Unipotent almost characters of p-adic groups

§1. Character sheaves in finite type

§2. Unipotent representations

§3. Unipotent almost characters

G almost simple, simply connected (adjoint)

§0. $G/\mathbb{F}_q$, $\mathbb{C}$

$$Sh_N\left(\frac{G}{G}\right)$$

Dream: Upgrade to

$$Sh_{LG^{tn}}\left(\frac{LG}{LG}\right)$$

& $Sh_{Lg^{tn}}\left(\frac{Lg}{LG}\right)$?

Unipotent:

$$Sh\left(I \backslash LG / I\right) \xrightarrow{uTr} Sh^{uni}_{LG^{tw}}\left(\frac{LG}{LG}\right)$$

§1. (Unipotent) character sheaves $/\mathbb{F}_q$

$$\begin{tikzcd} \overline{\mathbb{Q}}_\ell^\times & T \arrow[l, "\mathcal{L}"'] & \tilde{Y}_w \arrow[l, "\rho"'] \arrow[r, "/B"] & Y_w \arrow[r, "\pi"] & G \end{tikzcd}$$

$$Y_w = \{(g,g') \in G \times G \mid g g' g^{-1} \in BwB\}$$

$$K^w_{\mathcal{L}} := \pi_! \rho^* \mathcal{L} \qquad \Big| \quad \underline{\text{Ex.}}\ k^1_{\overline{\mathbb{Q}}_\ell} = \text{G-S sheaf}$$

Def. A character sheaf is any irreducible constituent of such $K^w_{\mathcal{L}}$, $w(\mathcal{L}) = \mathcal{L}$ and $\mathcal{L}$ runs over all characters.

It is unipotent, if it appears for $\mathcal{L} = \overline{\mathbb{Q}}_\ell$

Function - Sheaf dictionary:

$$\tilde{\mathcal{F}} \in Sh_W\left(\frac{G}{G}\right) \rightsquigarrow \chi_{\tilde{\mathcal{F}}}$$

$$\chi_{\tilde{\mathcal{F}}}(g) = \sum (-1)^i \operatorname{tr}(Fr, H^i \tilde{\mathcal{F}}_g)$$

$$\in \mathbb{C}[G^F]^{G^F}$$

For irreducible $\tilde{\mathcal{F}}$, almost characters of G^F.

Unipotent characters:

<u>Def</u>. $V \in \operatorname{Rep}(G^F)$ is unipotent if it appears in $[R_w, V] \neq 0$

$R_w = \sum (-1)^i H^i_c(X_w)$, where

$$X_w = \{ gB \mid g^{-1}F(g) \in BwB \}$$

Facts (Lusztig): 1) $\mathrm{Rep}^{uni}_{irr} \longleftrightarrow \mathrm{Irr}\left(\mathrm{Sh}^{uni}_{w}\left(\frac{G}{G}\right)\right)$

non-canonical

$$\updownarrow$$

$$\bigsqcup_{\text{families for } W} \mathcal{M}(\Gamma) \sim \mathrm{Irr}\left(\mathrm{Coh}\left(\frac{\Gamma}{\Gamma}\right)\right)$$

(BFO)

Γ some canonical finite gp

2) Bases for

$\mathrm{Span}\{ \chi_V, \ V \in \mathrm{Rep}(G^F)^{uni} \}$

given by χ_V, $V \in$ irreducible unip. rep

$\chi_{\mathcal{F}}$, $\mathcal{F} \in$ irr. unip. ch. sh.

Change of basis matrix is the "non-abelian FT"

Thm (Kazhdan) $\mathrm{tr}(u, H^*_c(X_w)) = \mathrm{tr}(\mathrm{Frob}_u, H^*(\mathcal{B}_u))$

u unipotent $\in G^F$

$\mathcal{B}_u$ Springer Fiber

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§2. Unip. reps of p-adic gps

quasi-split

G/F non-arch, interested in $\mathrm{Rep}(G)^{adm}$

$\mathrm{Irr}(\mathbb{C}[I\backslash G(F)/I]\text{-mod})$

I Iwahori

$\updownarrow$ Deligne-Langlands conj. (Kazhdan-Lusztig)

$G^\vee/\mathbb{C}$

$\{(s, n, \chi) \in G^{\vee ss} \times \mathfrak{g}^{nil} \times \overline{Z}(s,n)^\vee \mid$

$sns^{-1} = qn$, χ appears in perm. rep. on $\pi_0(\mathcal{B}_{s,n})\}$

Upgrade: Parametrize unipotent reps

Def. $V \in \mathrm{Irr}(\mathrm{Rep}(G(F))^{adm})$ is unip. if $\exists P \subset G(F)$ parahoric s.t. V^{U_P} contains an unip. rep of P/U_P

