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Character Sheaves I

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Contents.

This paper is an attempt to construct a geometric theory of characters of a reductive algebraic group G defined over an algebraically closed field. We are seeking a theory which is as close as possible to the theory of irreducible (complex) characters of the corresponding groups $G(F_q)$ over a finite field F_q and yet it should have a meaning over algebraically closed fields.

The basic objects in the theory are certain irreducible (l-adic) perverse sheaves (in the sense of [1]) on G; they are the analogues of the irreducible (l-adic) representations of $G(F_q)$ and are called the character sheaves of G. The definition of character sheaves is suggested by the following result [3, Corollary 7.7]: any irreducible representation of $G(F_q)$ appears in at least one of the virtual representations $R^{\theta}(w)$, defined by Deligne and Lusztig in [3, 1.9].

The virtual representations $R^{\theta}(w)$ have a geometric analogue $K_w^{\mathscr{L}}$, (here w is an element in the Weyl group and $\mathscr{L}$ is a "tame" local system of rank 1 on the maximal torus of G). We shall define here $K_w^{\mathscr{L}}$ only in the case where $\mathscr{L}$ is the constant local system $\mathscr{L}_0 = \bar{\mathbf{Q}}_l$.

Let Y_w be the variety of all pairs (g, B), where g is and element of G and B is a Borel subgroup of G such that B, gBg^{-1} are in relative position w; let $\pi_w: Y_w \to G$ be the morphism defined by $\pi_w(g, B) = g$. We define $K_w^{\mathscr{L}_0}$ to be the direct image with compact support $(\pi_w)_! \bar{\mathbf{Q}}_l$. Then, $K_w^{\mathscr{L}_0}$ is an object in the derived category of constructible l-adic sheaves on G. (The definition of $K_w^{\mathscr{L}}$ is given in 2.4.) The character sheaves of G are, by definition, those irreducible perverse sheaves which are constituents of a perverse cohomology sheaf ${}^pH^i(K_w^{\mathscr{L}})$ for some i, w, $\mathscr{L}$.

We note the similarity of $K_w^{\mathscr{L}_0}$ and $R^1(w)$: the virtual representation $R^1(w)$ is defined as the alternating sum of the $G(F_q)$-modules $H_c^i(X_w, \mathbf{Q}_l)$, where X_w is the variety of all Borel subgroups which are in relative position w with their transform under the Frobenius map. (Thus, Y_w is the analogue of X_w.)

Our objective in this paper and the ones following it is to classify the character sheaves of G and to compute their cohomology sheaves.

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The paper is organized as follows: Section 1 collects some of the basic results on perverse sheaves due to Beilinson–Bernstein–Deligne–Gabber [1]. Section 2 contains the definition of character sheaves. Apart from the definition in terms of $K_w^{\mathscr{L}}$ we also give an equivalent definition in terms of some compactification $\bar{\pi}_w$ of $\pi_w: Y_w \to G$. This compactification (which is analogous to the compactification [3, 9.10] of X_w) is essential to apply the deep results of [1, 2]. In Sections 3 and 4 we study the restriction and induction for character sheaves. (These are analogues of the familiar operations on representations of $G(F_q)$.) As a consequence of Theorem 4.4, the character sheaves of G are a special case of the "admissible complexes of G" defined in [4]; we hope to show elsewhere that these two classes of complexes on G coincide. Section 5 contains some technical preliminaries to Section 6. The most difficult result of this paper is Theorem 6.9(a) which asserts that the restriction functor carries a character sheaf to a direct sum of character sheaves.

1. Perverse Sheaves

1.1. The theory of perverse sheaves on algebraic varieties is due to Beilinson, Bernstein, Deligne, and Gabber. The basic reference is [1].

We shall review here some of the theory.

1.2. Let k be an algebraically closed field. Unless otherwise specified, all algebraic varieties will be over k.

We denote by $\mathscr{D}X = \mathscr{D}_c^b(X, \bar{\mathbf{Q}}_l)$ the bounded derived category of $\bar{\mathbf{Q}}_l$-(constructible) sheaves on X [1, 2.2.18]; here l is a fixed prime number such that $l^{-1} \in k$ and $\bar{\mathbf{Q}}_l$ is an algebraic closure of the field of l-adic numbers.

Objects of $\mathscr{D}X$ are referred to as "complexes." For a complex $K \in \mathscr{D}X$, we denote by $\mathscr{H}^i K$ the ith cohomology sheaf of K (a $\bar{\mathbf{Q}}_l$-sheaf on X); we denote by $DK \in \mathscr{D}X$ the Verdier dual of K.

1.3. Let $\mathscr{D}X^{\leqslant 0}$ be the full subcategory of $\mathscr{D}X$ whose objects are those K in $\mathscr{D}X$ such that, for any integer i, $\mathscr{H}^i K$ has support of dimension $\leqslant -i$. (In particular, we have $\mathscr{H}^i K = 0$ for $i > 0$.)

Let $\mathscr{D}X^{\geqslant 0}$ be the full subcategory of $\mathscr{D}X$ whose objects are those K in $\mathscr{D}X$ such that $DK \in \mathscr{D}X^{\leqslant 0}$. Let $\mathscr{M}X$ be the full subcategory of $\mathscr{D}X$ whose objects are those K in $\mathscr{D}X$ such that $K \in \mathscr{D}X^{\leqslant 0} \cap \mathscr{D}X^{\geqslant 0}$; the objects of $\mathscr{M}X$ are called *perverse sheaves* on X.

$\mathscr{M}X$ is an abelian category [1, 2.14, 1.3.6] in which all objects have finite length [1, 4.3.1].

1.4. The inclusion of $\mathscr{D}X^{\leqslant 0}$ in $\mathscr{D}X$ has a right adjoint ${}^p\tau_{\leqslant 0}$ and the inclusion of $\mathscr{D}X^{\geqslant 0}$ in $\mathscr{D}X$ has a left adjoint ${}^p\tau_{\geqslant 0}$, [1, 2.2.11, 1.3.3(i)]: we have natural morphisms ${}^p\tau_{\leqslant 0}K \to K \to {}^p\tau_{\geqslant 0}K$ ($K \in \mathscr{D}X$) and

$$\mathrm{Hom}(A, {}^p\tau_{\leqslant 0}B) = \mathrm{Hom}(A, B) \qquad \text{for all} \quad A \in \mathscr{D}X^{\leqslant 0},\ B \in \mathscr{D}X$$

and

$$\mathrm{Hom}({}^p\tau_{\geqslant 0}A', B') = \mathrm{Hom}(A', B') \qquad \text{for all} \quad A' \in \mathscr{D}X,\ B' \in \mathscr{D}X^{\geqslant 0}.$$

The functors ${}^p\tau_{\geqslant 0}{}^p\tau_{\leqslant 0}$, ${}^p\tau_{\leqslant 0}{}^p\tau_{\geqslant 0}$, ($\mathscr{D}X \to \mathscr{D}X$), are canonically isomorphic [1, 1.3.5]. Hence, for any $K \in \mathscr{D}X$, the complex ${}^p\tau_{\geqslant 0}{}^p\tau_{\leqslant 0}K$ is a perverse sheaf; it is denoted ${}^pH^0K$.

The functor ${}^pH^0: \mathscr{D}X \to \mathscr{M}X$ is a cohomological functor [1, 1.3.6], i.e., for any distinguished triangle (K, K', K'') in $\mathscr{D}X$ (notation of [1, 1.1.1]), the corresponding sequence ${}^pH^0K \to {}^pH^0K' \to {}^pH^0K''$ is exact.

We define ${}^pH^i: \mathscr{D}X \to \mathscr{M}X$ by ${}^pH^iK = {}^pH^0(K[i])$, where $[i]$ denotes "décalage," or shift. Then, it follows that for any distinguished triangle (K, K', K'') in $\mathscr{D}X$ we have a long exact sequence of perverse sheaves

$$\cdots \to {}^pH^iK \to {}^pH^iK' \to {}^pH^iK'' \to {}^pH^{i+1}K \to \cdots.$$

Moreover, for any $K \in \mathscr{D}X$, we have ${}^pH^iK = 0$ for all but a finite number of integers i.

1.5. The irreducible objects of $\mathscr{M}X$ can be described as follows [1, 4.3.1].

Let V be a locally closed, smooth, irreducible subvariety of X, of dimension d and let $\mathscr{L}$ be an irreducible $\bar{\mathbf{Q}}_l$-local system on V. Then $\mathscr{L}[d]$ is an ireducible perverse sheaf on V and there is a unique irreducible perverse sheaf $\widetilde{\mathscr{L}[d]}$ on the closure $\bar{V}$, whose restriction to V is $\mathscr{L}[d]$; we have $\widetilde{\mathscr{L}[d]} = \mathrm{IC}(\bar{V}, \mathscr{L})[d]$, where $\mathrm{IC}(\bar{V}, \mathscr{L})$ is the *intersection cohomology complex of* Deligne–Goresky–MacPherson of $\bar{V}$ with coefficients in $\mathscr{L}$. The extension of $\widetilde{\mathscr{L}[d]}$ to X (by 0 outside $\bar{V}$) is an irreducible perverse sheaf on X, and all irreducible perverse sheaves on X are obtained in this way.

1.6. Let X be a smooth irreducible variety of dimension d, and let $D_1, D_2, \ldots, D_r$ be smooth divisors with normal crossings in X. Let $\mathscr{L}$ be a one-dimensional, $\bar{\mathbf{Q}}_l$-local system on the open subset $X - (D_1 \cup \cdots \cup D_r)$, such that the corresponding representation of the fundamental group factors through a finite quotient of order invertible in k. The intersection complex $\mathrm{IC}(X, \mathscr{L})$ can be represented in $\mathscr{D}X$ as a single constructible $\bar{\mathbf{Q}}_l$-sheaf $\bar{\mathscr{L}}$ (in degree 0). Let I_0 be the set of $i \in [1, r]$ such that the local monodromy of $\mathscr{L}$ around D_i is nontrivial. Then $\bar{\mathscr{L}}$ restricted to the open subset $X - \bigcup_{i \in I_0} D_i$ is a local system of rank 1 and $\bar{\mathscr{L}}$ restricted to the closed subset $\bigcup_{i \in I_0} D_i$ is zero. (These statements can be reduced to the special case where $\dim X = 1$.)

1.7. Let $f: X \to Y$ be a morphism between the algebraic varieties X, Y. Let $f^*: \mathscr{D}Y \to \mathscr{D}X$ be the inverse image functor and let $f_!: \mathscr{D}X \to \mathscr{D}Y$ be the direct image with compact support. They admit adjoint functors $f_*: \mathscr{D}X \to \mathscr{D}Y, f^!: \mathscr{D}Y \to \mathscr{D}X$; for any $A \in \mathscr{D}X$, $B \in \mathscr{D}Y$, we have:

(1.7.1) $\operatorname{Hom}(f^*B, A) = \operatorname{Hom}(B, f_*A)$.

(1.7.2) $\operatorname{Hom}(f_!A, B) = \operatorname{Hom}(A, f^!B)$.

(1.7.3) If f is proper, then $f_* = f_!$.

(1.7.4) If f is smooth with connected fibres of dimension d, then $f^! = f^*[2d](d)$, where (d) denotes Tate twist; in this case, we set $\tilde{f} = f^*[d]$.

(1.7.5) Let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "h'"] \\ Z \arrow[r, "f'"'] & U \end{tikzcd}$$

be a cartesian diagram with f, f' smooth with connected fibres of dimension d. Then $h_!\tilde{f} = \tilde{f}'h'_!: \mathscr{D}Y \to \mathscr{D}Z$.

1.8. Assume that $f: X \to Y$ is smooth, with connected fibres of dimension d.

Here are some properties of $\tilde{f}$ (see (1.7.4)):

(1.8.1) If $K \in \mathscr{D}Y$, then we have

$$K \in \mathscr{D}Y^{\leqslant 0} \Leftrightarrow \tilde{f}K \in \mathscr{D}X^{\leqslant 0},$$
$$K \in \mathscr{D}Y^{\geqslant 0} \Leftrightarrow \tilde{f}K \in \mathscr{D}X^{\geqslant 0},$$
$$K \in \mathscr{M}Y \Leftrightarrow \tilde{f}K \in \mathscr{M}X,$$
$${}^pH^i(\tilde{f}K) = \tilde{f}({}^pH^iK).$$

(1.8.2) If $K \in \mathscr{D}Y^{\leqslant 0}$, $K' \in \mathscr{D}Y^{\geqslant 0}$, then $\operatorname{Hom}_{\mathscr{D}Y}(K, K') = \operatorname{Hom}_{\mathscr{D}X}(\tilde{f}K, \tilde{f}K')$.

(1.8.3) $\tilde{f}: \mathscr{M}Y \to \mathscr{M}X$ is fully faithful.

(1.8.4) If $K \in \mathscr{M}Y$ and $K' \in \mathscr{M}X$ is a subquotient of $\tilde{f}K \in \mathscr{M}X$, then K' is isomorphic to $\tilde{f}K_1$ for some $K_1 \in \mathscr{M}Y$.

(The proofs are in [1, 4.2.5, 4.2.6].)

1.9. Let $m: H \times Y \to Y$ be an action of a *connected* algebraic group H on the variety Y. Let $\pi: H \times Y \to Y$ be the second projection. Both m and π are smooth morphisms with fibres isomorphic to H. Hence, if $K \in \mathscr{M}Y$, then $\tilde{m}K$, $\tilde{\pi}K$ (see (1.7.4)) are perverse sheaves on $H \times Y$. We say that K is H-*equivariant* if $\tilde{m}K$, $\tilde{\pi}K$ are isomorphic as perverse sheaves on $H \times Y$. (This is equivalent to the definition in [4, Sect. 0].)

