}} & \mathrm{Fun}(G(\mathbb{F}_q)/\mathrm{Ad} G(\mathbb{F}_q)) \\ \uparrow & & \\ \text{group algebra} & & \end{array}$$

claim:

$$\begin{array}{ccc} D\text{-mod}(G_{\mathbb{C}})\text{-mod} & \xrightarrow{\mathrm{tr}} & D\text{-mod}(G_{\mathbb{C}}/\mathrm{Ad} G_{\mathbb{C}}) \\ \nearrow & & \end{array}$$

quick refresher on this:

objects:  $\mathcal{C}$  a complete dg-category  
w/ an action of  $D\text{-mod}(G)$

$$\begin{array}{ccc}
 D\text{-mod}(G) \otimes e & \longrightarrow & e \\
 \downarrow & & \downarrow \\
 \mathcal{M} \boxtimes c & \longmapsto & \mathcal{M} * c
 \end{array}$$

+ coherent homotopy for assoc

$$\begin{array}{ccc}
 \text{Fun}(G(\mathbb{F}_q)) \otimes V & \longrightarrow & V \\
 f \otimes v & \longmapsto & \sum f(g) g \cdot v
 \end{array}$$

morphisms:  $G \curvearrowright e \quad G \curvearrowright D$

$$\rightsquigarrow \text{Hom}_{D\text{-mod}(G)\text{-mod}}(e, D) \quad \text{1-category:}$$

$$\begin{array}{c}
 \varprojlim \\
 \leftarrow \\
 \text{Hom}_{D\&G\text{cat}_{\text{cont}}}(e, D) \rightleftarrows \text{Hom}(D(G) \otimes e, D) \rightleftarrows \text{Hom}(D(G)^{\otimes 2} \otimes e, D) \dots \\
 \uparrow \\
 \text{all cocomplete dg-categories} \\
 \cong \\
 \text{Hom}_{D\&G\text{cat}_{\text{cont}}}(e, D)^G
 \end{array}$$

$$\text{Hom}_{G(\mathbb{F}_2)}(V, W) \cong \text{Hom}_{\text{vect}}(V, W)^{G(\mathbb{F}_2)}$$

examples:  $G \curvearrowright X$  complex variety  $G \times X \xrightarrow{a} X$

$$\rightsquigarrow D\text{-mod}(G) \hookrightarrow D\text{-mod}(X)$$

w/ action map:

$$\begin{aligned}
 D\text{-mod}(G) \otimes D\text{-mod}(X) &\stackrel{\text{K\u00fcnneth formula}}{\simeq} D\text{-mod}(G \times X) \xrightarrow{a_*} D\text{-mod}(X) \\
 \mathcal{M} \otimes \mathcal{N} &\mapsto \pi_G^! \mathcal{M} \otimes \pi_X^! \mathcal{N} \mapsto a_* (\pi^! \mathcal{M} \otimes \pi^! \mathcal{N})
 \end{aligned}$$

$G \curvearrowright X \hat{=} Y$

$$\begin{aligned}
 \text{Hom}_{D\text{-mod}(G)\text{-mod}}(D\text{-mod}(X), D\text{-mod}(Y)) &\simeq \text{Hom}_{D\text{GrCat}_{\text{cont}}}(D\text{-mod}(X), D\text{-mod}(Y)) \\
 &\cong \text{Hom}_{D\text{-mod}(X \times Y)^G}(D\text{-mod}(X), D\text{-mod}(Y))
 \end{aligned}$$

given any  $\mathcal{K} \in D\text{-mod}(X \times Y)$ , get functor:  
(integral transforms)

$$D\text{-mod}(X) \xrightarrow{\pi_X^!} D\text{-mod}(X \times Y) \xrightarrow{- \otimes \mathcal{K}} D\text{-mod}(X \times Y) \xrightarrow{\pi_{Y,*}} D\text{-mod}(Y)$$

and all functors arise in this way.