Thm (Lusztig 1995)

$\mathrm{Irr}(\mathrm{Rep}^{uni}(G(F))) \longleftrightarrow \{(s, u, \mathcal{I}) \mid sus^{-1} = q u\}$

Proved using genzd. Springer:

cuspidal unip. rep. τ for some Levi L

$\mathrm{End}(\mathrm{Ind}_L^G \mathrm{Inf}_{L(\mathbb{F}_q)}^{L(\mathcal{O})} \tau) \leftarrow \mathcal{H}^{arith}$

$=$ AHA with unequal params

irreps $\longleftrightarrow$ irr. unipotent reps $G(F)$ "in this series"

$$\mathrm{End}\left(\mathrm{Ind}_{L^\vee}^{G^\vee}(\mathcal{D})\right) \cong \text{AHA from before}$$

Lusztig

cusp. loc. sys

$\mathcal{H}^{\text{geom}}$ Irreps $\longleftrightarrow \{(s, u, \mathcal{F}) \mid \text{some conditions}\}$

$q \longrightarrow 1$

<u>Def</u> $\Xi = \left\{(c, \tilde{\mathcal{F}}) \;\middle|\; \begin{array}{l} c \text{ conj. class} \subset G^\vee \\ \tilde{\mathcal{F}} \text{ irr. loc. sys. on } c \end{array}\right\}$

§3. Unipotent almost characters

Goal: construct class functions t_ξ on $G(F)_{rsc}$ = regular ss., compact, one for each $\xi \in \Xi$ s.t.

they give a basis for

Span $\{\chi_v \,,\ v \in \mathrm{Rep}^{uni}(G(F))\}$.

No sheaf in picture.

(Conj. $\exists\ \tilde{\mathcal{F}}_z \in \mathrm{loc}\ \mathrm{Sh}\left(\frac{G(F)}{G(F)}\right)$

with $\chi_{\tilde{\mathcal{F}}_z} = t_z$)

Affine Springer fibers.

$g \in G(F)_{rsc}$

$$\rightsquigarrow \quad X_g = \{ xP \mid x^{-1} g x \in NP \}$$

Fix cuspidal local system S on

$P/_{UP} = L(\mathbb{F}_q)$. Will have support C

$$\left(\begin{array}{c} \shortparallel \\ NP/_{UP} \end{array} \right) \overset{\pi}{\longleftarrow} P$$

$$X_g = \left\{ xP \;\middle|\; \begin{array}{l} x^{-1} g x \in NP \\ \pi(x^{-1} g x) \in C \end{array} \right\}$$

Pull back S to X_g $F = k((t))$

$$H_*(X_g, S) \circlearrowleft \mathrm{End}(\mathrm{Ind}_L^G(S)) \cong \mathbb{C}[\tilde{W}_J]$$

$P \Leftrightarrow J \subset I$

1/2

Def. A co-standard rep. of $\mathbb{C}[\tilde{W}_J]$ is an irrep

$$\check{E}_\xi = \mathrm{Ind}_{\tilde{W}_{J-S}}^{\tilde{W}_J}(\underbrace{t(\xi) \otimes \rho_\xi}_{\text{determined by gen'zed Springer}})$$

constructed from $\xi = (\underset{\text{conj. class in } G^\vee}{\epsilon}, \underset{\text{equiv. loc sys}}{\mathcal{L}})$

$$\mathrm{Hom}(H_*(X_g, S), \check{E}_\xi) =: A_g^\xi$$

Def. $t_\xi(g) := \mathrm{tr}(Fr, A_g^\xi)$ is an <u>unip. almost character</u>

of $G(F)$.

$$\begin{array}{ccc} [I/I] & \to & \left[\frac{LG}{LG}\right] \\ \downarrow & & \\ {[B/B]} & & \end{array}$$

Conj (Lusztig)

$t_\xi(g)$ is well-defined at every $g \in G(F)_{rsc}$ (finite generation of $H_*(\mathcal{F}_g, S)$)

Follows from Bezrukavnikov-Varshavsky

Conj (Lusztig)

$\exists$ $\tilde{\mathcal{F}}_\xi$ on $\{(g, B) \in G_c \times G(F)/I \mid g \in B\}$

s.t. $(\tilde{\mathcal{F}}_\xi)_{(g,B)} = A_g^\xi$

(Rather, I think one wants

$\tilde{\mathcal{F}}_\xi$ on G_c s.t. $(\tilde{\mathcal{F}}_\xi)_g = A_g^\xi$)

$\xi \overset{?}{\longleftrightarrow}$ Unip. reps

$c = s \cdot h$

"irreducible unip. affine char. sheaf."