Recall

G conn red $/k = \bar{k}$. $T \subseteq B \subseteq G$. $R = R(G,T) \supseteq R^+ \supseteq \Delta$ $\quad W = W(G,T)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup\!\!\!\mid\; S = \{s_\alpha ;\ \alpha \in \Delta\}$.

$G = \bigcup_{w \in W} G_w \qquad G_w = B\dot{w}B$.

Let $I \subseteq S$, $W_I = \langle I \rangle$. $\Delta_I \subseteq \Delta$. $P = \bigcup_{w \in W_I} B\dot{w}B$. $P = L \cdot U$, $T \subseteq L$.

Given $s \in S$ and $v_1, v_2 \in W$, set $Z := \bar{G}_s \cap \dot{v}_1^{-1} P \dot{v}_2$.

$$\begin{aligned} \varphi : Z &\to L \\ g &\mapsto \pi(\dot{v}_1 g \dot{v}_2^{-1}) \end{aligned} \qquad \text{where } \pi : P \to L.$$

We have shown

<u>Key Lemma</u>

$Z \neq \emptyset \Rightarrow v_1 = v_2$ or $v_1 s = v_2$. and one of the following holds:

<u>Case</u> a/ $v_1 s \notin {}^I W$. Then $v_1 = v_2 < v_1 s$ and $\sigma = v_1 s v_2^{-1} \in I$.

$$\varphi Z_s = L_\sigma, \ \varphi Z_e = L_e \text{ and } \varphi_s^* \mathcal{L}^L_{\xi,\sigma} = \mathcal{L}^G_{v_1^{-1}\xi, s}\big|_{Z_s}$$

<u>Case</u> b/ $v_1 s \in {}^I W$. $\varphi Z = L_e \ (= B_I)$.

bi/ $v_1 = v_2 < v_1 s$. Then $Z = Z_e$ and $\varphi_e^* \mathcal{L}^L_{\xi,e} \simeq \mathcal{L}^G_{v_1^{-1}\xi, e}\big|_{Z_e}$

bii/ $v_1 = v_2 > v_1 s$ Then $Z = Z_e \cup Z_s$ and $\varphi_s^* \mathcal{L}^L_{\xi,e} \simeq \mathcal{L}^G_{v_1^{-1}\xi, s}\big|_{Z_s}$ if $s \in W_{v_1^{-1}\xi}$

biii/ $v_1 s = v_2$ Then $Z = Z_s$ and $\varphi_s^* \mathcal{L}^L_{\xi,e} \simeq \mathcal{L}^G_{v_2^{-1}\xi, s}\big|_{Z_s}$

Moreover, $\varphi : Z \to \overline{L_\sigma}$ (case a/) or $\varphi : Z \to L_e$ (case b/)

is a fibration by affine spaces.

fibres = homogeneous space of $B \cap \dot{v}_1^{-1} U \dot{v}_1$

Now, consider

$$\begin{tikzcd}
 & \bigcup_{\underline{v}} Z_{\underline{v}} \arrow[r] \arrow[d, equal] & \bigcup_{\underline{\sigma}} \overline{Y}^L_{\underline{\sigma}} \arrow[dd] \\
G\times^B(\overline{G_{s_1}}\times^B\cdots\times^B\overline{G_{s_r}}) = \overline{Y}_{\underline{s}} \arrow[d] & \overline{Y}^P_{\underline{s}} \arrow[l] \arrow[d] & \\
G & P \arrow[l] \arrow[r] & L
\end{tikzcd}$$

- Fix $(v_1,\dots,v_r)\in({}^I W)^r$, set $v_0=v_r$ and

$$Z_{\underline{v}} = \{(g_0,g_1,\dots,g_r)\in\overline{Y}_{\underline{s}} ;\ g_0\cdots g_i\in P v_i B,\ g_0 g_1\cdots g_r g_0^{-1}\in P\}$$

Set $Z_i = \overline{G_{s_i}}\cap v_{i-1}^{-1}Pv_i$ for $i\in[1,r]$. $B_i = B\cap v_i^{-1}P v_i$

We have $\underline{B} := B_0\times B_1\times\cdots\times B_{r-1} \circlearrowright P\times Z_1\times\cdots\times Z_r$

$$\begin{tikzcd}
P\times Z_1\times\cdots\times Z_r \arrow[d, "\underline{B}\text{-torsor}"] & (p,g_1,\dots,g_r) \arrow[d, mapsto] \\
Z_{\underline{v}} & (pv_0,g_1,\dots,g_r)
\end{tikzcd}$$