$$\text{so: } \text{Hom}_{D\text{-mod}(G)\text{-mod}}(D\text{-mod}(X), D\text{-mod}(Y)) \simeq \left\{ \begin{array}{l} \Delta^G\text{-equivariant} \\ D\text{-modules} \\ \text{on } X \times Y \end{array} \right\}$$

12

$$D\text{-mod}(\mathbb{X}^{\times\mathbb{Y}} / \Delta G).$$

example:  $\mathfrak{g} = \text{Lie}(G)$  Lie algebra  
 $\mathfrak{g}\text{-mod}$  cat of representations

then  $D(G) \hookrightarrow \mathfrak{g}\text{-mod}$ .

(given  $x \in G(\mathbb{C})$ ,  $\delta_x * - : \mathfrak{g}\text{-mod} \xrightarrow{\sim} \mathfrak{g}\text{-mod}$ )  
 sends a rep  $\mathfrak{g} \xrightarrow{\rho} \text{End}(V)$   
 to "twist"  
 $\mathfrak{g} \xrightarrow{\text{Ad}_x^{-1}} \mathfrak{g} \xrightarrow{\rho} \text{End}(V)$

---

prop:  $\text{tr}(D\text{-mod}(G)\text{-mod}) \cong D\text{-mod}(G / \text{Ad}_G)$

proof: Wamup:  $A$  is a  $\mathbb{C}$ -algebra, then:

$$\text{tr}(A\text{-mod}) \cong \text{HH}_*(A) := \begin{array}{c} \mathbb{L} \\ A \otimes A \\ A \otimes A^{\text{op}} \end{array}$$

↑  
chain level  
version

Similarly, if  $\mathcal{M}$  is a monoidal dg-category /  $\mathbb{C}$ , then:

$$\text{tr}(\mathcal{M}\text{-mod}) \cong \text{HH}_*(\mathcal{M}) := \begin{array}{c} \mathcal{M} \otimes \mathcal{M} \\ \mathcal{M} \otimes \mathcal{M}^{\text{op}} \end{array}$$



Applying this to  $\mathcal{M} \simeq D\text{-mod}(G)$ , we get:

$$\text{tr} \left( D\text{-mod}(G)\text{-mod} \right) \simeq \frac{D\text{-mod}(G) \otimes D\text{-mod}(G)}{D\text{-mod}(G) \otimes D\text{-mod}(G)^{\text{op}}}$$

but  $D\text{-mod}(G)^{\text{op}} \xrightarrow{\text{inv}_*} D\text{-mod}(G)$ , + K\"{u}nneth formula

$$\simeq \frac{D\text{-mod}(G) \otimes D\text{-mod}(G)}{D\text{-mod}(G \times G)}$$

but we have:

$$e \otimes D \underset{D\text{-mod}(H)}{\simeq} (e \otimes D)_{\Delta H} \simeq (e \otimes D)^{\Delta H}$$

$$\simeq D\text{-mod}(G \times G)^{D\text{-mod}(G \times G)}$$

$$\simeq D\text{-mod} \left( \frac{G \times G}{G \times G} \right)$$

$$\simeq D\text{-mod} \left( \frac{G}{\text{Ad } G} \right)$$

this all completely parallels a computation for

$$\text{tr} \left( \text{Fun}(G(\mathbb{F}_q))\text{-mod} \simeq \text{Fun} \left( \frac{G(\mathbb{F}_q)}{\text{Ad } G(\mathbb{F}_q)} \right) \right)$$

