(Last updated September 8, 2022)

Let F be a non-archimedean field, $\mathcal{O}_F$ its ring of integers, k its residue field. We also write $q := \#k$, $p = \mathrm{char}(k)$. We fix $\ell \neq p$ and $\bar{\mathbb{Q}}_\ell$ from now on. We take F^s a separable closure and F^u the maximal unramified extension in it. We will be dealing with various reductive groups G over F and sometimes k. We always impose the assumption that $p > C \cdot \mathrm{rank}\, G$ for some large enough constant C that we are lazy to make precise here, but instead highlight some needed consequences of such assumption:

(1) The Jacobson-Morozov $\mathfrak{sl}_2$-triple theory holds for any reductive group / Lie algebra over k in discussion.
(2) The exponential map is always well-defined on nilpotent elements.
(3) Any torus in our group over F splits over a tamely ramified extension of F.

Fix a connected reductive group G over F. Write $\mathfrak{g} := \mathrm{Lie}\, G$, and we pick a G-invariant non-degenerate symmetric bilinear form β on $\mathfrak{g}$. Let $S \subset G$ be a maximal split torus. One may show that β induces a non-degenerate pairing on $X_*(S) \otimes \mathbb{Q}$. The **apartment** $\mathcal{A}(S)$ associated to S is a specific torsor of $X_*(S) \otimes \mathbb{R}$ on which there are certain affine hyperplanes with normal vectors in the root datum $\Phi(G, S)$, cutting $\mathcal{A}(S)$ into a locally finite configuration of polyhedrons, for which the boundary of any polyhedron is the union of (a finite number of) other smaller-dimensional polyhedrons in the configuration. Each polyhedron is called a **facet**. A **facet** is called an **alcove** if it has maximal dimension, i.e. of dimension equal to rank S. To each facet x, Bruhat-Tits theory associates to it a **parahoric subgroup** which we denote by $G_{x,0}$. Parahoric subgroups are open and compact in $G(F)$; when G is split, $G_{x,0}$ is the subgroup generated $S(\mathcal{O}_F)$ and the affine root subgroups. It has a canonical open normal subgroup $G_{x,0+}$ which is typically called the pro-unipotent radical of $G_{x,0}$. It is pro-nilpotent in the sense of abstract groups.

For $g \in G(F)$ we will use the abbreviation ${}^gU := gUg^{-1}$ for $U \subset G(F)$ and ${}^gU := \mathrm{Ad}(g)U$ for $U \subset \mathfrak{g}(F)$. The group $N_G(S)(F)$ acts on $\mathcal{A}(S)$, stabilizing the configuration. For any $n \in N_G(S)(F)$ we have ${}^nG_{x,0} = G_{n.x,0}$ and ${}^nG_{x,0+} = G_{n.x,0+}$. For any affine hyperplane on $\mathcal{A}(S)$, there exists an $n \in N_G(S)(F)$ which acts by orthogonal reflection about it. As a consequence, $N_G(S)(F)$ acts transitively on the set of alcoves.

More generally, all $\mathcal{A}(S)$ for all maximal split S can be glued together to form something that is typically called the Bruhat-Tits building $\mathcal{B}(G)$. It has a $G(F)$-action generalizing the aforementioned $N_G(S)(F)$-action, so that $g.\mathcal{A}(S) = \mathcal{A}({}^gS)$ and this respects the hyperplane/polyhedron configuration. We again have ${}^gG_{x,0} = G_{g.x,0}$ and ${}^gG_{x,0+} = G_{g.x,0+}$. Since all maximal split torus are conjugate under $G(F)$, we have that $G(F)$ acts transitively on the set of alcoves in $\mathcal{B}(G)$. In particular, the collection $\{G_{x,0} \mid x \subset \mathcal{B}(G) \text{ an alcove}\}$ forms a single conjugacy class of subgroups, called the **Iwahori subgroup**. In fact, we can fix an alcove $C \subset \mathcal{A}(S) \subset \mathcal{B}(G)$. By the same reasoning, every facet on $\mathcal{B}(G)$ is in the same $G(F)$-orbit of some facet on $\bar{C}$, the closure of C. Hence if we are happy to work modulo conjugation (in appropriate sense), we only have to work with facets in $\bar{C}$.

There is natural connected reductive group $\mathbb{G}_x$ over k with canonical isomorphism $G_{x,0}/G_{x,0+} \cong \mathbb{G}_x(k)$. Suppose we fix a maximal split torus S. Then for every $x \subset \mathcal{A}(S)$ the group $\mathbb{G}_x$ is equipped with a canonical split maximal torus $\mathbb{S}_x \subset \mathbb{G}_x$ and a canonical isomorphism $\mathrm{Hom}(\mathbb{S}_x, \mathbb{G}_m/_k) \cong \mathrm{Hom}(S, \mathbb{G}_m/_F)$. Suppose $x, y \in \mathcal{A}(S)$ are such that x is contained in the closure of y. We have a sequence of inclusions:

$$G_{x,0+} \subset G_{y,0+} \subset G_{y,0} \subset G_{x,0} \tag{1}$$

such that $G_{y,0}$ and $G_{y,0+}$ are respectively the preimage of $\mathbb{P}(k)$ and $\mathbb{U}(k)$ in $\mathbb{G}_x(k)$, where $\mathbb{P} \subset \mathbb{G}_x$ is a parabolic subgroup and $\mathbb{U}$ its unipotent radical. In particular $\mathbb{G}_y \cong \mathbb{P}/\mathbb{U}$ is our reductive group. Moreover, the torus $\mathbb{S}_x$ is contained in $\mathbb{P}$ and is mapped isomorphically to $\mathbb{S}_y \subset \mathbb{P}/\mathbb{U}$.