(1.9.1) *If $A \in \mathscr{M}Y$ is H-equivariant and $B \in \mathscr{M}Y$ is a subquotient of A, then B is again H-equivariant.*

(Apply (1.8.4) to $X = H \times Y, f = \pi$, $K = A$, $K' = \tilde{m}B$. We see that there exists $C \in \mathscr{M}Y$ such that $\tilde{m}B = \tilde{\pi}C$. Restricting this equality to $\{e\} \times Y \subset H \times Y$ we get $B = C$. Hence $\tilde{m}B = \tilde{\pi}B$.)

(1.9.2) Let $f: X \to Y$ be an H-equivariant morphism, with respect to actions of H on X and Y. If $K \in \mathscr{M}X$ is H-equivariant, then ${}^pH^if_!K$ is H-equivariant for all i. If $K' \in \mathscr{M}Y$ is H-equivariant, then ${}^pH^if^*K'$ is H-equivariant for all i. (The verification is left to the reader.)

(1.9.3) Assume that $f: X \to Y$ is as in (1.9.2), and that H acts freely on X and trivially on Y. Assume furthermore, that for each $y \in Y$, there is an open neighborhood $U \subset Y$, $(U \ni y)$, and an H-equivariant isomorphism $f^{-1}(U) \xrightarrow{\sim}{}^{i} H \times U$ (H acts on $H \times U$ by $h: (h', u) \to (hh', u)$) such that $pr_2 \circ i = f: f^{-1}U \to U$. Then the following conditions for $K \in \mathscr{M}X$ are equivalent.

(a) K is H-equivariant,

(b) K is isomorphic to $\tilde{f}(K_1)$, for some $K_1 \in \mathscr{M}Y$.

The implication (b) $\Rightarrow$ (a) is trivial, (see (1.9.2). Assume now that K is H-equivariant. Let $d = \dim H$. According to [1, 4.2.6], (b) is equivalent to the statement that the canonical map $K \to \tilde{f}({}^pH^{-d}f_* K)$ is an isomorphism. For this, we may assume that $X = H \times Y, f = pr_2$, and H acts on X by left translation on the first factor. Let $m, \pi: H \times H \times Y \to H \times Y$ be defined by $m(h, h', y) = (hh', y)$, $\pi(h, h', y) = (h', y)$ and let $i: H \times Y \to H \times Y \times Y$ be defined by $i(h, y) = (h, e, y)$. By our assumption, $m^*K \approx \pi^*K$, hence $i^*m^*K \approx i^*\pi^*K$ or equivalently, $K \approx f^*j^*K$, where $j: Y \to H \times Y$ is defined by $j(y) = (e, y)$. Let $K_1 = j^*K[-d] \in \mathscr{D}Y$. Then $K = \tilde{f}K_1$. It remains to show that $K_1 \in \mathscr{M}Y$. This follows from (1.8.1), since we know that $\tilde{f}K_1 \in \mathscr{M}X$.

1.10. Let X be an algebraic variety, let X' be an open subset of X and let X'' be the complement of X' in X. Let $j': X' \hookrightarrow X$, $j'': X'' \hookrightarrow X$ be the natural inclusions. For any $K \in \mathscr{D}X$, there is a canonical distinguished triangle in $\mathscr{D}X$: $(j'_! j'^*K, K, j''_! j''^*K)$. Hence, if $f: X \to Y$ is a morphism, then we have a canonical distinguished triangle $(f_!j'_! j^*K, f_!K, f_!j''_! j''^*K)$ in $\mathscr{D}Y$.

1.11. Let $n \geqslant 1$ be an integer invertible in k. Let $\mu_n = \{x \in k^* \mid x^n = 1\}$.

Consider the principal fibration $\rho_n: k^* \to k^*$ $(x \to x^n)$ with group μ_n. The finite group μ_n acts naturally on the direct image local system $(\rho_n)_* \bar{\mathbf{Q}}_l$; we denote by $\mathscr{E}_{n,\psi}$ the summand of $(\rho_n)_* \bar{\mathbf{Q}}_l$ on which μ_n acts according to the character $\psi: \mu_n \to \bar{\mathbf{Q}}_l$. Then $\mathscr{E}_{n,\psi}$ is a $\bar{\mathbf{Q}}_l$-local system of rank 1 on k^*. The following result is well known:

(1.11.1) If $m \geqslant 1$ is an integer not divisible by n and if ψ is injective, then $H_c^i(k^*, \mathscr{E}_{n,\psi}^{\otimes m}) = 0$ for all i.

1.12. A complex $K \in \mathscr{D}X$ is said to be split if K is isomorphic in $\mathscr{D}X$ to a direct sum $\bigoplus_i {}^pH^iK[-i]$.

If K is split, then $K[j]$ is split for any j. If $K' \in \mathscr{D}X$ is a direct summand of $K \in \mathscr{D}X$ with K split, then K' is split.

A complex $K \in \mathscr{D}X$ is said to be *semisimple* if it is split and each ${}^pH^iK$ is a semisimple object of $\mathscr{M}X$. If K is semisimple and $K' \in \mathscr{D}X$ is a direct summand of K, then K' is semisimple.

2. Definition of Character Sheaves.

2.1. Let G be a connected reductive algebraic group over k. We fix a Borel subgroup $B \subset G$ with unipotent radical U and a maximal torus $T \subset B$.

Let $R \subset \operatorname{Hom}(T, k^*)$ be the set of roots and $\check{R} \subset \operatorname{Hom}(k^*, T)$ the set of coroots; the canonical bijection $R \leftrightarrow \check{R}$ is denoted $\alpha \leftrightarrow \check{\alpha}$.

Let R^+ be the set of positive roots determined by B and let $R^- = R - R^+$.

Let $W = N_G(T)/T$ be the Weyl group. An element $w \in W$ may be regarded as an automorphism $w: T \to T$: $w(t) = \dot{w}t\dot{w}^{-1}$ $(t \in T)$. Here $\dot{w} \in N_G(T)$ is a representative for w in $N(T)$. Let S be the set of simple reflections in W (defined by R^+) and let $l: W \to \mathbb{N}$ be the corresponding length function.

2.2. Let $\mathscr{S}(T)$ be the set of isomorphism classes of $\bar{\mathbf{Q}}_l$-local systems of rank 1 on T which are of the form $\lambda^*(\mathscr{E}_{n,\psi})$, (see 1.12), for some character $\lambda \in \operatorname{Hom}(T, k^*)$, some integer $n \geqslant 1$ invertible in k, and some imbedding $\psi: \mu_n \hookrightarrow \bar{\mathbf{Q}}_l^*$; tensor product makes $\mathscr{S}(T)$ an abelian group.

We may (and shall) assume that ψ is the restriction to μ_n of a fixed injective homomorphism $\tilde{\psi}$: {group of roots of 1 in k^*} $\hookrightarrow \bar{\mathbf{Q}}_l^*$, which is independent of λ and n.

The choice of $\tilde{\psi}$ gives rise to a group isomorphisms $\lambda \otimes (1/n) \to \lambda^*\mathscr{E}_{n,\psi}$:

(2.2.1) $\operatorname{Hom}(T, k^*) \otimes (\mathbf{Q}'/\mathbb{Z}) \simeq \mathscr{S}(T)$, where $\mathbf{Q}' = \{m/n \in \mathbf{Q} \mid m \in \mathbb{Z}, n \in \mathbb{Z}, n \geqslant 1 \text{ invertible in } k\}$.

The Weyl group W operates on $\mathscr{S}(T)$ by $w: \mathscr{L} \to (w^{-1})^*\mathscr{L}$, where $(w^{-1})^*$ denotes inverse image under $w^{-1}: T \to T$; it also operates on $\operatorname{Hom}(T, k^*)$ by $w(\lambda)(t) = \lambda(w^{-1}(t))$, $t \in T$, $\lambda \in \operatorname{Hom}(T, k^*)$. These actions are compatible with (2.2.1).

For $\mathscr{L} \in \mathscr{S}(T)$, we set

$$W'_{\mathscr{L}} = \{w \in W \mid (w^{-1})^* \mathscr{L} = \mathscr{L}\}.$$

(2.2.2) *The following conditions on $w \in W$ and $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\psi}) \in \mathscr{S}(T)$ are equivalent*:

(a) *The local system $\mathscr{L}$ is T-equivariant for the action of T on T given by $t_0 : t \to w^{-1}(t_0)\, tt_0^{-1}$*

(b) *There exists a character $\lambda_1 \in \mathrm{Hom}(T, k^*)$ such that $w(\lambda) = \lambda_1^n \lambda$*

(c) $w \in W'_{\mathscr{L}}$.

2.3. For $\mathscr{L} \in \mathscr{S}(T)$ we define

$$R_{\mathscr{L}} = \{\alpha \in R \mid r_\alpha \in W'_{\mathscr{L}}\} = \{\alpha \in R \mid \langle \check{\alpha}, \lambda \rangle \equiv 0 \ (\mathrm{mod}\ n)\}$$

where r_α is the reflection in W corresponding to α, and $\langle\ ,\ \rangle$ is the natural pairing $\mathrm{Hom}(k^*, T) \times \mathrm{Hom}(T, k^*) \to \mathbb{Z}$. We define

$$W_{\mathscr{L}} = \text{subgroup of } W \text{ generated by the } r_\alpha,\ \alpha \in R_{\mathscr{L}}.$$

Then $R_{\mathscr{L}}$ is a root system with Weyl group $W_{\mathscr{L}}$. The set $R^+_{\mathscr{L}} = R_{\mathscr{L}} \cap R^+$ is a set of positive roots for $R_{\mathscr{L}}$; let $S_{\mathscr{L}}$ be the corresponding set of simple reflections for $W_{\mathscr{L}}$. (The set $S_{\mathscr{L}}$ is not in general contained in the set S.)

2.4. Let $\mathscr{B}$ be the variety of all Borel subgroups of G. For each $w \in W$, we consider the subvariety $O(w)$ of $\mathscr{B} \times \mathscr{B}$ defined by $O(w) = \{(B', B'') \in \mathscr{B} \times \mathscr{B} \mid \exists g \in G\colon gB'g^{-1} = B,\ gB''g^{-1} = \dot{w}B\dot{w}^{-1}\}$. We define a morphism

$$\pi_w : Y_w \to G$$

as follows:

$$Y_w = \{(g, B') \in G \times \mathscr{B} \mid (B', gB'g^{-1}) \in O(w)\}, \qquad \pi_w(g, B') = g.$$

Let $pr_{\dot{w}} : BwB \to T$ be the map defined by $pr_{\dot{w}}(u\dot{w}tu') = t$ $(u, u' \in U, t \in T)$. Let $\dot{Y}_w = \{(g, hU) \in G \times (G/U) \mid h^{-1}gh \in BwB\}$. The map $\dot{Y}_w \to T$ given by $(g, hU) \to pr_{\dot{w}}(h^{-1}gh)$ is T-equivariant with respect to the action $t_0 : (g, hU) \to (g, ht_0^{-1}(U))$ (of T on $\dot{Y}_w$) and $t_0 : t \to (\dot{w}^{-1}t_0\dot{w})\, tt_0^{-1}$ (of T on T).

Hence, if $\mathscr{L} \in \mathscr{S}(T)$ and $w \in W'_{\mathscr{L}}$, then the inverse image $\dot{\mathscr{L}}$ of $\mathscr{L}$ under $\dot{Y}_w \to T$ is T-equivariant. The map $\dot{Y}_w \to Y_w$ $((g, hU) \to (g, hBh^{-1}))$ is a principal fibration with group T, (the action of T on $\dot{Y}_w$ has been described above). It follows that there is a unique $\bar{\mathbb{Q}}_l$-local system of rank 1, $\tilde{\mathscr{L}}$ on Y_w whose inverse image under $\dot{Y}_w \to Y_w$ is $\dot{\mathscr{L}}$. It is easy to see that the isomorphism class of $\tilde{\mathscr{L}}$ is independent of the choice of representative $\dot{w}$.