$(b_0,b_1,\dots,b_{r-1})\cdot(p,g_1,\dots,g_r) = (p v_0 b_0^{-1}v_0^{-1}, b_0 g_1 b_1^{-1},\dots,b_{r-1}g_r b_0^{-1})$

Thus, $Z_{\underline{v}}=\emptyset$ unless $v_i\in\{v_{i-1}, v_{i-1}s_i\}$ $\forall i\in[1,r]$

• Assume $v_i \in \{v_{i-1}, v_{i-1}s_i\}$ $\forall i \in [1,r]$

For $i \in [1,r]$, set $\sigma_i = \begin{cases} v_{i-1}s_iv_{i-1}^{-1} \in I & \text{if } v_{i-1}s_iv_{i-1}^{-1} \notin {}^{I}W \text{ (case a)} \\ e & \text{if } v_{i-1}s_iv_{i-1}^{-1} \in {}^{I}W \text{ (case b)} \end{cases}$

$\underline{\sigma} = (\sigma_1, \dots, \sigma_r)$, $\quad \overline{Y}^L_{\underline{\sigma}} := L \times^{B_I} (\overline{L_{\sigma_1}} \times^{B_I} \cdots \times^{B_I} \overline{L_{\sigma_r}})$.

$$A_{s_2\cdots s_r J, s_1} \boxtimes \cdots \boxtimes A_{J, s_r} \qquad (p, g_1, \dots, g_r) \longmapsto (\pi(p), \pi(\dot v_0 g_1 \dot v_1^{-1}), \dots, \pi(\dot v_{r-1} g_r \dot v_0^{-1}))$$

$$\begin{tikzcd}
A_{J,\underline{s}} / G \times \overline{G}_{s_1} \times \cdots \times \overline{G}_{s_r} \arrow[d, "B^r\text{-torsor}"] & P \times Z_1 \times \cdots \times Z_r \arrow[l, hook'] \arrow[r, "\phi"] \arrow[d, "\underline{B}\text{-torsor}"'] & L \times \overline{L_{\sigma_1}} \times \cdots \times \overline{L_{\sigma_r}} \arrow[d, "(B_I)^r\text{-torsor}"] \\
\tilde{A}^G_{J,\underline{s}} / \overline{Y}^G_{\underline{s}} \arrow[d, "\gamma^G"] & \overline{Y}^P_{\underline{s}} \cong Z_{\underline{v}} \arrow[l, "\tilde{\iota}"] \arrow[l, bend right, "\mathfrak{z}_{\underline{v}}"'] \arrow[r, dashed, "\rho"] \arrow[d, bend right, "\gamma^P"'] \arrow[d, "\gamma^P_{\underline{v}}"] & \overline{Y}^L_{\underline{\sigma}} \arrow[d, "\gamma^L_{\underline{\sigma}}"] \\
G & P \arrow[l, "\iota"] \arrow[r, "\pi"] & L
\end{tikzcd}$$

$$\mathrm{res}_P^G C^G_{\mathfrak{z},\underline{s}} = \pi_! \, i^* \gamma_!^G \tilde{A}^G_{\mathfrak{z},\underline{s}} \cong \pi_! \, \gamma_!^P \tilde{i}^* A^G_{\mathfrak{z},\underline{s}} .$$

Let $A_{\underline{v}} = (\tilde{i}^* A^G_{\mathfrak{z},\underline{s}})|_{Z_{\underline{v}}} \in D_P(Z_{\underline{v}})$

1/ $(Z_{\underline{v}}, A_{\underline{v}})$ don't contribute to $\mathrm{res}_P^G C^G_{\mathfrak{z},\underline{s}}$.

2/ $(Z_{\underline{v}}, A_{\underline{v}})$ yields $\gamma_{\underline{\sigma}!} \rho_! A_{\underline{v}} \cong C^L_{v_0 \mathfrak{z},\underline{\sigma}}[-]$ (+ pure Weil str)

3/ By purity, we get $\mathrm{res}_P^G C^G_{\mathfrak{z},\underline{s}} = \bigoplus_{\underline{v}} C^L_{v_0 \mathfrak{z},\underline{\sigma}}[-]$.