For any topological space X we denote by $C_c^\infty(X)$ the space of locally constant compactly supported smooth $\bar{\mathbb{Q}}_\ell$-valued functions on X. When X is discrete and/or compact we suppress the superscript $^\infty$ and/or subscript $_c$. Let us define a category $\mathcal{C} = \mathcal{C}_C$ whose objects are facets $x \subset \bar{C}$. In $\mathcal{C}$ there is a unique morphism $x_1 \to x_2$ if $\bar{x}_1 \supset x_2$, and no morphism otherwise. We have that C is an initial object in $\mathcal{C}$. Consider the covariant functor $f : \mathcal{C} \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}$ for which $f(x) := C(\mathbb{G}_x(k))$ and $f(y \to x) : C(\mathbb{G}_y(k)) \to C(\mathbb{G}_x(k))$ is given by pulling back the function from $\mathbb{G}_y(k)$ to $\mathbb{P}(k)$ (in the setting after (1)), dividing it by $q^{\dim \mathbb{U}}$, and extending by 0 to $\mathbb{G}_x(k)$. Composing with the dual space functor $\mathrm{Vec}_{\bar{\mathbb{Q}}_\ell} \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}^{op}$, we have another contravariant functor $J : \mathcal{C} \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}$ for which $J(x) := C(\mathbb{G}_x(k))^*$ and $J(y \to x) : J(x) \to J(y)$ is the dual of $f(x \to y)$, i.e. the adjoint of the aforementioned linear map.

One may thus consider the colimit and limit

$$f'_{\mathcal{C}} = \lim_{\rightarrow} C(\mathbb{G}_x(k)),\ J'_{\mathcal{C}} = \lim_{\longleftarrow} C(\mathbb{G}_x(k), \bar{\mathbb{Q}}_\ell)^*.$$

Mimicking the definition of $f(y \to x) : C(\mathbb{G}_y(k), \bar{\mathbb{Q}}_\ell) \to C(\mathbb{G}_x(k), \bar{\mathbb{Q}}_\ell)$, we have for any $x \in \mathcal{C}$ a map $f_x : C(\mathbb{G}_x(k)) \to C_c^\infty(G(F))$, given by pulling back from $\mathbb{G}_x(k) = G(F)_{x,0}/G(F)_{x,0+}$, dividing the function by $q^{\dim \mathbb{G}_x/2}$, and extend by zero to $G(F)$. By the universal property, we have a linear map $f'_G : f'_{\mathcal{C}} \to C_c^\infty(G(F))$. We have its dual map $J'_G : C_c^\infty(G(F))^* \to J'_{\mathcal{C}}$.

Since characters of admissible representations of $G(F)$ are invariant distributions, i.e. elements in $J(G(F)) := (C_c^\infty(G(F))^*)^{G(F)}$, we would like to restrict J_G to such subspace. In this case, one would expect that $J_{\mathcal{C}}$ can be replaced by something similar. Let

$$f_{\mathcal{C}} = \lim_{\rightarrow} C(\mathbb{G}_x(k))_{\mathbb{G}_x(k)},\ J_{\mathcal{C}} = \lim_{\longleftarrow} \left(C(\mathbb{G}_x(k), \bar{\mathbb{Q}}_\ell)^*\right)^{\mathbb{G}_x(k)}.$$

Then f'_G induces a map

$$f_G : f_{\mathcal{C}} \to C_c^\infty(G(F))_{G(F)}.$$

Likewise we have

$$J_G = J'_G|_{J(G(F))} : J(G(F)) \to J_{\mathcal{C}}.$$

Define $G(F)^{cpct}$, the set of compact elements in $G(F)$, to be the union of all parahoric subgroups. It is a theorem of Deligne that

Lemma 1. *The character of a supercuspidal representations is supported on the union of normalizers of parahoric subgroups. When G is simply connected, every normalizer is its own normalizer, and thus supercuspidal characters are supported on $G(F)^{cpct}$.*

Apparently f_G has image in $C_c^\infty(G(F)^{cpct})_{G(F)}$ and likewise J_G factors through restriction to $J(G(F)^{cpct}) := \left(C_c^\infty(G(F)^{cpct})^*\right)^{G(F)}$. Consider a decomposition $J(G(F)^{cpct}) = J_0(G(F)^{cpct}) \oplus J_+(G(F)^{cpct})$, where $J_0(G(F)^{cpct})$ is the span of the images of depth-0 characters, and $J_+(G(F)^{cpct})$ is the closure of the span of the image of positive-depth characters, i.e.

$$J_+(G(F)^{cpct}) := \{D \in J(G(F)^{cpct}) \mid D(f) = 0 \text{ whenever } \Theta_\pi(f) = 0 \text{ for all } \pi \in \mathrm{Irr}_+(G(F))\}.$$

where $\mathrm{Irr}_+(G(F))$ is the set of positive-depth irreducible smooth representations of $G(F)$, i.e. those with $\pi^{G_{x,0+}} = 0$ for all $x \in \mathcal{B}(G)$. We can now state