We shall set for $w \in W'_{\mathscr{L}}$:

$$K_w^{\mathscr{L}} = (\pi_w)_! \tilde{\mathscr{L}} \in \mathscr{D}G.$$

2.5. More generally, let $\mathbf{w} = (w_1, w_2, ..., w_r)$ be a sequence in W and let $w = w_1 w_2 \cdots w_r$.

We define a morphism

$$\pi_{\mathbf{w}} : Y_{\mathbf{w}} \to G$$

as follows:

$$Y_{\mathbf{w}} = \{(g, B_0, B_1, ..., B_r) \in G \times \mathscr{B} \times \mathscr{B} \times \cdots \times \mathscr{B} \mid (B_{i-1}, B_i) \in O(w_i)\ (1 \leqslant i \leqslant r), B_r = gB_0 g^{-1}\},$$

$$\pi_{\mathbf{w}}(g, B_0, B_1, ..., B_r) = g.$$

Let $\dot{Y}_{\mathbf{w}} = (g, h_0 U, h_1 B, ..., h_r B)$: $h_{i-1}^{-1} h_i \in B w_i B$ $(1 \leqslant i \leqslant r)$, $h_r^{-1} g h_0 \in B\}$. Define a map $\dot{Y}_{\mathbf{w}} \to T$ by $(g, h_0 U, h_1 U, ..., h_r U) \to \dot{w}^{-1} n_1 n_2 \cdots n_r \tau$, where $n_i \in N_G(T)$ are defined by $h_{i-1}^{-1} h_i \in U n_i U$ and $\tau \in T$ is defined by $h_r^{-1} g h_0 \in \tau U$. This map is T-equivariant with respect to the action $t_0 : (g, h_0 U, h_1 B, ..., h_r B) \to (g, h_0 t_0^{-1} U, h_1 B, ..., h_r B)$ (of T on $\dot{Y}_{\mathbf{w}}$) and $t_0 : t \to (\dot{w}^{-1} t_0 \dot{w})\, t t_0^{-1}$ (of T on T). Hence, if $\mathscr{L} \in \mathscr{S}(T)$ and $w \in W'_{\mathscr{L}}$, then the inverse image $\dot{\mathscr{L}}$ of $\mathscr{L}$ under $\dot{Y}_{\mathbf{w}} \to T$ is T-equivariant. The map $\dot{Y}_{\mathbf{w}} \to Y_{\mathbf{w}}$ given by $(g, h_0 U, h_1 B, ..., h_r B) \to (g, h_0 B h_0^{-1}, h_1 B h_1^{-1}, ..., h_r B h_r^{-1})$ is a principal fibration with group T. It follows that there is a unique $\bar{\mathbb{Q}}_l$-local system of rank 1, $\tilde{\mathscr{L}}$ on $Y_{\mathbf{w}}$ whose inverse image under $\dot{Y}_{\mathbf{w}} \to Y_{\mathbf{w}}$ is $\dot{\mathscr{L}}$. We shall set

$$K_{\mathbf{w}}^{\mathscr{L}} = (\pi_{\mathbf{w}})_! \tilde{\mathscr{L}} \in \mathscr{D}G.$$

(This is defined only when $w_1 w_2 \cdots w_r \in W'_{\mathscr{L}}$.)

(2.5.1) When $\mathbf{w}$ reduces to a single element w, the variety $Y_{\mathbf{w}}$ may be identified with the variety Y_w in 2.4: $(g, B_0, B_1) \in Y_{\mathbf{w}}$ corresponds to $(g, B_0) \in Y_w$. This is compatible with the maps $\pi_{\mathbf{w}}$, π_w and with the local systems $\tilde{\mathscr{L}}$ (if $w \in W'_{\mathscr{L}}$). Hence $K_{\mathbf{w}}^{\mathscr{L}} = K_w^{\mathscr{L}}$.

(2.5.2) In general, $Y_{\mathbf{w}}$ is smooth and connected.

An equivalent statement is $\{(g, x_0, x_1, ..., x_r) \in G \times G \times \cdots \times G \mid x_{i-1}^{-1} x_i \in B w_i B\ (1 \leqslant i \leqslant r),\ x_r^{-1} g x_0 \in B\}$ is smooth and connected. By the substitution $b = x_r^{-1} g x_0$, $x_{i-1}^{-1} x_i = y_i$ $(1 \leqslant i \leqslant r)$, we are reduced to showing that $\{(b, x_0, y_1, ..., y_r) \in B \times G \times \cdots \times G \mid y_i \in B w_i B\ (1 \leqslant i \leqslant r)\}$ is smooth and connected, and this is clear.

2.6. For any sequence $\mathbf{s} = (s_1, s_2, \ldots, s_r)$ in $S \cup \{e\}$ (e = neutral element of W) we define a *proper* morphism

$$\bar{\pi}_{\mathbf{s}}: \bar{Y}_{\mathbf{s}} \to G$$

as follows:

$$\bar{Y}_{\mathbf{s}} = \{(g, B_0, B_1, \ldots, B_r) \in G \times \mathscr{B} \times \mathscr{B} \times \cdots \times \mathscr{B} \mid (B_{i-1}, B_i) \in \overline{O(s_i)}\ (1 \leqslant i \leqslant r),\ B_r = gB_0g^{-1}\}$$

$$\bar{\pi}_{\mathbf{s}}(g, B_0, B_1, \ldots, B_r) = g.$$

Here, $\overline{O(s_i)}$ denotes the Zariski closure of $O(s_i)$ in $\mathscr{B} \times \mathscr{B}$. It is $O(s_i) \cup O(e)$ if $s_i \in S$, and it is $O(e)$ if $s_i = e$.

Let $J_0 = \{j \in [1, r] \mid s_j \in S\}$. For such subset $J \subset J_0$, we consider the r element sequence $\mathbf{s}_J$ in $S \cup \{e\}$ whose ith term is s_i if $i \notin J$ and e if $i \in J$. Then $Y_{\mathbf{s}_J}$ (see 2.6) may be identified with the locally closed subvariety of $\bar{Y}_{\mathbf{s}}$ defined by the conditions $B_{i-1} = B_i$ if $i \in J$, $(B_{i-1}, B_i) \in O(s_i)$ if $i \in [1, r] - J$. The sets $Y_{\mathbf{s}_J}$ $(J \subset J_0)$ form a partition of $\bar{Y}_{\mathbf{s}}$. We have $\mathbf{s}_\varnothing = \mathbf{s}$ and the corresponding piece $Y_{\mathbf{s}_\varnothing} = Y_{\mathbf{s}}$ is open dense in $\bar{Y}_{\mathbf{s}}$. For each $j \in J_0$, we write $\mathbf{s}_j$ instead of $\mathbf{s}_{\{j\}}$.

LEMMA 2.7. *$\bar{Y}_{\mathbf{s}}$ is smooth, connected. The closures of $Y_{\mathbf{s}_j}$ (for various $j \in J_0$) are smooth divisors on $\bar{Y}_{\mathbf{s}}$ with normal crossings.*

Proof. An equivalent statement is: the variety

$$\{(g, x_0, x_1, \ldots, x_r) \in G \times G \times \cdots \times G \mid x_{i-1}^{-1}x_i \in \overline{Bs_iB}\ (1 \leqslant i \leqslant r), x_r^{-1}gx_0 \in B\}$$

is smooth and connected and its subvarieties

$$\{(g, x_0, x_1, \ldots, x_r) \in G \times G \times \cdots \times G \mid x_{i-1}^{-1}x_i \in \overline{Bs_iB}\ (1 \leqslant i \leqslant r, i \neq j_0), x_{j_0-1}^{-1}x_{j_0} \in B, x_r^{-1}gx_0 \in B\}$$

$(j_0 \in J_0)$, are smooth divisors with normal crossings. By the substitution $b = x_r^{-1}gx_0$, $x_{i-1}^{-1}x_i = y_i$ $(1 \leqslant i \leqslant r)$, we are reduced to the following statement: the variety

$$\{(b, x_0, y_1, y_2, \ldots, y_r) \in B \times G \times \cdots \times G \mid y_i \in \overline{Bs_iB}\ (1 \leqslant i \leqslant r)\}$$

is smooth and connected and its subvarieties

$$\{(b, x_0, y_1, y_2, \ldots, y_r) \in B \times G \times \cdots \times G \mid y_i \in \overline{Bs_iB}\ (1 \leqslant i \leqslant r, i \neq j_0), y_{j_0} \in B\}$$

($j_0 \in J_0$) are smooth divisors with normal crossings. This is, in turn, a consequence of the following obvious statement: if $s_i \in S$, then $\overline{Bs_iB} = (Bs_iB) \cup B$ is smooth and connected and B is a smooth divisor on it.

2.8. Assume now that $\mathbf{s}$ is such that $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$. By 1.6 and 2.7, there is a well-defined (constructible) $\bar{\mathbf{Q}}_l$-sheaf on $\bar{\mathscr{L}}$ on $\bar{Y}_{\mathbf{s}}$ such that $\bar{\mathscr{L}} = \mathrm{IC}(\bar{Y}_{\mathbf{s}}, \tilde{\mathscr{L}})$ in $\mathscr{D}\bar{Y}_{\mathbf{s}}$; (here we regard $\tilde{\mathscr{L}}$ as a $\bar{\mathbf{Q}}_l$-local system on $Y_{\mathbf{s}}$ as in 2.5, and we identify $Y_{\mathbf{s}}$ with an open dense subset of $\bar{Y}_{\mathbf{s}}$, as in 2.6).

We shall set

$$\bar{K}_{\mathbf{s}}^{\mathscr{L}} = (\bar{\pi}_{\mathbf{s}})_! \bar{\mathscr{L}} \in \mathscr{D}G.$$

(Here $\bar{\mathscr{L}}$ is regarded as an object in $\mathscr{D}\bar{Y}_{\mathbf{s}}$, concentrated in degree 0.)

We can now state the following result.

PROPOSITION 2.9. *Let $\mathscr{L} \in \mathscr{S}(T)$ and let A be an irreducible perverse sheaf on G. The following conditions on A are equivalent*:

(a) *A is a constituent of ${}^pH^i(K_w^{\mathscr{L}})$ for some $w \in W'_{\mathscr{L}}$ and some $i \in \mathbb{Z}$.*

(b) *A is a constituent of ${}^pH^i(K_{\mathbf{w}}^{\mathscr{L}})$ for some sequence $\mathbf{w} = (w_1, w_2,..., w_r)$ in W such that $w_1 w_2 \cdots w_r \in W'_{\mathscr{L}}$ and for some $i \in \mathbb{Z}$.*

(c) *A is a constituent of ${}^pH^i(K_{\mathbf{s}}^{\mathscr{L}})$ for some sequence $\mathbf{s} = (s_1, s_2,..., s_r)$ in $S \cup \{e\}$ such that $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$ and for some $i \in \mathbb{Z}$.*

(d) *A is a constituent of ${}^pH^i(\bar{K}_{\mathbf{s}}^{\mathscr{L}})$ for some sequence $\mathbf{s} = (s_1, s_2,..., s_r)$ in $S \cup \{e\}$ such that $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$ and for some $i \in \mathbb{Z}$.*

The proof will be given in 2.11–2.16.

2.10. DEFINITION. For $\mathscr{L} \in \mathscr{S}(T)$, we denote by $\hat{G}_{\mathscr{L}}$ the set of isomorphism classes of irreducible perverse sheaves A on G which satisfy the equivalent conditions 2.9(a)–(d) with respect to $\mathscr{L}$.

A character sheaf on G is an irreducible perverse sheaf on G, which is in $\hat{G}_{\mathscr{L}}$ for some $\mathscr{L} \in \mathscr{S}(T)$. The set of isomorphism classes of character sheaves on G is denoted by $\hat{G}$.

Note that the character sheaves of the torus T are the perverse sheaves $\mathscr{L}[d]$ ($\mathscr{L} \in \mathscr{S}(T)$), where $d = \dim T$.

2.11. We now begin the proof of 2.9. The implication (a) $\Rightarrow$ (b) follows from (2.5.1). The implication (c) $\Rightarrow$ (*b*) is trivial. We now prove the implication (b) $\Rightarrow$ (c). Let $\mathbf{w} = (w_1,..., w_r)$ be a sequence in W, and let, for some i ($1 \leqslant i \leqslant r$), w_i', w_i'' be elements of W such that $w_i = w_i' w_i''$ and $l(w_i) = l(w_i') + l(w_i'')$. Let $\tilde{\mathbf{w}} = (w_1,..., w_{i-1}, w_i', w_i'', w_{i+1},..., w_r)$. The map

$(g, B_0, B_1, ..., B_{i-1}, B_i, B_{i+1}, ..., B_{r+1}) \to (g, B_0, B_1, ..., B_{i-1}, B_{i+1}, ..., B_{r+1})$ defines an isomorphism $Y_{\tilde{\mathbf{w}}} \simeq Y_{\mathbf{w}}$. It is compatible with the maps $\pi_{\tilde{\mathbf{w}}}$, $\pi_{\mathbf{w}}$ and with the local systems $\tilde{\mathscr{L}}$ defined on $Y_{\mathbf{w}}$, $Y_{\tilde{\mathbf{w}}}$ in terms of $\mathscr{L}$ as in 2.5 (assuming $w_1 w_2 \cdots w_r \in W'_{\mathscr{L}}$). Hence

$$K^{\mathscr{L}}_{\tilde{\mathbf{w}}} = K^{\mathscr{L}}_{\mathbf{w}}.$$

Applying this repeatedly, we see that $K^{\mathscr{L}}_{\mathbf{w}}$ is equal to $K^{\mathscr{L}}_{\mathbf{s}}$ for some sequence $\mathbf{s}$ in $S \cup \{e\}$. This proves the implication (b) $\Rightarrow$ (c).

For the proof of the equivalence (c) $\Leftrightarrow$ (d) we shall need the following result.

LEMMA 2.12. *Let* $\mathbf{s} = (s_1, ..., s_r)$ *be a sequence in* $S \cup \{e\}$ *such that* $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$. *Let* $I_{\mathbf{s}} = \{j \in [1, r] \mid s_j \in S, s_r s_{r-1} \cdots s_j \cdots s_{r-1} s_r \in W_{\mathscr{L}}\}$.

(a) $\bar{\mathscr{L}}$ *is a* $\bar{\mathbf{Q}}_l$-*local system of rank* 1 *on the open subset* $\bigcup_{j \in I_{\mathbf{s}}} Y_{\mathbf{s}_j}$ *of* $\bar{Y}_{\mathbf{s}}$ (*see* 2.6) *and is zero on its complement.*

(b) *If* $J \subset I_{\mathbf{s}}$, *the restriction of* $\bar{\mathscr{L}}$ *to* $Y_{\mathbf{s}_J}$ *is isomorphic to the local system* $\tilde{\mathscr{L}}$ *on* $Y_{\mathbf{s}_J}$ (*defined in* 2.5 *for* $\mathbf{s}_J$ *instead of* $\mathbf{s}$); *note that, for* $J \subset I_{\mathbf{s}}$, *the product of the elements in the sequence* $\mathbf{s}_J$ *belongs to* $W'_{\mathscr{L}}$, *hence* $\tilde{\mathscr{L}}$ *is defined on* $Y_{\mathbf{s}_J}$.