Set $I_{\underline{s}} = \{ i \in [1,r] \; ; \; s_r \cdots s_{i+1} s_i s_{i+1} \cdots s_r \in W_{\mathfrak{z}} \}$

$J_{\underline{v}} = \{ j \in [1,r] \; ; \; v_i = v_{i-1} , \; v_{i-1} s_i \in {}^I W \}$ case b:/ b::/

<u>Lem</u>. Assume $v_i \in \{v_{i-1}, v_{i-1} s_i\} \; \forall i$, so that $Z_{\underline{v}} \neq \emptyset$.

If $J_{\underline{v}} \not\subset I_{\underline{s}}$, then $\rho_! A_{\underline{v}} = 0$

Pf. Via the above diagram, it suffices to show that for $v_1, v_2 \in {}^I W$ and $s \in S$ s.t. $Z = \bar{G}_s \cap \dot{v}_1^{-1} P \dot{v}_2 \neq \emptyset$, $v_1 = v_2$, $v_1 s \in {}^I W$

and $s \notin W_I$, we have $\varphi ! A_{\xi, s}|_Z = 0$.

- Suppose $v_1 s > v_1$. Then $Z = B \cap \dot{v}_1^{-1} P \dot{v}_1^{-1}$

but $s \notin W_I$ implies $\mathcal{L}_{\xi, s} / \bar{G}_s$ is clean on B. $\rightsquigarrow A_{\xi, s}|_Z = 0$.

- Suppose $v_1 s < v_1$. Then $A_{\xi, s}|_{Z_e} = 0$ as previous case.

$$v_1\alpha \in \Delta \setminus \Delta_I \subseteq \text{rad}.$$

$$s = s_\alpha$$

$$\begin{tikzcd} & \underset{\dot{s} B \dot{s}^{-1}}{Z_s \cup} = T \cdot (X_{-\alpha} \setminus \{e\}) \, U' \arrow[dl, "pr^G"'] \arrow[d, "\varphi_s"] \\ T & L_e \end{tikzcd}$$

$U' = U_B \cap \dot{v}_1^{-1} U_B \dot{v}_1$.

There is an iso $X_{-\alpha} \setminus \{e\} \overset{\psi}{\cong} \mathbb{G}_m$ s.t. $pr^G u = \alpha^\vee(u)$ for $u \in X_{-\alpha} \setminus \{e\}$.

Then, $tuu' \in T \cdot (X_{-\alpha} \setminus \{e\}) U' \longmapsto pr^G(tuu') = s(t) \cdot \alpha^\vee(\psi(u)) \in T$.

$\varphi_s^{-1}(e) = (X_{-\alpha} \setminus \{e\}) \times (\dot{v}_1^{-1} U_P \dot{v}_1) \Rightarrow \mathcal{L}^G_{\xi, s}|_{\varphi_s^{-1}(e)} \neq \text{triv}$.

$\Rightarrow \varphi_*(A_{\underline{v}})\big|_e = 0 \Rightarrow \varphi_* A_{\underline{v}} = 0$ by equivariance of φ. #

__Lem__. Suppose $\mathcal{J}_{\underline{v}} \subseteq \mathcal{J}_{\underline{s}}$. Then $p_! A_{\underline{v}} \cong A^L_{v\circ\underline{j},\underline{\sigma}}[-2d(\underline{v})]$, where

$$d(\underline{v}) = \dim U_p + \#\{i \in [1,r] ; v_i = v_{i-1},\ v_{i-1}s_i < v_i\} \quad (\text{case } b\ \text{ii})$$

Pf. Since p is an affine-space fibration of rel. dim. $d(\underline{v})$,

it suffices to show $p^* A^L_{j,\underline{\sigma}}[-2d(\underline{v})] \cong A_{\underline{v}}$.

Moreover, since both sides are simple perverse sheaves and $A^L_{j,\underline{\sigma}}$ has full support, it suffices to show the isom on any open subvar of $Z_{\underline{v}}$

Consider $J'_{\underline{v}} = \{ i \in J_{\underline{v}} \,;\, v_{i-1}s_i > v_{i-1} \}$ (case 6i/).