③ baby version of depth filtration:

$$\begin{array}{c}
 B \subset G \quad \text{Borel} \quad | \quad N \rightarrow B \rightarrow T \rightarrow 1 \\
 D\text{-mod}(G) \hookrightarrow D\text{-mod}(G/B) \supset \text{End}_{D\text{-mod}(G)\text{-mod}}(D\text{-mod}(G/B), D\text{-mod}(G/B)) \\
 \cong D\text{-mod}(G/B \times G/B)^{\Delta G} \\
 \cong D\text{-mod}(B \backslash G/B) := \mathcal{H}
 \end{array}$$

we get functors:

$$\begin{array}{ccc}
 \mathcal{N} & \xrightarrow{\quad} & D\text{-mod}(G/B) \otimes_{\mathcal{H}} \mathcal{N} \\
 \mathcal{H}\text{-mod} & \xrightarrow{\quad} & D\text{-mod}(G)\text{-mod} \\
 & \xleftarrow{\quad} & \\
 \text{Hom}_{D\text{-mod}(G)\text{-mod}}(D\text{-mod}(G/B), \mathcal{M}) & \xleftarrow{\quad} & \mathcal{M} \\
 \cong & & \\
 \text{Hom}_{D\text{-mod}(B)\text{-mod}}(D\text{-mod}(B/B), \text{Res}(\mathcal{M})) & & \\
 \cong & & \\
 \mathcal{M}^B & &
 \end{array}$$

next time: see this gives a fully faithful embedding on traces...

$$\text{tr}(\mathcal{H}\text{-mod}) \hookrightarrow D\text{-mod}(G/A_G^{\text{Ad}})$$

(remainder from post-talk discussion)

$$D\text{-mod}(G/A^1_G)^{\text{nilp-ss}} \cong \bigoplus_{\substack{\text{inf} \\ \text{char} \\ \lambda}} (\text{---})$$

$$\cong \bigoplus_{\lambda} \text{tr } D\text{-mod}_{\lambda}(G/B)$$

$$\bigoplus_{x \in A^1(\mathbb{F})} \text{Qcoh}(A^1)_{\mathbb{F}} \hookrightarrow \text{Qcoh}(A^1)$$

$$D\text{-mod}(N \backslash G/N)\text{-mod} \cong D\text{-mod}(G)\text{-mod}$$

Ben-Zvi — Gunningham — Orem

$$Z(U(N \backslash G/N)) \cong D\text{-mod}(G/A^1_G)$$

$$D\text{-mod}(G)^{\Delta G} \xrightarrow{\cap} g\text{-mod} \otimes g\text{-mod}^{\Delta G}$$

$$D\text{-mod}_k(LG)^{\Delta LG} \xrightarrow{\pi} \hat{\mathfrak{g}}_k^{-\text{mod}} \otimes \hat{\mathfrak{g}}_{-k-2h^\vee}^{-\text{mod}}{}^{\Delta LG}$$

12

---


$$\text{HCh-bimodules}_{-k-2h^\vee}$$

$$D_G \cong \mathcal{U}(\mathfrak{g}) \otimes \mathcal{O}_G$$

$$\text{CDO}_G = \text{Dist}(L^+G \subset LG)$$

$$\text{ind}_{\mathfrak{g}[[z]]}^{\hat{\mathfrak{g}}_k} \text{Fun}(L^+G) \approx \mathbb{V}_{\mathfrak{g}} \otimes \mathcal{O}_{L^+G}$$

---


$$\hat{\mathfrak{g}}_k^{-\text{mod}} \xleftrightarrow{\text{local Langlands}} D_{-k^L}^{-\text{mod}} \left( \frac{LG^L}{LN_{1,\psi}^L} \right)$$

$$D\text{-mod} \left( \frac{LG}{LN_{1,\psi}} \right)^{\text{gen}} \subset D\text{-mod}(LG)^{-\text{mod}}$$

10 11

10 11

$|\mathcal{L}^{\perp}$  $\mathcal{Q}(\text{coh}(\text{LocSys}_{G^{\vee}}))$  $\subset$  $|\mathcal{L}^{\perp}$  $\mathcal{Q}\text{-IndCoh}_{\text{nilp}}(\text{LocSys}_{G^{\vee}})$