Proof. We first prove (a). Let $j \in [1, r]$ be such that $s_j \in S$. Then $Y_{\mathbf{s}_j}$ is a smooth divisor in the smooth variety $Y_{\mathbf{s}} \cup Y_{\mathbf{s}_j}$ (see 2.7). By a computation which takes place essentially in SL_2, we see that the local monodromy of the local system $\mathscr{L}$ (on $Y_{\mathbf{s}}$) along the divisor $Y_{\mathbf{s}_j}$ is the same as the monodromy of the local system $\mathscr{E}^{\otimes m}_{n,\psi}$ on k^* at 0, where $m = \langle \check{\beta}_j, \lambda \rangle$, β_j is the root corresponding to the reflection $s_r s_{r-1} \cdots s_j \cdots s_{r-1} s_r$ and $\mathscr{L} = \lambda^* \mathscr{E}_{n,\psi}$, as in 2.2. Hence, this local monodromy is trivial if and only if $\langle \check{\beta}_j, \lambda \rangle \equiv 0 \pmod n$, i.e., if $s_r s_{r-1} \cdots s_j \cdots s_{r-1} s_r \in W_{\mathscr{L}}$. Hence (a) follows from 1.6.

To prove (b), we may assume that J consists of a single element $j \in I_{\mathbf{s}}$. Then $\mathbf{s}_j$ has the same entries as $\mathbf{s}$ except for the jth entry which is e for $\mathbf{s}_j$ and s_j for $\mathbf{s}$.

Let $\tilde{G} \to G$ be a surjective homomorphism of algebraic groups whose kernel is a central torus in $\tilde{G}$ and such that $\tilde{G}$ is a reductive group with simply connected derived group. The varieties $Y_{\mathbf{s}}$, $\bar{Y}_{\mathbf{s}}$ for $\tilde{G}$ are locally trivial fibrations over the corresponding varieties for G with connected smooth fibres (isomorphic to a torus). Hence if (b) is true for $\tilde{G}$, then it is also true for G. Thus, we may assume that G has simply connected derived group.

Let $\dot{w}_1$, $\dot{s}_j$, $\dot{w}_2$ be representatives in $N_G(T)$ for $s_1 s_2 \cdots s_{j-1}$, s_j, $s_{j+1} \cdots s_{r-1} s_r$ respectively, and let $\dot{w} = \dot{w}_1 \dot{s}_j \dot{w}_2$. We shall assume (as we may) that $\dot{s}_j$ is a product of three unipotent elements in $(Bs_jB)\cup B$. Consider the smooth variety

$$Z=\{(b,x_0,y_1,\ldots,y_r)\in B\times G\times\cdots\times G\mid y_i\in Bs_iB$$
$$(1\leqslant i\leqslant r, i\neq j), y_j\in(Bs_jB)\cup B\}$$

and the smooth divisor $D\subset Z$ defined by the equation $y_j\in B$. Let $f\colon Z-D\to k^*, f'\colon D\to k^*$ be the maps given by

$$f(b,x_0,y_1,\ldots,y_r)=\lambda(\dot{w}_2^{-1}\dot{s}_j^{-1}\dot{w}_1^{-1}n_1n_2\cdots n_r\tau),$$
$$f'(b,x_0,y_1,\ldots,y_r)=\lambda(\dot{w}_2^{-1}\dot{w}_1^{-1}n_1n_2\cdots n_r\tau),$$

where $n_i\in N_G(T)$ are defined by $y_i\in Un_iU$ and $\tau\in T$ is defined by $b\in\tau U$. As in the proof of 2.7, we are reduced to proving the following statement

(2.12.1) If $\langle\check{\beta_j},\lambda\rangle\equiv 0\ (\mathrm{mod}\ n)$, then there exists a local system on Z whose restriction to $Z-D$ is $f^*\mathscr{E}_{n,\psi}$ and whose restriction to D is $f'^*\mathscr{E}_{n,\psi}$.

We can write $\langle\check{\beta_n},\lambda\rangle=nn_1$, where n_1 is an integer. Since G has simply connected derived group, there exists $\lambda'\in\mathrm{Hom}(T,k^*)$ such that $\langle\check{\beta_j},\lambda'\rangle=n_1$. Then $\langle\check{\beta_j},(\lambda')^{-n}\lambda\rangle=0$. Replacing λ by $(\lambda')^{-n}\lambda$ does not change $f^*\mathscr{E}_{n,\psi}, f'^*\mathscr{E}_{n,\psi}$. Hence, we may assume in (2.12.1) that $\langle\check{\beta_j},\lambda\rangle=0$. In this case, there is a unique homomorphism of algebraic groups $\gamma\colon(Bs_jB)\cup B\to k^*$ such that

$$\gamma(t)=\lambda(\dot{w}_2^{-1}t\dot{w}_2)\qquad\text{for all}\quad t\in T.$$

Since $\dot{s}_j$ is a product of unipotent elements in $(Bs_jB)\cup B$, we must have $\gamma(\dot{s}_j)=1$. We define a morphism $\tilde{f}\colon Z\to k^*$ by

$$\tilde{f}(b,x_0,y_1,\ldots,y_r)=\gamma(\dot{s}_j^{-1}\dot{w}_1^{-1}(n_1n_2\cdots n_{j-1})\,y_j(n_{j+1}\cdots n_r\tau)\,\dot{w}_2^{-1}),$$

where $n_i\in N_G(T)$ are defined by $y_i\in Un_iU$ $(i\neq j)$, and $\tau\in T$ is defined by $b\in\tau U$.

We show that $f=\tilde{f}\mid Z-D, f'=\tilde{f}\mid D$. If $y_j\in Bs_jB$, we write $y_j\in Un_jU$, $n\in N(T)$ and we have

$$\begin{aligned}\tilde{f}(b,x_0,y_1,\ldots,y_r)&=\gamma(\dot{s}_j^{-1}\dot{w}_1^{-1}(n_1n_2\cdots n_{j-1})\,n_j(n_{j+1}\cdots n_r\tau)\,\dot{w}_2^{-1})\\&=\lambda(\dot{w}_2^{-1}\dot{s}_j^{-1}\dot{w}_1^{-1}n_1n_2\cdots n_r)\\&=f(b,x_0,y_1,\ldots,y_r).\end{aligned}$$

If $y_j\in B$, we write $y_j\in n_jU$, $n_j\in T$, and we have

$$\begin{aligned}\tilde{f}(b,x_0,y_1,\ldots,y_r)&=\gamma(\dot{w}_1^{-1}(n_1n_2\cdots n_{j-1})\,n_j(n_{j+1}\cdots n_r)\,\dot{w}_2^{-1})\\&=\lambda(\dot{w}_2^{-1}\dot{w}_1^{-1}n_1n_2\cdots n_r\tau)\\&=f'(b,x_0,y_1,\ldots,y_r).\end{aligned}$$

It follows that the local system $\tilde{f}^*\mathscr{E}_{n,\psi}$ on Z has the property required in (2.12.1). The lemma is proved.

2.13. Let $\mathbf{s}$, $\mathscr{L}$ be as in 2.9(d). Consider the sequence of closed subsets $Z^{(i)} \subset \bar{Y}_{\mathbf{s}}$ defined by

$$Z^{(i)} = \bigcup_{\substack{J \subset J_0 \\ |J| \geqslant i}} Y_{\mathbf{s}_J} \qquad (i \in \mathbb{Z})$$

(see 2.6). We have $Z^{(i+1)} \subset Z^{(i)}$. If $\phi^{(i)}: Z^{(i)} \hookrightarrow \bar{Y}_{\mathbf{s}}$, $\psi^{(i)}: Z^{(i)} - Z^{(i+1)} \hookrightarrow \bar{Y}_{\mathbf{s}}$ are the inclusion maps, we have (by 1.10) a natural distinguished triangle in G:

$$((\bar{\pi}_{\mathbf{s}})_! \psi_!^{(i)}(\psi^{(i)})^* \bar{\mathscr{L}}, (\bar{\pi}_{\mathbf{s}})_! \phi_!^{(i)}(\phi^{(i)})^* \bar{\mathscr{L}}, (\bar{\pi}_s)_! \phi_!^{(i+1)}(\phi^{(i+1)})^* \bar{\mathscr{L}}).$$

It gives rise to a long exact sequence in $\mathscr{M}G$ (for each i):

$$(2.13.1) \quad \cdots \to {}^pH^{j-1}((\bar{\pi}_{\mathbf{s}})_! \phi_!^{(i+1)}(\phi^{(i+1)})^* \bar{\mathscr{L}}) \to \bigoplus_{\substack{J \subset J_{\mathbf{s}} \\ |J| = i}} {}^pH^j(K_{\mathbf{s}_J}^{\mathscr{L}})$$
$$\to {}^pH^j((\bar{\pi}_{\mathbf{s}})_! \phi_!^{(i)}(\phi^{(i)})^* \bar{\mathscr{L}}) \to {}^pH^j((\bar{\pi}_s)_! \phi_!^{(i+1)}(\phi^{(i+1)})^* \bar{\mathscr{L}})$$
$$\to \bigoplus_{\substack{J \subset J_{\mathbf{s}} \\ |J| = i}} {}^pH^{j+1}(K_{\mathbf{s}_J}^{\mathscr{L}}) \to \cdots.$$

Here we have used the isomorphism

$$(\bar{\pi}_s)_! \psi_!^{(i)}(\psi^{(i)})^* \bar{\mathscr{L}} = \bigoplus_{\substack{J \subset I_{\mathbf{s}} \\ |J| = i}} K_{\mathbf{s}_J}^{\mathscr{L}}$$

which follows from Lemma 2.12. Note that

$$(2.13.2) \quad {}^pH^j((\bar{\pi}_s)_! \phi_!^{(i)}(\phi^{(i)})^* \bar{\mathscr{L}}) = \begin{cases} {}^pH^j(\bar{K}_{\mathbf{s}}) & \text{for } i \leqslant 0, \\ 0 & \text{for } i > |I_{\mathbf{s}}|. \end{cases}$$

2.14. We now prove the equivalence of (c) and (d) in 2.9. Let A be as in 2.9. For a sequence $\mathbf{s} = (s_1, \ldots, s_r)$ in $S \cup \{e\}$, we denote by $m(\mathbf{s})$ the number of $i \in [1, r]$ such that $s_i \in S$. If $m(\mathbf{s}) = 0$, then $Y_{\mathbf{s}} = \bar{Y}_{\mathbf{s}}$ and $K_{\mathbf{s}}^{\mathscr{L}} = \bar{K}_{\mathbf{s}}^{\mathscr{L}}$ (if defined) hence A is a constituent of ${}^pH^i(K_{\mathbf{s}}^{\mathscr{L}})$, if and only if it is a constituent of ${}^pH^i(\bar{K}_{\mathbf{s}}^{\mathscr{L}})$. It is enough to prove the following statement.

(2.14.1) Assume that $\mathbf{s}$ satisfies $m(\mathbf{s}) = m \geqslant 1$, $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$, and that for any sequence $\mathbf{s}'$ in $S \cup \{e\}$, with product in $W'_{\mathscr{L}}$ and with $m(\mathbf{s}') < m$, and any integer j, A is not a constituent of ${}^pH^j(K_{\mathbf{s}'}^{\mathscr{L}})$. Then, for any j, A is a constituent of ${}^pH^j(\bar{K}_{\mathbf{s}}^{\mathscr{L}})$ if and only if it is a constituent of ${}^pH^j(K_{\mathbf{s}'}^{\mathscr{L}})$.

Using our hypothesis and (2.13.1) for $i > 0$, we see that for any $i > 0$, and any j, we have:

> A is a constituent of ${}^pH^j((\bar{\pi}_s)_!\phi_!^{(i)}(\phi^{(i)})^*\bar{\mathscr{L}})$ if and only if A is a constituent of ${}^pH^j((\bar{\pi}_s)_!\phi_!^{(i+1)}(\phi^{(i+1)})^*\bar{\mathscr{L}})$.

Applying this repeatedly for $i = |I_s|, |I_s| - 1, ..., 1$ and using (2.13.2), we see that for any j, A is not a constituent of ${}^pH^j((\bar{\pi}_s)_!\phi_!^{(1)}(\phi^{(1)})^*\bar{\mathscr{L}})$.

This, together with (2.13.1) for $i = 0$, implies that A is a constituent of ${}^pH^j(K_s^{\mathscr{L}})$, if and only if A is a constituent of ${}^pH^j((\pi_s)_!\phi_!^{(0)}(\phi^{(0)})^*\tilde{\mathscr{L}})$ which, by (2.13.2) is the same as ${}^pH^j(\bar{K}_s^{\mathscr{L}})$. Thus, (2.14.1) and hence the equivalence (c) $\Leftrightarrow$ (d) in 2.9, are verified.