Theorem 2. *(Waldspurger, DeBacker) For any G, the image of f_G is perpendicular to $J_+(G(F)^{cpct})$, i.e. $J_G|_{J_+(G(F)^{cpct})} \equiv 0$. On the other hand $J_G|_{J_0(G(F)^{cpct})} : J_0(G(F)^{cpct}) \to J_{\mathcal{C}}$ is injective. Moreover, when G is simply connected the map f_G is injective and J_G is surjective.*

In order to drop the assumption of G being simply connected, we need to enrich the indexing category $\mathcal{C}$ into another category $\mathcal{C}'$ with the same objects but more morphisms (and it's sort of me). Suppose there is an element $n \in N_G(S)(F)$ such that $n.C = C$. Then whenever $\bar{x}_1 \supset n.x_2$ we add a morphism $x_1 \xrightarrow{n} x_2$. For the functor f, one puts $f(y \xrightarrow{n} n.x) := f(x \xrightarrow{n} n.x) \circ f(y \to x)$ where $f(x \xrightarrow{n} n.x) : C(\mathbb{G}_x(k)) \to C(\mathbb{G}_{n.x}(k))$ is given by conjugation by n. Similar for the functor J. With this fix, the assumption of G being simply connected can be dropped in Theorem 2 and also later in Theorem 7.

Remark 3. The simply connected-ness assumption in Lemma 1 cannot be dropped by the adjustment in the previous paragraph; in principle it can be tackled by replacing $G(F)_{x,0}$ by its own normalizer, so that $\mathbb{G}_x$ becomes some disconnected reductive group with $\mathbb{G}_x^o$ being the original group. However, there is the issue that $\mathbb{G}_y$ might not be a subgroup of $\mathbb{G}_x$ even though $\mathbb{G}_y^o$ is a Levi subgroup of $\mathbb{G}_x^o$. One needs to change the indexing category $\mathcal{C}$ to remedy this, and we will not going into the mess.

For any $g \in G(F)^{cpct}$, there exists a unique pair $(s, u) \in G(F)^2$ such that (i) $g = su = us$, (ii) s has finite prime-to-p order, and (iii) $u^{p^n} \to 1$ as $n \to +\infty$. Such (s, u) is called the **topological Jordan decomposition** of g, and we say s (resp. u) is topologically semisimple (resp. topologically unipotent). For any algebraic group H over F, let us denote by $H(F)^{ts}$ (resp. $H(F)^{tu}$) the subset of topologically semisimple (resp. topological unipotent) elements. We have

$$G(F)^{cpct} = \bigsqcup_{s \in G(F)^{ts}} s \cdot Z_G(s)(F)^{tu}$$

This gives

$$C_c^\infty(G(F)^{cpct})_{G(F)} = \bigoplus_{s \in G(F)^{ts}/\sim} C_c^\infty(Z_G(s)(F)^{tu})_{Z_G(s)(F)}. \tag{2}$$

where $C_c^\infty(Z_G(s)(F)^{tu})_{Z_G(s)(F)} \hookrightarrow C_c^\infty(G(F)^{cpct})_{G(F)}$ is given by $f \mapsto f_s$, $f_s(u) := f(su)$. Similarly we have

$$J(G(F)^{cpct}) = \bigoplus_{s \in G(F)^{ts}/\sim} J(Z_G(s)(F)^{tu}), \tag{3}$$

where $J(Z_G(s)(F)^{tu}) := \left(C(Z_G(s)(F)^{tu})^*\right)^{Z_G(s)(F)}$. In particular, this suggests that we can replace $G(F)^{cpct}$ by $G(F)^{tu}$. We have

Lemma 4. *The set $G(F)^{tu}$ is the union of all $G(F)_{x,0+}$, equivalently the union of all $G(F)_{x,0+}$ for $x \subset \mathcal{B}(G)$ an alcove. For any $x \in \mathcal{B}(G)$, the projection $G(F)_{x,0} \twoheadrightarrow \mathbb{G}_x(k)$ sends $G(F)^{tu} \cap G(F)_{x,0}$ to $\mathbb{G}_x(k)^{uni}$, the set of unipotent elements in $\mathbb{G}_x(k)$.*

This leads us to consider

$$f_{\mathcal{C}}^{uni} = \lim_{\rightarrow} C(\mathbb{G}_x(k)^{uni})_{\mathbb{G}_x(k)}, \; J_{\mathcal{C}}^{uni} = \lim_{\longleftarrow} \left(C(\mathbb{G}_x(k)^{uni}, \bar{\mathbb{Q}}_\ell)^*\right)^{\mathbb{G}_x(k)}.$$

Write $J_0(G(F)^{tu}) := J_0(G(F)^{cpct}) \cap J(G(F)^{tu})$. We have the following corollary of Theorem 2:

Corollary 5. *We have $J_G|_{J_0(G(F)^{tu})} : J_0(G(F)^{tu}) \hookrightarrow J_{\mathcal{C}}^{uni}$ and this is an isomorphism when G is simply connected or if we take the fix in the paragraph after Theorem 2.*