Set $\underline{s}' = (s'_1, \dots, s'_r)$ $\quad s'_i = \begin{cases} s_i & i \notin J'_{\underline{v}} \\ e & i \in J'_{\underline{v}} \end{cases}$

Then
$$\begin{array}{ccc} Z_{\underline{v}} \cap Y^G_{\underline{s}'} & \subseteq & Z_{\underline{v}} \\ \downarrow \phi & & \downarrow \phi \\ Y^L_{\underline{\sigma}} & \subseteq & \bar{Y}^L_{\underline{\sigma}} \end{array}$$
which is open & dense by the Key Lemma

Moreover,
$$\begin{array}{ccc} P \times \mathring{Z}_1 \times \cdots \times \mathring{Z}_r & \longrightarrow & L \times L_{\sigma_1} \times \cdots \times L_{\sigma_r} \\ \downarrow B_0 \times \cdots \times B_{r-1}\text{-torsor} & & \downarrow (B_L)^r\text{-torsor.} \\ Z_{\underline{v}} \cap Y^G_{\underline{s}'} & \longrightarrow & Y^L_{\underline{\sigma}} \end{array}$$

$\mathring{Z}_i = G_{s'_i} \cap v_{i-1}^{-1} P v_i$

$$A^L_{v_0 j, \underline{\sigma}} \Big|_{L \times L_{\sigma_1} \times \cdots \times L_{\sigma_r}} = \mathcal{L}^L_{\sigma_2 \cdots \sigma_r v_0 j, \sigma_1} \boxtimes \cdots \boxtimes \mathcal{L}^L_{v_0 j, \sigma_r} [\dim Y^L_{\underline{\sigma}}]$$

$$A_{\underline{v}} \Big|_{P \times \mathring{Z}_1 \times \cdots \times \mathring{Z}_r} = \mathcal{L}^G_{s'_2 \cdots s'_r j, s'_1}\Big|_{\mathring{Z}_1} \boxtimes \cdots \boxtimes \mathcal{L}^G_{j, s'_r}\Big|_{\mathring{Z}_r} [\dim \bar{Y}^G_{\underline{s}}]$$

Key lemma $\Rightarrow$ $\varphi^* \mathcal{L}^L_{\sigma_{i+1}\cdots\sigma_r v_0 \mathfrak{J}, \sigma_i} \simeq \mathcal{L}^G_{s_{i+1}\cdots s_r \mathfrak{J}, s_i}$

Thus $\phi_! A_{\underline{v}} \simeq A^L_{v_0 \mathfrak{J}, \underline{\sigma}} [\dim Y^G_{\underline{s}} - \dim Y^L_{\underline{\sigma}} - 2d(\underline{v})]$ #.

Back to the proof of the theorem.

we simply notice that $A^L_{v_0\mathfrak{J},\underline{\sigma}}$ has a natural

Weil structure of wt 0 and $\phi_! A_{\underline{v}} \simeq A^L_{v_0 \mathfrak{J}, \underline{\sigma}}[-2d(\underline{v})](-d(\underline{v}))$.

$\Rightarrow$ dévissage $\operatorname{res}^G_P C^G_{\mathfrak{J},\underline{s}} \simeq \bigoplus_{\underline{v}} r^L_* A^L_{v_0\mathfrak{J},\underline{\sigma}}[\dim Y^G_{\underline{s}} - \dim Y^L_{\underline{\sigma}} - 2d(\underline{v})] \cong \bigoplus_{\substack{\underline{v} \\ \mathfrak{J}_{\underline{v}} \subseteq \mathfrak{J}_0}} C^L_{v_0\mathfrak{J},\underline{\sigma}}[\dim \cdot - 2d(\underline{v})]$

proving i/.

#

now, $$[\mathrm{res}^G_P\, C^G_{\zeta,s}] = \sum_{f_{\underline{v}} \subseteq \mathcal{J}_s} t^{-n_+(\underline{v})+n_-(\underline{v})} \tau^L(C_{\sigma_2 \cdots \sigma_r v_0 \zeta, \sigma_1} \cdots C_{v_0\zeta,\sigma_r})$$

$$n_\pm(\underline{v}) = \#\{i \in [1,r] \,;\, v_{i-1}s_i \in {}^I W,\ v_{i-1}s_i \gtrless v_{i-1}\}$$

$$n(\underline{v}) = n_+(\underline{v}) - n_-(\underline{v}).$$

On the other hand, $v \in {}^I W$.