2.15. Let $\mathbf{s} = (s_1, s_2, ..., s_r)$ be a sequence in $S \cup \{e\}$ such that $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$. Assume that, for some h $(2 \leqslant h \leqslant r)$, we have $s_{h-1} = s_h \in S$. We have a partition $Y_{\mathbf{s}} = Y'_{\mathbf{s}} \cup Y''_{\mathbf{s}}$ where $Y'_{\mathbf{s}}$ (resp. $Y''_{\mathbf{s}}$) is the subvariety of $Y_{\mathbf{s}}$ defined by $(B_{h-2}, B_h) \in O(s_h)$ (resp. by $B_{h-2} = B_h$). Then $Y'_{\mathbf{s}}$ is open in $Y_{\mathbf{s}}$ and $Y''_{\mathbf{s}}$ is closed in $Y_{\mathbf{s}}$, so that, if we denote $\pi'_{\mathbf{s}}$ (resp. $\pi''_{\mathbf{s}}$) the restriction of $\pi_{\mathbf{s}}$ to $Y'_{\mathbf{s}}$ (resp. $Y''_{\mathbf{s}}$) we have a natural distinguished triangle (1.10) in $\mathscr{D}G$:

$$((\pi'_{\mathbf{s}})_!\tilde{\mathscr{L}}, K_{\mathbf{s}}^{\mathscr{L}}, (\pi''_{\mathbf{s}})_!\tilde{\mathscr{L}}).$$

Here, we denote the restriction of $\tilde{\mathscr{L}}$ from $Y_{\mathbf{s}}$ to $Y'_{\mathbf{s}}$ or $Y''_{\mathbf{s}}$ again by $\tilde{\mathscr{L}}$. It follows that we have a natural long exact sequence in $\mathscr{M}G$:

$$\text{(2.15.1)} \quad \cdots \to {}^pH^i((\pi'_{\mathbf{s}})_!\tilde{\mathscr{L}}) \to {}^pH^i(K_{\mathbf{s}}^{\mathscr{L}}) \to {}^pH^i((\pi''_{\mathbf{s}})_!\tilde{\mathscr{L}}) \to {}^pH^{i+1}((\pi'_{\mathbf{s}})_!\tilde{\mathscr{L}}) \to \cdots .$$

Let $\mathbf{s}'$ be the sequence $(s_1, s_2, ..., s_{h-1}, s_{h+1}, ..., s_r)$ and let $\mathbf{s}''$ be the sequence $(s_1, s_2, ..., s_{h-2}, s_{h+1}, ..., s_r)$. Then $(g, B_0, B_1, ..., B_r) \to (g, B_0, B_1, ..., B_{h-2}, B_h, B_{h+1}, ..., B_r)$ makes $Y'_{\mathbf{s}}$ into a locally trivial k^*-bundle over $Y_{\mathbf{s}'}$ and $(g, B_0, B_1, ..., B_r) \to (g, B_0, B_1, ..., B_{h-2}, B_{h+1}, ..., B_r)$ makes $Y''_{\mathbf{s}}$ into a locally trivial affine line bundle over $Y_{\mathbf{s}''}$.

The local system $\tilde{\mathscr{L}}$ on $Y''_{\mathbf{s}}$ is just the inverse image of the local system $\tilde{\mathscr{L}}$ on $Y_{\mathbf{s}''}$ (obtained by the construction in 2.5 applied to $\mathbf{s}''$, whose product is in $W'_{\mathscr{L}}$). The local system $\tilde{\mathscr{L}}$ on $Y'_{\mathbf{s}}$ is the inverse image of the local system $\tilde{\mathscr{L}}$ on $Y_{\mathbf{s}'}$, if $h \in I_{\mathbf{s}}$ (by the argument in the proof of (a) in 2.12); if $h \notin I_{\mathbf{s}}$, the direct image with compact support of $\tilde{\mathscr{L}}$ under $Y'_{\mathbf{s}} \to Y_{\mathbf{s}'}$ is zero, (using 1.11.1). It follows that

$$(\pi''_{\mathbf{s}})_!\tilde{\mathscr{L}} = K_{\mathbf{s}''}^{\mathscr{L}}[-2](-1),$$

and, if $h \notin I_{\mathbf{s}}$, we have $(\pi'_{\mathbf{s}})_!\tilde{\mathscr{L}} = 0$. If $h \in I_{\mathbf{s}}$, we have a natural distinguished triangle (1.10) in $\mathscr{D}G$:

$$((\pi'_{\mathbf{s}})_!\tilde{\mathscr{L}}, K_{\mathbf{s}'}^{\mathscr{L}}[-2)(-1), K_{\mathbf{s}'}^{\mathscr{L}}).$$

Hence, we have long exact sequences in $\mathscr{M}G$:

$$(2.15.2) \quad \cdots \to {}^pH^i((\pi'_{\mathbf{s}})_!\tilde{\mathscr{L}}) \to {}^pH^i(K_{\mathbf{s}}^{\mathscr{L}}) \to {}^pH^{i-2}(K_{\mathbf{s}''}^{\mathscr{L}})(-1) \to {}^pH^{i+1}((\pi'_{\mathbf{s}})_!\tilde{\mathscr{L}}) \to \cdots$$

$$(2.15.3) \quad \cdots \to {}^pH^i((\pi'_{\mathbf{s}})_!\tilde{\mathscr{L}}) \to {}^pH^{i-2}(K_{\mathbf{s}'}^{\mathscr{L}})(-1) \to {}^pH^i(K_{\mathbf{s}'}^{\mathscr{L}}) \to {}^pH^{i+1}((\pi'_{\mathbf{s}})_!\tilde{\mathscr{L}}) \to \cdots,$$

if $h \in I_{\mathbf{s}}$, and isomorphisms

$$(2.15.4) \quad {}^pH^i(K_{\mathbf{s}}^{\mathscr{L}}) \simeq {}^pH^{i-2}(K_{\mathbf{s}''}^{\mathscr{L}})(-1), \qquad \text{if} \quad h \notin I_{\mathbf{s}}.$$

2.16. We now prove the implication (c) $\Rightarrow$ (a) in 2.9.

Assume that A is a constituent of ${}^pH^i(K_{\mathbf{s}}^{\mathscr{L}})$ for some sequence $\mathbf{s} = (s_1, s_2,..., s_r)$ in $S \cup \{e\}$ such that $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$ and some i. We may assume that r is minimum possible (for A), which implies that all s_j are in S. We want to prove that A is a constituent of ${}^pH^j(K_w^{\mathscr{L}})$ for some $w \in W'_{\mathscr{L}}$ and some j.

If $l(s_1 s_2 \cdots s_r) = r$, then $K_{\mathbf{s}}^{\mathscr{L}} = K_w^{\mathscr{L}}$, where $w = s_1 s_2 \cdots s_r$ (see 2.11, (2.5.1)) and the desired conclusion follows. Hence we may assume that $l(s_1 s_2 \cdots s_r) < r$. We shall show that this contradicts the minimality of r. We can find h $(2 \leqslant h \leqslant r)$ such that $s_h \cdots s_{r-1} s_r$ is a reduced expression and $s_{h-1} s_h \cdots s_r$ is not a reeduced expression. We can find $s'_h,..., s'_{r-1}, s'_r$ in S such that $s'_h \cdots s'_{r-1} s'_r = s_h \cdots s_{r-1} s_r = y$ and $s'_h = s_{h-1}$. Let $\boldsymbol{\sigma} = (s_1, s_2,..., s_{h-1}, s'_h,..., s'_{r-1}, s'_r)$, $\boldsymbol{\tau} = (s_1, s_2,..., s_{h-1}, y)$. From 2.11, we see that $K_{\mathbf{s}}^{\mathscr{L}} = K_{\boldsymbol{\tau}}^{\mathscr{L}}$, $K_{\boldsymbol{\tau}}^{\mathscr{L}} = K_{\boldsymbol{\sigma}}^{\mathscr{L}}$. Hence $K_{\mathbf{s}}^{\mathscr{L}} = K_{\boldsymbol{\sigma}}^{\mathscr{L}}$. Hence we may assume that $s_{h-1} = s_h$, so that the discussion in 2.15 is applicable.

If $h \notin I_{\mathbf{s}}$, then (2.15.4) shows that A is a constituent of ${}^pH^{i-2}(K_{\mathbf{s}''}^{\mathscr{L}})$; the sequence $\mathbf{s}''$ in S has length $r-2$. This contradicts the minimality of r.

Assume now that $h \in I_{\mathbf{s}}$. By the minimality of r, A is not a constituent of ${}^pH^j(K_{\mathbf{s}'}^{\mathscr{L}})$ (see 2.15) for any i. From (2.15.3) it then follows that A is not a constituent of ${}^pH^j((\pi'_{\mathbf{s}})_!\tilde{\mathscr{L}})$ for any j. This, together with (2.15.2) shows that A is a constituent of ${}^pH^{i-2}(K_{\mathbf{s}''}^{\mathscr{L}})$. This again contradicts the minimality of r.

Thus, the implication (c) $\Rightarrow$ (a) in 2.9 is proved. This completes the proof of 2.9.

PROPOSITION 2.17. *Let $\mathbf{s} = (s_1, s_2,..., s_r)$ be a sequence in $S \cup \{e\}$ such that $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$, $\mathscr{L} \in \mathscr{S}(T)$. Let m be the number of indices $i \in [1, r]$ such that $s_i \in S$, and let $m' = m + \dim G$.*

(a) *$\bar{K}_{\mathbf{s}}^{\mathscr{L}} \in \mathscr{D}G$ is semisimple (see 1.12).*

(b) *${}^pH^j(\bar{K}_{\mathbf{s}}^{\mathscr{L}})$ is isomorphic to ${}^pH^{2m'-j}(\bar{K}_{\mathbf{s}}^{\mathscr{L}})$ (in $\mathscr{M}G$) for any j.*

Proof. (a) is a special case of the "decomposition theorem" [1, 6.2.5], and (b) is a special case of the "relative hard Lefschetz theorem" [1, 6.2.10] applied to the projective morphism $\bar{\pi}_{\mathbf{s}}: \bar{Y}_{\mathbf{s}} \to G$ and to the perverse sheaf $\bar{\mathscr{L}}(m'\rbrack$ on $\bar{Y}_{\mathbf{s}}$.

PROPOSITION 2.18. (a) *If $K \in \hat{G}$, then K is G-equivarant for the conjugation action of G on G.*

(b) *More precisely, if $K \in \hat{G}_{\mathscr{L}}$, $\mathscr{L} = \lambda^* \mathscr{E}_{n,\psi}$ (see 2.2), and $\mathscr{Z}_G^0$ is the connected centre of G, then K is $G \times \mathscr{Z}_G^0$- equivariant for the action $(g_0, z): g \to z^n g_0 g g_0^{-1}$ of $G \times \mathscr{Z}_G^0$ on G.*

Proof. Define an action of $G \times \mathscr{Z}_G^0$:

(i) On T by $(g_0, z): t \to z^n t$.

(ii) On $\dot{Y}$ by (g_0, z): $(g, hU) \to (z^n g_0 g g_0^{-1}, g_0 hU)$ (see 2.4).

(iii) On Y_w by (g_0, z): $(g, B') \to (z^n g_0 g g_0^{-1}, g_0 B')$.

If $\mathscr{L}$ is as in (b), then $\mathscr{L}$ is T-equivariant for the action of T on itself given by $t_0: t \to t_0^n t$. Hence, it is $\mathscr{Z}_G^0$-equivariant since $\mathscr{Z}_G^0$ is a subgroup of T and G acts trivially on T. With the notation in 2.4, $\dot{Y}_w \to T$ is $G \times \mathscr{Z}_G^0$-equivariant hence the local system $\dot{\mathscr{L}}$ on $\dot{Y}_w$ is $G \times \mathscr{Z}_G^0$-equivariant. Since $\dot{Y}_w \to Y_w$ is $G \times \mathscr{Z}_G^0$-equivariant, the local system $\mathscr{L}$ on Y_w is $G \times \mathscr{Z}_G^0$-equivariant. Now using (1.9.2). it follows that ${}^pH^i(K_w^{\mathscr{L}})$ is $G \times \mathscr{Z}_G^0$-equivariant for all i hence, by (1.9.1) any subquotient of ${}^pH^i(K_w^{\mathscr{L}})$ (in $\mathscr{M}G$) is $G \times \mathscr{Z}_G^0$-equivariant). Thus (b) is proved; (a) is a special case of (b).

2.19. Consider a sequence $\mathbf{s} = (s_1, s_2, \ldots, s_r)$ in $S \cup \{e\}$ such that $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$. Let $\mathbf{s}' = (s_2, s_3, \ldots, s_r, s_1)$; we have $s_2 s_3 \cdots s_r s_1 \in W'_{\mathscr{L}'}$, where $\mathscr{L}' = s_1^* \mathscr{L}$. We have a natural isomorphism $Y_{\mathbf{s}} \to Y_{\mathbf{s}'}$ (over G) defined by $(g, B_0, B_1, \ldots, B_r) \to (g, B_1, B_2, \ldots, B_r, g B_1 g^{-1})$. One can verify that this isomorphism carries $\tilde{\mathscr{L}}$ on $Y_{\mathbf{s}}$ to $\tilde{\mathscr{L}}'$ on $Y_{\mathbf{s}'}$ ($\tilde{\mathscr{L}}'$ is defined in terms of $\mathscr{L}'$ in the same way as $\tilde{\mathscr{L}}$ in defined in terms of $\mathscr{L}$). It follows that $K_{\mathbf{s}}^{\mathscr{L}} = K_{\mathbf{s}'}^{\mathscr{L}'}$. Applying this property r times, we obtain the following result.

(2.19.1) Let $\mathbf{s} = (s_1, s_2, \ldots, s_r)$, $\mathbf{s}' = (s'_1, s'_2, \ldots, s'_{r'})$ be two sequences in $S \cup \{e\}$ such that $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$, $s'_1 s'_2 \cdots s'_{r'} \in W'_{\mathscr{L}}$. Let $\mathbf{ss}'$ be the sequence $(s_1, s_2, \ldots, s_r, s'_1, s'_2, \ldots, s'_{r'})$ and let $\mathbf{s}'\mathbf{s}$ be the sequence $(s'_1, s'_2, \ldots, s'_{r'}, s_1, s_2, \ldots, s_r)$. Then $K_{\mathbf{ss}'}^{\mathscr{L}} = K_{\mathbf{s}'\mathbf{s}}^{\mathscr{L}}$.