We will later simplify $J_G|_{J_0(G(F)^{tu})}$ to J_G. Recall that $\mathfrak{g} = \operatorname{Lie} G$. We have also parahoric subalgebra $\mathfrak{g}(F)_{x,0}$ and their pro-nilpotent radical $\mathfrak{g}(F)_{x,0+}$. The quotient $\mathfrak{g}(F)_{x,0}/\mathfrak{g}(F)_{x,0+}$ is a Lie algebra over k canonically isomorphic to $\operatorname{Lie} \mathbb{G}(k)$. There is a notion of topologically nilpotent elements $\mathfrak{g}(F)^{tn} \subset \mathfrak{g}(F)$, which may be taken to be the union of all $\mathfrak{g}(F)_{x,0+}$ over facets $x \subset \mathcal{B}(G)$ or alcoves $x \subset \mathcal{B}(G)$, i.e. the union of all conjugates of Iwahori subalgebra $\mathfrak{g}(F)_{C,0+}$. We have that $\mathfrak{g}(F)^{tn}_{x,0}$ is exactly the preimage of nilpotent elements $\operatorname{Lie}\mathbb{G}(k)^{nil}$ under $\mathfrak{g}(F)_{x,0} \twoheadrightarrow \operatorname{Lie}\mathbb{G}(k)$. Let us assume the following hypothesis from now on.

Hypothesis 6. *There exists a $G(F)$-equivariant homeomorphism $\mathsf{e} : \mathfrak{g}(F)^{tn} \xrightarrow{\sim} G(F)^{tu}$ sending $\mathfrak{g}(F)^{tn}_{x,0}$ to $G(F)^{tu}_{x,0}$, $\mathfrak{g}(F)_{x,0+}$ to $G(F)_{x,0+}$, and induces the usual exponential map $\operatorname{Lie}\mathbb{G}(k)^{nil} \xrightarrow{\sim} \mathbb{G}(k)^{uni}$.*

The hypothesis holds when either the usual p-adic exponential map converges, or when there is a "nice" substitute: for classical group there is the Caylay transform $X \mapsto (1+X/2)/(1-X/2)$. Thanks to the hypothesis we have a commutative diagram

$$\begin{array}{ccc} J_0(G(F)^{tu}) & \xrightarrow{J_G} & \varprojlim \left(C(\mathbb{G}_x(k)^{uni},\bar{\mathbb{Q}}_\ell)^*\right)^{\mathbb{G}_x(k)} \\ \Big\downarrow{\scriptstyle \mathsf{e}^*} & & \Big\downarrow{\scriptstyle \mathsf{e}^*} \\ J(\mathfrak{g}(F)^{tn}) & \xrightarrow{J_{\mathfrak{g}}} & \varprojlim \left(C(\operatorname{Lie}\mathbb{G}_x(k)^{nil},\bar{\mathbb{Q}}_\ell)^*\right)^{\mathbb{G}_x(k)} \end{array} \tag{4}$$

where $J_{\mathfrak{g}}$ is defined via parahoric restriction as J_G. It is obvious that the right vertical arrow is an isomorphism. On the other hand, as the above diagram is, the bottom-left object $J(\mathfrak{g}(F)^{tn})$ is too large; it is the only ∞-dimensional object in the diagram. To remedy this, let $\mathfrak{g}(F)^{nil}$ be the nilpotent (not topologically nilpotent) cone of $\mathfrak{g}(F)$, and $J(\mathfrak{g}(F)^{nil})$ be those elements in $J(\mathfrak{g}(F))$ with supports contained in $\mathfrak{g}(F)^{nil}$. For $D \in J(\mathfrak{g}(F))$, define its Fourier transform $\hat{D}$ by $\hat{D}(f) := D(\hat{f})$ for any $f \in C_c^\infty(\mathfrak{g}(F))$. We are ready to define

$$\mathrm{QC}(\mathfrak{g}(F)) := \{\hat{D} \mid D \in J(\mathfrak{g}(F)^{nil})\} \subset J(\mathfrak{g}(F)^{tn}).$$
$$\mathrm{QC}(\mathfrak{g}(F)^{tn}) := \{\hat{D}|_{\mathfrak{g}(F)^{tn}} \mid D \in J(\mathfrak{g}(F)^{nil})\} \subset J(\mathfrak{g}(F)^{tn}).$$

Here QC stands for "quasi-characters." The highly non-trivial harmonic analysis is

Theorem 7. *(Waldspurger, DeBacker, Tsai) Replacing/restricting the bottom-left of (4) by/to $\mathrm{QC}(\mathfrak{g}(F)^{tn})$, we still have a commutative diagram*

$$\begin{array}{ccccc} & & J_0(G(F)^{tu}) & \xrightarrow{J_G} & \varprojlim \left(C(\mathbb{G}_x(k)^{uni},\bar{\mathbb{Q}}_\ell)^*\right)^{\mathbb{G}_x(k)} \\ & & \Big\downarrow{\scriptstyle \mathsf{e}^*} & & \Big\downarrow{\scriptstyle \mathsf{e}^*} \\ \mathrm{QC}(\mathfrak{g}(F)) & \xrightarrow{\sim} & \mathrm{QC}(\mathfrak{g}(F)^{tn}) & \xrightarrow{J_{\mathfrak{g}}} & \varprojlim \left(C(\operatorname{Lie}\mathbb{G}_x(k)^{nil},\bar{\mathbb{Q}}_\ell)^*\right)^{\mathbb{G}_x(k)} \end{array}$$

Moreover, both vertical arrows are isomorphisms, both horizontal arrows are injective, and they are surjective if G is simply connected or if we take the fix in the paragraph after Theorem 2.