$$u_v C_{\zeta,s} = \begin{cases} C_{vs\zeta,e}\, u_{vs} & vs \in {}^I W,\ s \notin W_\zeta \\ t^{-1} C_{v\zeta,e}(u_v + u_{vs}) & vs \in {}^I W,\ s \in W_\zeta,\ vs > v \\ t\, C_{v\zeta,e}(u_v + u_{vs}) & vs < v \\ C_{v\zeta,\sigma}\, u_v & vsv^{-1} = \sigma \in I \end{cases}$$

$$\Rightarrow \mathrm{tr}_I(C_{s_2 \cdots s_r \zeta, s_1} \cdots C_{\zeta, s_r}) = \sum_{f_{\underline{v}} \subseteq \mathcal{J}_s} t^{-n_{+,\zeta}(\underline{v}) + n_{-,\zeta}(\underline{v})}.$$

Statement ii/ follows from $n_+(\underline{v}) - n_-(\underline{v}) = n_{+,\zeta}(\underline{v}) - n_{-,\zeta}(\underline{v})$ provided $f_{\underline{v}} \subseteq \mathcal{J}_s$ and $s_1 \cdots s_r \in W_\zeta'$.

(proof omitted).

#

§10 Mackey's formula & duality

Let $P, Q \subseteq G$ be parabolic subgroups. $\Sigma \subseteq G$: rep of $Q \backslash G / P$.
$L \subseteq P$ $M \subseteq Q$ Levi component. ${}^xL \cap M$ contains a max torus $\forall x \in \Sigma$.

Prop. A : admissible cpx $\Rightarrow$ $\mathrm{res}^G_Q \,\mathrm{ind}^G_P A \cong \bigoplus_{x \in \Sigma} \mathrm{ind}^M_{M \cap {}^xP} \, {}^x(\mathrm{res}^L_{L \cap Q^x} A)$.

Pf

$$\begin{tikzcd}
M \cap {}^xL & M \cap {}^xP \arrow[l] & M \times M \cap {}^xP \arrow[l] \arrow[r, "\text{torsor}"] & M \times^{M \cap {}^xP} M \cap {}^xP \arrow[r] & M \\
Q \cap {}^xL \arrow[u] \arrow[d, "(-)^x"'] \arrow[ur, phantom, "\ulcorner"] & Q \cap {}^xP \arrow[l] \arrow[u] \arrow[d] & Q \times Q \cap {}^xP \arrow[l] \arrow[u] \arrow[d] \arrow[r, "\text{torsor}"] & Q \times^{Q \cap {}^xP} Q \cap {}^xP \subseteq V \arrow[u] \arrow[d] \arrow[r] \arrow[dr, phantom, "\lrcorner"] & Q \arrow[u] \arrow[d] \\
L & P \arrow[l] & G \times P \arrow[l] \arrow[r, "\text{torsor}"] & G \times^P P \arrow[r] & G
\end{tikzcd}$$

Prove via proper base change (for the two cartesian squares), equivariant induction (for the torsor arrows)

#.

Define $CG \subseteq K_0(\mathrm{Perv}_G(G)) \otimes \mathbb{C}$.
$\quad\quad\quad\quad \parallel$
$\quad\quad\quad \langle$ adm perv. shvs $\rangle$

$\langle\ ,\ \rangle_G : CG \times CG \to \mathbb{C}$ $\quad\quad \langle A, B\rangle_G = \dim \mathrm{Hom}(A, B)$.

We have $\quad \langle \mathrm{ind}_P^G A, B\rangle = \langle A, \mathrm{res}_P^G B\rangle$

Then, Mackey's formula $\Rightarrow$

$$\langle \mathrm{ind}_P^G A, \mathrm{ind}_Q^G B\rangle = \sum_{x \in \Sigma} \langle \mathrm{ind}_{M \cap {}^xP}^M {}^x\mathrm{res}_{L \cap Q^x}^L A, B\rangle_M$$

Take $Q \supseteq L$ (parab.), $B = A$

$\Rightarrow \langle \mathrm{ind}_P^G A - \mathrm{ind}_{P'}^G A, \mathrm{ind}_P^G A - \mathrm{ind}_{P'}^G A\rangle_G = 0 \quad \Rightarrow \mathrm{ind}_P^G A \cong \mathrm{ind}_{P'}^G A$.

Duality : $\quad d := \sum_{I \subseteq S} (-1)^{|I|} i_I^S r_I^S$. $\quad \langle d-, -\rangle = \langle -, d-\rangle$. $\quad d^2 = \mathrm{id}$.