3. RESTRICTION

3.1. We now fix a parabolic P of G such that $P \supset B$ and we denote by U_P its unipotent radical and by L the Levi subgroup of P containing T. We denote by π_P the canonical homomorphism of P onto L. Let $B^* = B \cap L$; it is a Borel subgroup of L. We shall denote by R^*, W^*, S^*, $W^*_{\mathscr{L}}$, $S^*_{\mathscr{L}}$, $W'^*_{\mathscr{L}}$, $\mathscr{B}^*$, $O^*(w)$ the objects defined by replacing G by L in the definition of R, W, S, $W_{\mathscr{L}}$, $S_{\mathscr{L}}$, $W'_{\mathscr{L}}$, $\mathscr{B}$, $O(w)$. (We regard T also as a maximal torus of L.) Let

$$W_* = \{y \in W \mid y \text{ has minimal length among elements in } W^*y\}.$$

The correspondence $y \to W^*y$ is a 1–1 correspondence $W_* \simeq W^*\backslash W$. The set $W^*\backslash W$ is also in 1–1 correspondence with the set of P-orbits on $\mathscr{B}$: to the coset W^*y ($y \in W$), corresponds the P-orbit of $\dot{y}B\dot{y}^{-1}$; we denote this P-orbit by $v(y)$.

3.2. If v is a P-orbit on $\mathscr{B}$ and $w \in W$, we define a new P-orbit vw by: $vw = v(yw)$, where $v = v(y)$.

We may assume here that $y \in W_*$. If $s \in S$, there are three possibilities for vs:

(a) $ys \in W_*$ and $ys > y$; then $v \subset \overline{vs} - vs$,

(b) $ys \in W_*$ and $ys < y$; then $vs \subset \bar{v} - v$,

(c) $ys \notin W_*$; then $ysy^{-1} \in S^*$ and $vs = v$.

3.3. Let $\mathbf{s} = (s_1, s_2, \dots, s_r)$ be a sequence in S such that $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$ ($\mathscr{L} \in \mathscr{S}(T)$). Let $\bar{Y}'$ be the closed subvariety of $\bar{Y}_s$ defined by

$$\bar{Y}' = \{(g, B_0, B_1, \dots, B_r) \in \bar{Y}_{\mathbf{s}} \mid g \in P\},$$

let $\bar{\mathscr{L}}'$ be the restriction of $\bar{\mathscr{L}}$ (see 2.8) from $\bar{Y}_{\mathbf{s}}$ to $\bar{Y}'$ and let $\bar{\pi}' \colon \bar{Y}' \to L$ be the map defined by $\bar{\pi}'(g, B_0, B_1, \dots, B_r) = \pi_P(g)$.

Any sequence $\mathbf{v} = (v_0, v_1, \dots, v_r)$ of P-orbits on $\mathscr{B}$ defines a locally closed subvariety $\bar{Y}'_{\mathbf{v}}$ of $\bar{Y}'$:

$$\bar{Y}'_{\mathbf{v}} = \{(g, B_0, B_1, \dots, B_r) \in \bar{Y}_{\mathbf{s}} \mid g \in P, B_i \in v_i \ (0 \leqslant i \leqslant r)\}.$$

It is clear that $\bar{Y}'_{\mathbf{v}}$ is empty unless $\mathbf{v}$ satisfies

(3.3.1) $\quad v_i = v_{i-1}$ or $v_{i-1}s_i$ for all i, $1 \leqslant i \leqslant r$, and $v_0 = v_r$.

Let $\bar{\mathscr{L}}'_{\mathbf{v}}$ be the restriction of $\bar{\mathscr{L}}'$ to $\bar{Y}'_{\mathbf{v}}$ and let $\bar{\pi}'_{\mathbf{v}}$ be the restriction of $\bar{\pi}'$ to $\bar{Y}'_{\mathbf{v}}$. We associate with $\mathbf{v}$ (satisfying (3.3.1)) the sequence $\tilde{\mathbf{s}} = (\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_r)$ in $S \cup \{e\}$ defined by

$$(3.3.2) \quad \tilde{s}_i = \begin{cases} s_i & \text{if } v_i = v_{i-1}s_i, \\ e & \text{if } v_i \neq v_{i-1}s_i. \end{cases}$$

We then have $v_i = v_{i-1}\tilde{s}_i$ $(1 \leqslant i \leqslant r)$, hence $v_0\tilde{s}_1\tilde{s}_2 \cdots \tilde{s}_r = v_0$.

(3.3.3) Let $y_i \in W_*$ be defined by $v_i = v(y_i)$ $(0 \leqslant i \leqslant r)$. We define

$$(3.3.4)\quad t_i = y_{i-1}\tilde{s}_i y_i^{-1} = \begin{cases} y_{i-1}s_i y_{i-1}^{-1}, & \text{if } v_{i-1}s_i = v_{i-1}, \\ e, & \text{if } v_{i-1}s_i \neq v_{i-1} \end{cases}$$

$(1 \leqslant i \leqslant r)$. Then $\mathbf{t} = (t_1, t_2, ..., t_r)$ is a sequence in $S^* \cup \{e\}$.

3.4. The formula $(g, B_0, B_1, ..., B_r) \to (\pi_P(g), \pi_P(B_0 \cap P), \pi_P(B_1 \cap P), ..., \pi_P(B_r \cap P))$ defines a morphism $\rho: \bar{Y}'_{\mathbf{v}} \to \bar{Y}^*_{\mathbf{t}}$, where

$$\bar{Y}^*_{\mathbf{t}} = \{(l, B_0^*, B_1^*, ..., B_r^*) \in L \times \mathscr{B}^* \times \cdots \times \mathscr{B}^* \mid (B_{i-1}^*, B_i^*) \in \overline{O^*(t_i)}, B_r^* = lB_0^* l^{-1}\}.$$

This morphism is a locally trivial fibration. Its fibre over any point $(l, B_0^*, B_1^*, ..., B_r^*) \in \bar{Y}^*_{\mathbf{t}}$ is isomorphic to the affine space of dimension

$$(3.4.1)\quad d(\mathbf{v}) = \dim U_p + \#\{i \in [1, r] \mid v_i s_i \subset \bar{v}_i - v_i\}.$$

Indeed, the set of all $B_0 \in v_0$ such that $\pi_P(B_0 \cap P) = B_0^*$ is an affine space of dimension $l(y_0)$. If $(B_0, B_1, ..., B_{i-1})$ are already determined, the set of all $B_i \in v_i$ such that $(B_{i-1}, B_i) \in \overline{O(s_i)}$ and $\pi_P(B_i \cap P) = B_i^*$ is an affine line if $v_i s_i \subset \bar{v}_i - v_i$ and is a point, otherwise. Finally, if $B_0, B_1, ..., B_r$ are already determined, the set of all $g \in \pi_P^{-1}(l)$ such that $B_r = gB_0 g^{-1}$ is an affine space of dimension $\dim U_P - l(y_0)$, (since $y_0 = y_n$). Hence our fibre is an affine space of dimension $l(y_0) + \#\{i \in [1, r] \mid v_i s_i \subset \bar{v}_i - v_i\} + (\dim U_P - l(y_0)) = d(\mathbf{v})$. We now state

LEMMA 3.5. *Let notations be as in* 3.3; *we assume that* $\mathbf{v}$ *satisfies* (3.3.1). *Let* $I_{\mathbf{s}} \subset [1, r]$ *be as in* 2.12 *and let* $J = J_{\mathbf{v}} = \{i \in [1, r] \mid \tilde{s}_i = e\}$:

(a) *If* $J \not\subset I_{\mathbf{s}}$, *then* $\bar{\pi}'_{\mathbf{v}}(\bar{\mathscr{L}}'_{\mathbf{v}}) = 0$.

(b) *If* $J \subset I_{\mathbf{s}}$, *then* $\pi'_{\mathbf{v}}(\bar{\mathscr{L}}'_{\mathbf{v}}) = \bar{K}_{\mathbf{t}}^{\mathscr{L}_1}[-2d(\mathbf{v})](-d(\mathbf{v}))$ *where* $\mathscr{L}_1 = (y_0^{-1})^*\mathscr{L}$ *and* $K_{\mathbf{t}}^{\mathscr{L}} \in \mathscr{D}L$ *is defined as in* 2.8, *with respect to* L.

Proof. Let $\pi_{\mathbf{t}}$ be the canonical projection $\bar{Y}^*_{\mathbf{t}} \to L$. We have $(\bar{\pi}'_{\mathbf{v}})_! = (\bar{\pi}_{\mathbf{t}})_! \rho_!$ (ρ is as in 3.4). Hence it is enough to prove:

(a′) If $J \not\subset I_{\mathbf{s}}$, then $\bar{\mathscr{L}}'_{\mathbf{v}} = 0$.

(b′) If $J \subset I_{\mathbf{s}}$, then $\rho_!(\mathscr{L}'_{\mathbf{v}}) = \bar{\mathscr{L}}_1[-2d(\mathbf{v})](-d(\mathbf{v}))$, where $\bar{\mathscr{L}}_1$ is the constructible sheaf on $\bar{Y}^*_{\mathbf{t}}$ defined in the same way as $\bar{\mathscr{L}}$ in 2.8, but replacing $\mathbf{s}$, $\mathscr{L}$, G by $\mathbf{t}$, $\mathscr{L}_1$, L.

(For $\bar{\mathscr{L}}_1$ to be defined, we must know that $t_1 t_2 \cdots t_r \in W'^*_{\mathscr{L}_1}$ or that $y_0^{-1}t_1 t_2 \cdots t_r y_0 \in W'_{\mathscr{L}}$. We have $y_0^{-1}t_1 t_2 \cdots t_r y_0 = (y_0^{-1}t_1 y_1)(y_1^{-1}t_2 y_2) \cdots (y_{r-1}^{-1}t_r y_r) = \tilde{s}_1\tilde{s}_2 \cdots \tilde{s}_r$ (since $y_0 = y_r$), and this is in $W'_{\mathscr{L}}$, since $J \subset I_{\mathbf{s}}$.)

Since ρ is a locally trivial fibration with fibres $\cong k^{d(\mathbf{v})}$ (see 3.4) we see that (b′) is a consequence of

(b″) $\bar{\mathscr{L}}'_{\mathbf{v}} = \rho^*(\bar{\mathscr{L}}_1)$, if $J \subset I_{\mathbf{s}}$.

First, assume that $J \not\subset I_{\mathbf{s}}$ and let j be an index in $J - I_{\mathbf{s}}$. If $(g, B_0, B_1, ..., B_r) \in Y'_{\mathbf{v}}$, we have $B_{j-1} = B_j$ (since $j \in J$) and $(B_{i-1}, B_i) \in \overline{O(s_i)}$ for all $i \in [1, r] - \{j\}$. Since $j \notin I_{\mathbf{s}}$, from 2.12(a) it follows that the stalk of $\bar{\mathscr{L}}$ (or $\bar{\mathscr{L}}'_{\mathbf{v}}$) at $(g, B_0 B_1, ..., B_r)$ is 0, hence $\bar{\mathscr{L}}'_{\mathbf{v}} = 0$, proving (a′).

Next, we assume that $J \subset I_{\mathbf{s}}$. Let $H_0 = \{i \in [1, r] \mid t_i \neq e\}$. Then $J \cap H_0 = \varnothing$. For any subset $H \subset H_0$ we have the locally closed subvariety of $\bar{Y}^*_{\mathbf{t}}$,

$$Y^*_{\mathbf{t}_H} = \{(l, B^*_0, B^*_1, ..., B^*_r) \in \bar{Y}^*_{\mathbf{t}} \mid (B^*_{i-1}, B^*_i) \in O^*(t_i) \text{ if } i \notin H,\ B^*_{i-1} = B^*_i \text{ if } i \in H\}.$$

These form a partition of $\bar{Y}^*_{\mathbf{t}}$. Define $Y'_{\mathbf{v},H} = \rho^{-1} \bar{Y}^*_{\mathbf{t}_H}$, for all $H \subset H_0$. The subvarieties $Y'_{\mathbf{v},H}$ form a partition of $\bar{Y}'_{\mathbf{v}}$ into locally closed pieces.

Let $I_{\mathbf{t}}$ be the set of all $j \in [1, r]$ such that $t_j \neq e$ and $t_r t_{r-1} \cdots t_j \cdots t_{r-1} t_r \in \underline{W}^*_{\mathscr{L}_1}$. Then $I_{\mathbf{t}} = \mathbf{I}_s \cap H_0$. Applying 2.12(a) to $\bar{\mathscr{L}}_1$, t, and L, we see that $\bar{\mathscr{L}}_1$ is a local system of rank 1 on the open subset $\bigcup_{H \subset I_{\mathbf{t}}} Y^*_{\mathbf{t}_H}$ of $\bar{Y}^*_{\mathbf{t}}$ and is zero on its complement.

It follows that $\rho^* \bar{\mathscr{L}}_1$ is a local system of rank 1 on the open subset $\bigcup_{H \subset I_{\mathbf{t}}} Y'_{\mathbf{v},H}$ of $\bar{Y}'_{\mathbf{v}}$ and is zero on its complement in $\bar{Y}'_{\mathbf{v}}$.