A philosophy to be taken away is that "characters of $G(F)$ looks locally like quasi-characters on $\mathfrak{g}(F)$," with a satisfying definition of what "locally" means when we restrict to depth-0 characters. At this stage, Cheng-Chiang will be confused why the bottom-right of the diagram isn't described by quasi-characters. It turns out that it is possible too. Let us define

$$\mathrm{QC}(\operatorname{Lie}\mathbb{G}_x(k)) := \{\hat{D} \mid D \in \left(C(\operatorname{Lie}\mathbb{G}_x(k),\bar{\mathbb{Q}}_\ell)^*\right)^{\mathbb{G}_x(k)}\}$$
$$\mathrm{QC}(\operatorname{Lie}\mathbb{G}_x(k)^{nil}) := \{\hat{D}|_{\operatorname{Lie}\mathbb{G}_x(k)^{nil}} \mid D \in \left(C(\operatorname{Lie}\mathbb{G}_x(k),\bar{\mathbb{Q}}_\ell)^*\right)^{\mathbb{G}_x(k)}\}$$

We have

Lemma 8. *The natural restriction* $\mathrm{QC}(\mathrm{Lie}\,\mathbb{G}_x(k)) \to \mathrm{QC}(\mathrm{Lie}\,\mathbb{G}_x(k)^{nil})$ *and the natural map* $\mathrm{QC}(\mathrm{Lie}\,\mathbb{G}_x(k)^{nil}) \to \big(C(\mathrm{Lie}\,\mathbb{G}_x(k)^{nil}, \bar{\mathbb{Q}}_\ell)^*\big)^{\mathbb{G}_x(k)}$ *are both isomorphisms.*

Theorem 9. *We have a commutative diagram*

$$\begin{array}{ccccc}
\mathrm{QC}(\mathfrak{g}(F)^{tn}) & \xrightarrow{J_\mathfrak{g}} & \varprojlim \mathrm{QC}(\mathrm{Lie}\,\mathbb{G}_x(k)^{nil}) & \xrightarrow{\sim} & \varprojlim \Big(C(\mathrm{Lie}\,\mathbb{G}_x(k)^{nil}, \bar{\mathbb{Q}}_\ell)^*\Big)^{\mathbb{G}_x(k)} \\
\sim\Big\uparrow & & \sim\Big\uparrow & & \\
\mathrm{QC}(\mathfrak{g}(F)) & \xrightarrow{J_\mathfrak{g}} & \varprojlim \mathrm{QC}(\mathrm{Lie}\,\mathbb{G}_x(k)) & &
\end{array}$$

Combining the two diagrams in Theorem 7 and 9, and also the Fourier transform, one arrives at

Corollary 10. *We have the following commutative diagram, where the first pair of vertical arrows are in Theorem 7, the second pair in Theorem 9, and the third pair being Fourier transforms.*

$$(5)\qquad \begin{array}{ccc}
J_0(G(F)^{tu}) & \xrightarrow{J_G} & \varprojlim \big(C(\mathbb{G}_x(k)^{uni}, \bar{\mathbb{Q}}_\ell)^*\big)^{\mathbb{G}_x(k)} \\
\Big\downarrow \mathsf{e}^* & & \Big\downarrow \mathsf{e}^* \\
\mathrm{QC}(\mathfrak{g}(F)^{tn}) & \xrightarrow{J_\mathfrak{g}} & \varprojlim \mathrm{QC}(\mathrm{Lie}\,\mathbb{G}_x(k)^{nil}) \\
\sim\Big\uparrow & & \sim\Big\uparrow \\
\mathrm{QC}(\mathfrak{g}(F)) & \xrightarrow{J_\mathfrak{g}} & \varprojlim \mathrm{QC}(\mathrm{Lie}\,\mathbb{G}_x(k)) \\
\Big\downarrow \sim & & \Big\downarrow \sim \\
J(\mathfrak{g}(F)^{nil}) & \xrightarrow{J_\mathfrak{g}} & \varprojlim \Big(C(\mathrm{Lie}\,\mathbb{G}_x(k)^{nil}, \bar{\mathbb{Q}}_\ell)^*\Big)^{\mathbb{G}_x(k)}
\end{array}$$

Let us look at the objects on the right of the third row and the fourth row in (5). For simplicity **from now on we assume G is split.** Let $\bar{k}$ be an algebraic closure of our residue field k. We have a standard categorification for the vector spaces $\big(C(\mathrm{Lie}\,\mathbb{G}_x(k)^{nil}, \bar{\mathbb{Q}}_\ell)^*\big)^{\mathbb{G}_x(k)}$ and $\mathrm{QC}(\mathrm{Lie}\,\mathbb{G}_x(k))$. Namely we categorify $\big(C(\mathrm{Lie}\,\mathbb{G}_x(k)^{nil}, \bar{\mathbb{Q}}_\ell)^*\big)^{\mathbb{G}_x(k)}$ via $\mathbb{G}_x$-equivariant perverse sheaves on $\mathbb{G}_x^{nil}$. Performing Fourier transforms on both sides, we categorify $\mathrm{QC}(\mathrm{Lie}\,\mathbb{G}_x(k))$ via character sheaves. By generalized Springer theory [Lus84], we have

$$(6)\qquad \mathrm{QC}(\mathrm{Lie}\,\mathbb{G}_x(k)) \cong \bigoplus_{(\mathbb{L},\mathcal{F})} K_0(\mathrm{Rep}(N_{\mathbb{G}_x}(\mathbb{L})/\mathbb{L})) \otimes \bar{\mathbb{Q}}_\ell$$

is isomorphic to a direct sum of Grothendieck groups of representations of relative Weyl groups, where the sum is over the set of isomorphism classes of $(\mathbb{L}, \mathcal{F})$ where $\mathbb{L} \subset \mathbb{G}_x$ is a standard Levi subgroup and $\mathcal{F}$ is an irreducible cuspidal character sheaf on $\mathrm{Lie}\,\mathbb{L}/_{\bar{k}}$ that is isomorphic to its own Frobenius pullback. The induction $\mathrm{ind}_{\mathbb{L}}^{\mathbb{G}_x} \mathcal{F}$ breaks into a direct sum of simple perverse sheaves indexed by $\mathrm{Irr}(N_{\mathbb{G}_x}(\mathbb{L})/\mathbb{L})$. Each simple perverse sheaf arising this way gives a vector in $\mathrm{QC}(\mathrm{Lie}\,\mathbb{G}_x(k))$ up to constant in $\bar{\mathbb{Q}}_\ell^\times$ so that all such vectors form a basis of $\mathrm{QC}(\mathrm{Lie}\,\mathbb{G}_x(k))$. Now, in light of the third row of (5), we look at the limit of (6) as

$$\begin{aligned}\mathrm{QC}(\mathfrak{g}(F)) &\cong \varprojlim \mathrm{QC}(\mathrm{Lie}\,\mathbb{G}_x(k)) \cong \varprojlim \bigoplus_{(\mathbb{L},\mathcal{F})} K_0(\mathrm{Rep}(N_{\mathbb{G}_x}(\mathbb{L})/\mathbb{L}))\otimes\bar{\mathbb{Q}}_\ell \\ &\cong \bigoplus_{(\mathbb{G}_y,\mathcal{F})} \varprojlim K_0(\mathrm{Rep}(N_{\mathbb{G}_x}(\mathbb{G}_y)/\mathbb{G}_y))\otimes\bar{\mathbb{Q}}_\ell.\end{aligned} \tag{7}$$

Here we are using that parabolic restrictions of character sheaves preserve blocks. In the last term in (7), $\mathbb{G}_y$ is taken over all facets y on the boundary of our alcove C modulo the action of $N_{G(F)}(G_{C,0})$, and the limit is taken over those x with a morphism from y to x (see the paragraph after Theorem 2 for a necessary fix when G is not simply connected). In the special case when $y=C$ and $\mathcal{F}$ is the constant sheaf, we have a limit of $K_0(\mathrm{Rep}(W_x))\otimes\bar{\mathbb{Q}}_\ell$. All these W_x glue together to be the affine Weyl group, and any element in the affine Weyl group is conjugate to some element in W_x if and only if it has finite order. That is, the limit $\varprojlim K_0(\mathrm{Rep}(W_x))\otimes\bar{\mathbb{Q}}_\ell$ is the space of $\bar{\mathbb{Q}}_\ell$-valued class functions on the affine Weyl group that are only supported on finite order elements. In general, for a facet y on the boundary of C, we consider all hyperplanes on $\mathcal{A}(S)$ that contains y. Each hyperplane is associated to some root in $\Phi(G,S)$ and let $\mathfrak{z}_y\subset\mathrm{Lie}\,S$ be the zero locus of the differentials of all such roots, and let $L_y=Z_G(\mathfrak{z}_y)\subset G$ be a standard Levi subgroup (over F). We denote by $\tilde{W}^y=??$[1] the relative affine Weyl group. We have

Lemma 11. *For a facet y and a cuspidal character sheaf $\mathcal{F}$ on $\mathrm{Lie}\,\mathbb{G}_y/_{\bar{k}}$, the limit*

$$\varprojlim K_0(\mathrm{Rep}(N_{\mathbb{G}_x}(\mathbb{G}_y)/\mathbb{G}_y))\otimes\bar{\mathbb{Q}}_\ell \cong \varprojlim K_0(\mathrm{Rep}(N_{W_x}(W_y)/W_y))\otimes\bar{\mathbb{Q}}_\ell$$

is the space of $\bar{\mathbb{Q}}_\ell$-valued class functions on $\tilde{W}^y$ supported on its finite order elements. We will denote this space by $C_{\mathrm{fin}}^{\tilde{W}^y}(\tilde{W}^y)$.