With the notations in 2.6, we have $Y'_{\mathbf{v},H} = Y_{\mathbf{s}_{H \cup J}} \cap \bar{Y}'_{\mathbf{v}}$. For a set $H \subset H_0$, the conditions $H \subset I_{\mathbf{t}}$ and $H \cup J \subset I_{\mathbf{s}}$ are equivalent. By 2.12(a), $\bar{\mathscr{L}}$ (and hence $\bar{\mathscr{L}}'_{\mathbf{v}}$) is a local system of rank 1 on the open subset $\bigcup_{H \subset I_{\mathbf{t}}} Y'_{\mathbf{v},H}$ of $\bar{Y}'_{\mathbf{v}}$ and is zero on its complement in $\bar{Y}'_{\mathbf{v}}$. To prove (b'') it is then enough to show that the local systems on $\bigcup_{H \subset I_t} Y'_{\mathbf{v},H}$ defined by $\bar{\mathscr{L}}$ and $\rho^*(\bar{\mathscr{L}}')$ are isomorphic. Since $Y'_{\mathbf{v},\varnothing}$ is open dense in the smooth variety $\bigcup_{H \subset I_{\mathbf{t}}} Y'_{\mathbf{v},H}$ it is even enough to show that the local systems on $Y'_{\mathbf{v},\varnothing}$ defined by $\bar{\mathscr{L}}$ and $\rho^*(\bar{\mathscr{L}}')$ are isomorphic.

The local system defined by $\bar{\mathscr{L}}$ on $Y'_{\mathbf{v},0}$ is the restriction of the local system $\tilde{\mathscr{L}}$ from $Y_{\mathbf{s}_J}$ to $Y'_{\mathbf{v},\varnothing}$, ($\tilde{\mathscr{L}}$ is constructed explicitly in 2.5).

The local system defined $\rho^*(\bar{\mathscr{L}}')$ on $Y'_{\mathbf{v},\varnothing}$ is the inverse image under ρ: $Y'_{\mathbf{v},0} \to Y^*_{\mathbf{t}_\varnothing}$ of the local system $\tilde{\mathscr{L}}'$, which is explicitly constructed as in 2.5 (for $\mathbf{t}$, $\mathscr{L}'$, L instead of $\mathbf{s}$, $\mathscr{L}$, G). From these explicit constructions, we get immediately an explicit isomorphism between our two local system. Thus, (b'') follows, completing the proof of the lemma.

3.6. We consider the sequence of closed subsets of $\bar{Y}'$ defined by

$$Z_i = \bigcup_{c(\mathbf{v}) \leqslant i} \bar{Y}'_{\mathbf{v}},$$

where $\mathbf{v}$ runs over all sequences $\mathbf{v} = (v_0, v_1, \ldots, v_r)$ of P-orbits on $\mathscr{B}$ satisfying (3.3.1) and $c(\mathbf{v}) \leqslant i$, where

$$c(\mathbf{v}) = \dim v_0 + \dim v_1 + \cdots + \dim v_r.$$

If β_i is the inclusion $Z_i' \hookrightarrow \bar{Y}'$, and γ_i is the inclusion $Z_i' - Z_{i-1}' \subset \bar{Y}'$, then we have a natural distinguished triangle (1.10) in $\mathscr{D}L$:

$$(\bar{\pi}_!'(\gamma_i)_! \gamma_i^* \bar{\mathscr{L}}', \bar{\pi}_!'(\beta_i)_! \beta_i^* \bar{\mathscr{L}}', \bar{\pi}_!'(\beta_{i-1})_! \beta_{i-1}^* \bar{\mathscr{L}}').$$

It gives rise to a long exact sequence in $\mathscr{M}L$ (for each i)

$$\text{(3.6.1)} \quad \cdots \to {}^pH^{j-1}(\bar{\pi}_!'(\beta_{i-1})_! \beta_{i-1}^* \bar{\mathscr{L}}') \xrightarrow{\delta} \bigoplus_{\substack{\mathbf{v} \\ c(\mathbf{v}) = i}} {}^pH^j((\bar{\pi}_{\mathbf{v}}')_! \bar{\mathscr{L}}_{\mathbf{v}}')$$
$$\to {}^pH^j(\bar{\pi}_!'(\beta_i)_! \beta_i^* \bar{\mathscr{L}}') \to {}^pH^j(\bar{\pi}_!'(\beta_{i-1})_! \beta_{i-1}^* \bar{\mathscr{L}}')$$
$$\xrightarrow{\delta} \bigoplus_{\substack{\mathbf{v} \\ c(\mathbf{v}) = i}} {}^pH^{j+1}((\bar{\pi}_{\mathbf{v}}')_! \bar{\mathscr{L}}_{\mathbf{v}}').$$

Here we have used the isomorphism $\bar{\pi}_!'(\gamma_i)_! \gamma_i^* \bar{\mathscr{L}}' = \bigoplus_{\mathbf{v}, c(\mathbf{v}) = i} ((\bar{\pi}_{\mathbf{v}}')_! \bar{\mathscr{L}}_{\mathbf{v}}')$. Note that

$$\text{(3.6.2)} \quad \bar{\pi}_!'(\beta_i)_! \beta_i^* \bar{\mathscr{L}}' = \begin{cases} \bar{\pi}_!' \bar{\mathscr{L}}' & \text{for large } i, \\ 0 & \text{for } \ i < 0. \end{cases}$$

We now prove the following result.

LEMMA 3.7. (a) *For each integer i, the maps* δ *in* (3.6.1) *are zero.*

(b) *For each integer i, the complex* $\bar{\pi}_!'(\beta_i)_! \beta_i^* \bar{\mathscr{L}}' \in \mathscr{D}L$ *is semisimple* (1.12); *it is isomorphic in* $\mathscr{D}L$ *to the direct sum* $\bigoplus_{\mathbf{v}, c(\mathbf{v}) \leqslant i} ((\bar{\pi}_{\mathbf{v}}')_! \bar{\mathscr{L}}_{\mathbf{v}}')$.

(c) *The complex* $\bar{\pi}_!' \bar{\mathscr{L}}' \in \mathscr{D}L$ *is semisimple*; *it is isomorphic in* $\mathscr{D}L$ *to the direct sum* $\bigoplus_{\mathbf{v}} ((\bar{\pi}_{\mathbf{v}}')_! \bar{\mathscr{L}}_{\mathbf{v}}')$.

Proof. From (3.6.2) we see that (c) is a special case of (b), (for large i). Assuming that (a) and the first assertion of (b) are proved, we prove the second assertion of (b) as follows. Since both complexes in question are semisimple (see 3.5 and 2.17), it is enough to show that they have the same ${}^pH^j$ for all j. Using (a) we see that (3.6.1) decomposes into short exact sequences of semisimple objects in $\mathscr{M}(L)$. Hence

$$\begin{aligned} &{}^pH^j(\bar{\pi}_!'(\beta_i)_! \beta_i^* \bar{\mathscr{L}}') \\ &\quad \cong {}^pH^j(\bar{\pi}_!'(\beta_{i-1})_! \beta_{i-1}^* \bar{\mathscr{L}}') \oplus \Bigg(\bigoplus_{\substack{\mathbf{v} \\ c(\mathbf{v}) = i}} {}^pH^j((\bar{\pi}_{\mathbf{v}}')_! \bar{\mathscr{L}}_{\mathbf{v}}') \Bigg). \end{aligned}$$

This proves the desired equality for ${}^pH^j$ by induction on i. (The case $i < 0$ is trivial by (3.6.2).)

It remains to prove (a) and the first assertion of (b). By general principles [1, Sect. 6] it is enough to prove them in the case where the ground field k is the algebraic closure of a finite field. In this case, we can realize (3.6.1), (3.6.2) in the category of mixed perverse sheaves over G_0 (a split F_q-form of G with B, T, P defined over F_q) for sufficiently large $F_q \subset k$, depending on $\mathscr{L}$. The isomorphisms in Lemma 3.5 can also be realized in that category (possibly with an even larger F_q). Now $\bar{K}_{\mathbf{t}}^{\mathscr{L}_1}$ in that lemma is a pure complex of weight 0 (by Deligne's theorem [2, 6.2.6] applied to the proper map $\bar{\pi}_{\mathbf{s}}: \bar{Y}_{\mathbf{s}} \to G$ and to $\bar{\mathscr{L}}$ which is pure of weight 0, as we can see either directly, or from Gabber's purity theorem [1, 5.3.4]); after applying to it $[-2d(\mathbf{v})](-d(\mathbf{v}))$, it remains pure of weight 0, see [1, 6.1.4]. Hence, by 3.5, $(\pi'_{\mathbf{v}})_!(\bar{\mathscr{L}}'_{\mathbf{v}})$ are pure complexes of weight 0; it follows that

(3.7.1) $\bigoplus {}^pH^j((\pi'_{\mathbf{v}})_!\bar{\mathscr{L}}'_{\mathbf{v}})$ in (3.6.1) are pure complexes of weight j.

We now show by induction on i that ${}^pH^j(\pi'_!(\beta_i)_!\beta_i^*\bar{\mathscr{L}}')$ is a pure complex of weight j for any i. This is obvious for $i < 0$, by (3.6.2). If we assume that this is true for $i-1$, the statement for i follows from (3.6.1), using (3.7.1), the statement for $i-1$ and the following fact: if $K_1 \to K_2 \to K_3$ is an exact sequence of mixed perverse sheaves with K_1, K_3 pure of weight j, then K_2 is also pure of weight j.

Now using [1, 5.4.4] it follows that $\pi'_!(\beta_i)_!\beta_i^*\bar{\mathscr{L}}'$ is pure of weight 0. Using the "decomposition theorem" [1, 5.4.5, 5.3.8] it follows that $\pi'_!(\beta_i)_!\beta_i^*\bar{\mathscr{L}}'$ is semisimple.

The vanishing of δ in (3.6.1) follows from the fact that δ is a morphism between two pure perverse sheaves of different weights. This completes the proof of the lemma.

3.8. We define a functor $\operatorname{res}: \mathscr{D}G \to \mathscr{D}L$ by $\operatorname{res} A = (\pi_P)_! i^*A(\alpha)$, where $i: P \hookrightarrow G$ is the inclusion and $\alpha = \dim U_P$. It is clear that, with the notation in 3.3, we have

(3.8.1) $$\operatorname{res} \bar{K}_{\mathbf{s}}^{\mathscr{L}} = \bar{\pi}'_!\bar{\mathscr{L}}'(\alpha) \in \mathscr{D}L.$$

Hence 3.7(c) and 3.5 imply

(3.8.2) $\operatorname{res} \bar{K}_{\mathbf{s}}^{\mathscr{L}} \in \mathscr{D}L$ is semisimple; more precisely it is a direct sum of finitely many complexes of the form $A'[i]$, where $A' \in \hat{L}$ and i is an integer.

We can now state

Proposition 3.9. *If $A \in \hat{G}$, then $\operatorname{res} A \in \mathscr{D}L$ is semisimple; more* *precisely, it is a direct sum of finitely many complexes of form $A'[i]$, where $A' \in \hat{L}$ and i is an integer.*

Proof. We can find a sequence $\mathbf{s} = (s_1, s_2, ..., s_r)$ in S and $\mathscr{L} \in \mathscr{S}(T)$ such that $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$ and such that A is a constituent of ${}^pH^j(\bar{K}_{\mathbf{s}}^{\mathscr{L}})$. From 2.17(a), it follows that $A[-j]$ is a direct summand of $\bar{K}_{\mathbf{s}}^{\mathscr{L}}$. Since res transforms direct sums into direct sums, it follows that $\operatorname{res}(A)[-j]$ is a direct summand of $\operatorname{res}(\bar{K}_{\mathbf{s}}^{\mathscr{L}})$ which is semisimple by (3.8.2). By 1.12, $\operatorname{res} A[-j]$ must be also semisimple. Now ${}^pH^i(\operatorname{res} A)$ is a direct summand of ${}^pH^{i+j}(\operatorname{res} \bar{K}_{\mathbf{s}}^{\mathscr{L}})$ which, by (3.8.2) has all its irreducible subquotients in $\hat{L}$; hence all irreducible subquotients of ${}^pH^i(\operatorname{res} A)$ are in $\hat{L}$. The proposition follows.

DEFINITION 3.10. A character sheaf $A \in \hat{G}$ is said to be *cuspidal* if for any parabolic subgroup $P \subsetneq G$ containing B (with Levi subgroup $L \supset T$), we have $\operatorname{res} A[-1] \in \mathscr{D}L^{\leqslant 0}$ (with res defined with respect to P), or, equivalently, $\dim \operatorname{supp} \mathscr{H}^i(\operatorname{res} A) < -i$ for all i. The cuspidal character sheaves form a subset $\hat{G}^{(0)}$ of $\hat{G}$.

3.11. For any $g \in G$, we denote by g_s the semisimple part of g and we define $H_G(g)$ to be the centralizer in G of the connected centre of $Z_G^0(g_s)$. Then $H_G(g)$ is the smallest Levi subgroup of a parabolic subgroup containing $Z_G^0(g_s)$. We say that g (or its conjugacy class) is isolated if $H_G(g) = G$. (When G is semisimple, it has only finitely many isolated classes.)

Following [4, 3.1], we now define a partition of G into finitely many locally closed smooth irreducible subvarieties stable by conjugation. The pieces in the partition are parametrized by pairs (L, Σ) up to G-conjugacy, where L is a subgroup of G, which is the Levi subgroup of some parabolic subgroup of G, and Σ is a subset of L, which is the inverse image under $L \to L/\mathscr{Z}_L^0$ ($\mathscr{Z}_L^0 =$ connected centre of L) of an isolated conjugacy class of $L/\mathscr{Z}_L^0$. For such (L, Σ), we define

$$\Sigma_{\mathrm{reg}} = \{ g \in \Sigma \mid H_G(g) = L \} = \{ g \in \Sigma \mid Z_G^0(g_s) \subset L \}$$

and $Y_{(L,\Sigma)} = \bigcup_{x \in G} x(\Sigma_{\mathrm{reg}}) x^{-1}$.