Thanks to this lemma, we have an embedding

$$\iota_{y,\mathcal{F}}:\varprojlim C_{\mathrm{fin}}^{\tilde{W}^y}(\tilde{W}^y)\to\mathrm{QC}(\mathfrak{g}(F)) \tag{8}$$

from the rightmost term in (7) to the leftmost term in (7). Let us also write $C^{W^y}(W^y)\subset C_{\mathrm{fin}}^{\tilde{W}^y}(\tilde{W}^y)$ the subspace of those class functions that factors through $\tilde{W}^y\twoheadrightarrow W^y$, so that $\iota_{y,\mathcal{F}}(C^{W^y}(W^y))$ is an even smaller subspace of $\mathrm{QC}(\mathfrak{g}(F))$. For the study of $\mathrm{QC}(\mathfrak{g}(F))$, an subspace of great interest in Langlands program is that of stable distributions. Since the space of stable distributions does not change under isogeny, let us now assume G is adjoint. Let us recall the definition:

Definition 12. *(i) Two regular semisimple element $X,X'\in\mathfrak{g}(F)^{rs}$ are stable conjugate, or say they are in the same stable orbit, if $\mathrm{Ad}(G(F^u))X=\mathrm{Ad}(G(F^u))X'$, or equivalently $\mathrm{Ad}(G(F^s))X=\mathrm{Ad}(G(F^s))X'$ (ii) A function $f\in C_c^\infty(\mathfrak{g}(F))$ is called* **unstable** *if $I_X^{st}(f)=0$ for any $X\in\mathfrak{g}(F)^{rs}$, where I_X^{st} is the integral over the stable orbit of X. (iii) An invariant distribution $D\in(C_c^\infty(\mathfrak{g}(F))^*)^{G(F)}$ is called* **stable** *if and only if $D(f)=0$ for all unstable f.*

The notion of stable distribution can be pretty subtle. For example, it was shown by DeBacker and Kazhdan that when $G=G_2$, for any *rational* subregular nilpotent orbit n, the distribution $I_n(-)$ is stable, despite that the $G(F^u)$-orbit of n have many other rational orbits.

Let us write $\mathrm{QC}(\mathfrak{g}(F))^{st}\subset\mathrm{QC}(\mathfrak{g}(F))$ the subspace of stable distributions in $\mathrm{QC}(\mathfrak{g}(F))$. We can now state

[1] Cheng-Chiang couldn't figure out some essential detail here ... Oh no ...

Theorem 13. *(Waldspurger) (i) The space* $\mathrm{QC}(\mathfrak{g}(F))^{st}$ *is a direct sum as:*

$$\mathrm{QC}(\mathfrak{g}(F))^{st} = \bigoplus_{(y,\mathcal{F})/\sim} \left(\iota_{y,\mathcal{F}}\left(C_{\mathrm{fin}}^{\tilde{W}^y}(\tilde{W}^y)\right) \cap \mathrm{QC}(\mathfrak{g}(F))^{st}\right).$$

(ii) A summand above is non-zero iff the same datum $(\mathbb{G}_y, \mathcal{F})$ *gives rise to a stable distribution in* $\mathrm{QC}(\mathfrak{l}_y(F))$ *where* $\mathfrak{l}_y := \mathrm{Lie}\, L_y = Z_{\mathfrak{g}}(\mathfrak{z}_y)$ *(see the paragraph before Lemma 11). In this case, we have*

$$\iota_{y,\mathcal{F}}\left(C^{W^y}(W^y)\right) = \iota_{y,\mathcal{F}}\left(C_{\mathrm{fin}}^{\tilde{W}^y}(\tilde{W}^y)\right) \cap \mathrm{QC}(\mathfrak{g}(F))^{st}$$

In other words, the study of $\mathrm{QC}(\mathfrak{g}(F))^{st}$ has been reduced to the case when $G = L_y$, in which case $\mathbb{G}_y$ is a smallest-dimensional facet. Equivalently in the indexing category $\mathcal{C}^{(L_y)}$ for G replaced by L_y, the object y becomes "maximal" in the sense that any morphism $y' \to y$ is an isomorphism. We note that in this case, the image of $\iota_{y,\mathcal{F}}$ is 1-dimensional. Now the real breakthrough in Waldspurger's articles in 2019 and 2020 is:

Theorem 14. *(Waldspurger) For* $y \in \mathcal{C}$ *maximal and* $\mathcal{F}$ *a cuspidal character sheaf on* $\mathrm{Lie}\,\mathbb{G}_y$ *as above, there is an algorithm to compute whether the* 1*-dimensional image of* $\iota_{y,\mathcal{F}}$ *consists of stable distributions. By applying the algorithm case-by-case, it happens that* $\iota_{y,\mathcal{F}}$ *gives stable distributions iff the data* $(y, \mathcal{F})$ *corresponds to a unipotent cuspidal character sheaves on* $G/_{\bar{k}}$.

This algorithm is based on the dual point of view:

Theorem 15. *Let* $FC(\mathfrak{g}(F)) \subset C_c^\infty(\mathfrak{g}(F))$ *be the subspace of functions* f *with the property that both* f *and* $\hat{f}$ *are supported on topologically nilpotent elements, and let* $FC(\mathfrak{g}(F))_{G(F)} \subset C_c^\infty(\mathfrak{g}(F))_{G(F)}$ *be its image in the coinvariant. Then we have a perfect pairing*

$$FC(\mathfrak{g}(F))_{G(F)} \times \bigoplus_{\substack{(y,\mathcal{F}) \\ y \in \mathcal{C} \text{ maximal}}} \mathrm{Im}(\iota_{y,\mathcal{F}}) \to \bar{\mathbb{Q}}_\ell.$$

for which a distribution in the second object is stable iff it is perpendicular to all unstable functions in $FC(\mathfrak{g}(F))_{G(F)}$.

Recall that we have reduced the question about stability to those maximal y. Hence it suffices to determine unstable functions in $FC(\mathfrak{g}(F))_{G(F)}$. We demonstrate the idea of the algorithm in Theorem 14 in the following example.

Example 16. Suppose $G = SO_5$. There are three maximal $y \in \mathcal{C}$. Let us denote them by y_1, y_2, y_3, corresponding to the three vertices of the alcove C, which is a right-angle triangle for $G = SO_5$. Suppose y_2 is the vertex at the right-triangle. We have $\mathbb{G}_{y_1} \cong \mathbb{G}_{y_3} \cong SO_5$ and $\mathbb{G}_{y_2} \cong SO_4 \cong SL_2 \times SL_2/\mu_2$. In fact, the parahoric subalgebra $\mathrm{Lie}\, G_{y_2}$ is of the form

$$\begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathfrak{m} & \mathfrak{m} & 0 \\ \mathcal{O} & \mathcal{O} & \mathfrak{m} & 0 & \mathfrak{m} \\ \mathcal{O} & \mathcal{O} & 0 & \mathfrak{m} & \mathfrak{m} \\ \mathfrak{m}^{-1} & 0 & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ 0 & \mathfrak{m}^{-1} & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix}$$

while Lie $G_{y_2} \twoheadrightarrow \mathfrak{so}_4(k)$ is given by

$$\begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathfrak{m} & \mathfrak{m} & 0 \\ \mathcal{O} & \mathcal{O} & \mathfrak{m} & 0 & \mathfrak{m} \\ \mathcal{O} & \mathcal{O} & 0 & \mathfrak{m} & \mathfrak{m} \\ \mathfrak{m}^{-1} & 0 & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ 0 & \mathfrak{m}^{-1} & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix} \twoheadrightarrow \begin{bmatrix} \mathcal{O}/\mathfrak{m} & \mathcal{O}/\mathfrak{m} & 0 & \mathfrak{m}/\mathfrak{m}^2 & 0 \\ \mathcal{O}/\mathfrak{m} & \mathcal{O}/\mathfrak{m} & 0 & 0 & \mathfrak{m}/\mathfrak{m}^2 \\ 0 & 0 & 0 & 0 & 0 \\ \mathfrak{m}^{-1}/\mathcal{O} & 0 & 0 & \mathcal{O}/\mathfrak{m} & \mathcal{O}/\mathfrak{m} \\ 0 & \mathfrak{m}^{-1}/\mathcal{O} & 0 & \mathcal{O}/\mathfrak{m} & \mathcal{O}/\mathfrak{m} \end{bmatrix} \tag{9}$$

We note that there is no non-trivial local system on the nilpotent cone of $\mathfrak{so}_5$; the number 5 is not a square. On the other hand, there is a cuspidal equivariant local system $\mathcal{F}$ on the nilpotent cone of $\mathfrak{so}_4$; it is the non-trivial equivariant local system on the regular nilpotent orbit. Under function-sheaf correspondence, this local system corresponds to a function that takes Legendre symbol value $\left(\frac{ab}{q}\right)$ on elements of the form

$$\begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b & 0 & 0 & -a \\ 0 & -b & 0 & 0 \end{bmatrix} \in \mathfrak{so}_4(k). \tag{10}$$

Let $t \in F$ be a uniformizer. The above elements have lifts of the form

$$\begin{bmatrix} 0 & a & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ bt^{-1} & 0 & 0 & 0 & -a \\ 0 & -bt^{-1} & 0 & 0 & 0 \end{bmatrix} \in \mathfrak{so}_5(F). \tag{11}$$

with $a, b, c \in \mathcal{O}_F^{\times}$. One notes that any element above is **inertially elliptic**, i.e. their centralizer in G is a torus that is anisotropic over F^u. The point is that for any element in (11), the orbital integral on its orbit is non-zero on the function in $FC(\mathfrak{g}(F))$ corresponding to (10) using (9). One may furthermore computes that

Lemma 17. *Any element in (11) has its stable orbit equal to its rational orbit.*

The lemma shows that the function in $FC(\mathfrak{g}(F))$ corresponding to (10) using (9), essentially created by y_2 and $\mathcal{F}$ above, is **not** unstable. This shows that in Theorem 15 we get a 1-dimensional space of stable distribution in the image of $\iota_{y_2,\mathcal{F}}$, while there is another 5-dimensional space of stable distributions in the image of (y, triv) for $y = C$ the alcove; by Theorem 13 it comes from the 5-dimensional space $C^{W^y}(W^y)$ where W^y is just the Weyl group for SO_5.

We note that this picture is completely analogous to the situation for unipotent character sheaves for SO_5: Among the 6 unipotent character sheaves, there are 5 that comes from the group version of Grothendieck-Springer alternation, and these 5 aren't cuspidal. There is one more that is cuspidal. It is supported on elements of the form $su \in SO_5$ for which $Z_G(s)^o \cong SO_4$ and u is regular unipotent in $Z_G(s)$, so that the character sheaf corresponds to the cuspidal local system on the $Z_G(s)^o$-orbit of u.