The $Y_{(L,\Sigma)}$ form the required partition of G.

PROPOSITION 3.12. *Let $A \in \hat{G}_{\mathscr{L}}$ be cuspidal, where $\mathscr{L} = \lambda^* \mathscr{E}_{n,\psi}$ is as in* 2.2. *Consider the action of $G \times \mathscr{Z}_G^0$ on G defined in* 2.18(b). *Then there is a unique $G \times \mathscr{Z}_G^0$-orbit $\Sigma_0 \subset G$ and a unique irreducible, $G \times \mathscr{Z}_G^0$–equivariant $\bar{\mathbf{Q}}_l$-local system $\mathscr{E}$ on Σ_0 such that $A = IC(\bar{\Sigma}_0, \mathscr{E})[d]$, where $d = \dim \Sigma_0$.*

Moreover, the image of Σ_0 in $G/\mathscr{Z}_G^0$ is an isolated conjugacy class (see 3.11) *of $G/\mathscr{Z}_G^0$. If $g \in \Sigma_0$ and H is the centralizer of g in G, then $H^0/\mathscr{Z}_G^0$ is a unipotent group.*

Proof. Let V be a locally closed smooth irreducible subvariety of G which is dense in the support of A and is such that $A \mid V$ is isomorphic to $\mathscr{E}[d]$, where $\mathscr{E}$ is an irreducible $\bar{\mathbf{Q}}_l$-local system on V and $d = \dim V$. We shall assume (as we may be 2.18(b)) that V is $G \times \mathscr{Z}_G^0$-stable and that $\mathscr{E}$ is $G \times \mathscr{Z}_G^0$-equivariant.

Since V is irreducible, there is a unique piece $Y_{(L,\Sigma)}$ in the finite partition of G described in 3.11 such that $V \cap Y_{(L,\Sigma)}$ is open dense in $Y_{(L,\Sigma)}$. Since $Y_{(L,\Sigma)}$ is $G \times \mathscr{Z}_G^0$-stable, we may assume (by replacing V by $V \cap Y_{(L,\Sigma)}$) that $V \subset Y_{(L,\Sigma)}$. We may also assume that $L \supset T$ and is the Levi subgroup of a parabolic subgroup P of G containing B. Let U_P, π_P be defined as in 3.1 and let i be the inclusion $P \hookrightarrow G$. Let $g \in \sum_{\text{reg}} \cap V \subset L$. The orbit of g under the conjugation action of U_P is closed (it is an orbit of a unipotent group acting on an affine variety) and is contained in gU_P (since $\pi_P(g) = \pi_P(ugu^{-1})$ for $u \in U_P$). The isotropy group of g in U_P is contained in $U_P \cap Z_G(g) \subset U_P \cap Z_G^0(g_s)$; hence it is trivial since $g \in \Sigma_{\text{reg}}$. Hence the dimension of the U_P-orbit of G is equal to dim U_P; this orbit being closed in gU_P, it must be equal to gU_P. In particular, we have $gU_P \subset V$ (since V is stable by conjugation). The restriction of $\mathscr{E}$ to gU_P is a U_P-equivariant local system (for the conjugation action of U_P) on the U_P-orbit gU_P, with trivial isotropy group. It follows that $\mathscr{E}$ is a constant nonzero local system on gU_P, hence $H_c^{2\alpha}(gU_P, \mathscr{E}) \neq 0$, $(\alpha = \dim U_P)$. This means that the stalk of the cohomology sheaf $\mathscr{H}^{2\alpha-d}((\pi_P)_! i^*A)$ at g is nonzero. Thus, we have shown that

$$\text{(3.12.1)} \qquad \Sigma_{\text{reg}} \cap V \subset \operatorname{sup} \mathscr{H}^{2\alpha-d}((\pi_P)_! \, i^*A).$$

Let $G_1 = \{g \in N_G(L) \mid g\Sigma g^{-1} = \Sigma\} = \{g \in N_G(L) \mid g\Sigma_{\text{reg}}\, g^{-1} = \Sigma_{\text{reg}}\} = \{g \in N_G(L) \mid g\Sigma g^{-1} \cap \Sigma \neq \varnothing\}$. The group G_1 acts on $G \times (\Sigma_{\text{reg}} \cap V)$ by $g_1 : (g, \sigma) \to (gg_1^{-1}, g_1\sigma g_1^{-1})$. The map $G \times (\Sigma_{\text{reg}} \cap V) \to V$ defined by $(g, \sigma) \to g\sigma g^{-1}$ is surjective (since $V \subset Y_{(L,\Sigma)}$) and its fibres are precisely the orbits of the G_1-action just described. It follows that

$$\begin{aligned}\dim(\Sigma_{\text{reg}} \cap V) + \dim G = \dim V + \dim G_1 &= \dim V + \dim L \\ &= d + \dim G - 2\alpha;\end{aligned}$$

hence $\dim(\Sigma_{\text{reg}} \cap V) = d - 2\alpha$. From (3.12.1) it now follows that $\dim \operatorname{supp} \mathscr{H}^{2\alpha-d}((\pi_p)_! i^*A) \geqslant d - 2\alpha$. Since A is assumed to be cuspidal, it follows that $L = P = G$. In this case, $Y_{(L,\Sigma)} = \Sigma$ is a single $G \times \mathscr{Z}_G^0$-orbit on

G, and therefore V must also be a single $G \times \mathscr{Z}_G^0$-orbit; the image of $V = \Sigma$ in $G/\mathscr{Z}_G^0$ is isolated. The last assertion follows from [4, 2.8], since $(\Sigma, \mathscr{E})$ is a *cuspidal pair* for G in the sense of [4, 2.4].(The condition [4, 2.4(a)] follows from $G \times \mathscr{Z}_G^0$-equivariance; the condition [4, 2.4(b)] follows from the fact that A is cuspidal.)

4. Induction

4.1. Let P, L, U_P, π_P be as in 3.1. Consider the diagram

$$L \xleftarrow{\pi} V_1 \xrightarrow{\pi'} V_2 \xrightarrow{\pi''} G,$$

where

$$V_1 = \{(g, h) \in G \times G \mid h^{-1}gh \in P\},$$
$$V_2 = \{(g, h) \in G \times (G/P) \mid h^{-1}gh \in P\},$$
$$\pi(g, h) = \pi_P(h^{-1}gh), \quad \pi'(g, h) = (g, hP), \quad \pi''(g, hP) = g.$$

Then π, π' are smooth morphisms with connected fibres.

We associate with any perverse sheaf $K \in \mathscr{M}L$ (which is L-equivariant for the conjugation action of L on L) a complex $\operatorname{ind} K \in \mathscr{D}G$, as follows. The perverse sheaf $\tilde{\pi}K \in \mathscr{M}V_1$ is P-equivariant for the action p: $(g, h) \to (g, hp^{-1})$ of P on V_1 and the action p: $l \to \pi_P(p)\, l\pi'_P(p)^{-1}$ of P on L. Since π' is a locally trivial principal P-bundle, there is (1.9.3) a well-defined perverse sheaf $K_1 \in \mathscr{M}V_2$ such that $\tilde{\pi}K = \tilde{\pi}'K_1$. We define $\operatorname{ind} K = (\pi'')_! K_1$. We shall sometimes write $\operatorname{ind}_P^G K$ instead of $\operatorname{ind} K$.

In the case where $L = P = G$, we have $\operatorname{ind}_P^G K = K$, as we see immediately from the definition of G-equivariance of K. From (1.9.2) it follows easily that

(4.1.1) ${}^pH^i(\operatorname{ind}_P^G K)$ is a G-equivariant perverse sheaf on G (for the conjugation action), for all i.

We shall now state a transitivity property of induction. Let Q be a parabolic subgroup of L containing $B^* = B \cap L$, let M be the Levi subgroup of Q containing T, and let $\pi_Q : Q \to M$ be the canonical projection.

Let $\tilde{Q} = QU_P$; it is parabolic subgroup of G, $B \subset \tilde{Q} \subset P$.

Proposition 4.2. *Let $K \in \mathscr{M}(M)$ be M-equivariant (for the conjugation action). Assume that* $\operatorname{ind}_Q K$ *is in* $\mathscr{M}L$. *Then* $\operatorname{ind}_P^G(\operatorname{ind}_Q^L K) = \operatorname{ind}_{\tilde{Q}}^G K$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
 & M & \\
X_0 \arrow[ur, "d_0"] \arrow[d, "f_0"'] & X_1 \arrow[l, "h_1"'] \arrow[r, "h_2"] \arrow[u, "d_1"'] \arrow[d, "f_1"] & X_2 \arrow[ul, "d_2"'] \arrow[d, "f_2"] \\
Y_0 \arrow[d, "g_0"'] & Y_1 \arrow[l, "e_1"'] \arrow[r, "e_2"] \arrow[d, "g_1"] & Y_2 \arrow[d, "g_2"] \\
L & V_1 \arrow[l, "\pi"'] \arrow[r, "\pi'"] & V_2 \arrow[d, "\pi''"] \\
 & & G
\end{tikzcd}$$

where

$$X_0 = \{(x, y) \in L \times L \mid y^{-1}xy \in Q\},$$
$$X_1 = \{(y, z, v') \in P \times G \times G \mid v'^{-1}zv' \in \tilde{Q}\},$$
$$X_2 = \{(z, v') \in G \times G \mid v'^{-1}zv' \in \tilde{Q}\},$$

$$Y_0 = \{(x, yQ) \in L \times L/Q \mid y^{-1}xy \in \tilde{Q}\},$$
$$Y_1 = \{(y, z, v') \in P \times G \times G \mid v'^{-1}zv' \in \tilde{Q}\} \text{ mod. action of } \tilde{Q}\text{:}$$
$$q\colon (y, z, v') \to (yq^{-1}, z, v'q^{-1}),$$
$$Y_2 = \{(z, v'\tilde{Q}) \in G \times G/\tilde{Q} \mid v'^{-1}zv' \in \tilde{Q}\},$$

$V_1, V_2, \pi, \pi', \pi''$ are as in 4.1,

$$e_1(y, z, v') = (yv'^{-1}zv'y^{-1}, yQ),$$
$$e_2(y, z, v') = (z, v'\tilde{Q}),$$

$$h_1(y, z, v') = (yv'^{-1}zv'y, y),$$
$$h_2(y, z, v') = (z, v'),$$

$$d_0(x, y) = \pi_Q(y^{-1}xy),$$
$$d_1(y, z, v') = \pi_P(v'^{-1}zv')$$
$$d_2(z, v') = \pi_P(v'^{-1}zv')$$

$$f_0(x, y) = (x, yQ),$$
$$f_1(y, z, v') = \tilde{Q}\text{-orbit of } (y, z, v'),$$
$$f_2(z, v') = (z, v'\tilde{Q}),$$

$$g_0(x, yQ) = x$$
$$g_1(y, z, v') = (z, v'y^{-1}),$$
$$g_2(z, v'\tilde{Q}) = (z, v'P).$$

The two lowest squares (e_1, g_0, g_1, π) and (e_2, g_2, g_1, π') are cartesian and the maps e_i, f_i, π, π' are smooth with connected fibres. It follows that

(4.2.1) $(g_1)_! \tilde{e}_0 = \tilde{\pi}(g_0)_!$,

(4.2.2) $(g_1)_! \tilde{e}_1 = \tilde{\pi}(g_2)_!$,

(4.2.3) $\tilde{f}_i : \mathcal{M}Y_i \to \mathcal{M}X_i$ is fully faithful ($i = 0, 1, 2$).

Since K is M-equivariant, $\tilde{d}_0 K \in \mathcal{M}X_0$ is in the image of $\tilde{f}_0 : \mathcal{M}Y_0 \to \mathcal{M}X_0$. We shall write $(\tilde{f}_0)^{-1}\tilde{d}_0 K$ for the object in $\mathcal{M}Y_0$ which maps under $\tilde{f}_0$ to $\tilde{d}_0 K$. Again, since K is M-equivariant, $\tilde{d}_2 K \in \mathcal{M}X_2$ is in the image of $\tilde{f}_2 : \mathcal{M}Y_2 \to \mathcal{M}X_2$; we write $(\tilde{f}_2)^{-1}\tilde{d}_2 K$ for the object in $\mathcal{M}Y_2$ which maps under $\tilde{f}_2$ to $\tilde{d}_2 K$.

Let $K' = (g_2)_!(\tilde{f}_2)^{-1}\tilde{d}_2 K \in \mathcal{D}V_2$. It is enough to prove the following three statements:

(a) $K' \in \mathcal{M}V_2$,

(b) $\tilde{\pi}(g_0)_!(\tilde{f}_0)^{-1}\tilde{d}_0 K = \tilde{\pi}' K'$,

(c) $\pi_! K' = \pi_!(g_2)_!(\tilde{f}_2)^{-1}\tilde{d}_2 K$.

Property (c) is obvious from the definition of K'. We now prove (b).

From $d_1 = d_0 h_1 = d_2 h_2$, we see that

$$\tilde{h}_2 \tilde{d}_2 K = \tilde{h}_1 \tilde{d}_0 K \in \mathcal{M}X_1.$$

This can also be written as

$$\tilde{h}_2 \tilde{f}_2 (\tilde{f}_2)^{-1} \tilde{d}_2 K = \tilde{h}_1 \tilde{f}_0 (\tilde{f}_0)^{-1} \tilde{d}_0 K \in \mathcal{M}X_1.$$

Now, using $f_2 h_2 = e_2 f_1$, $f_0 h_1 = e_1 f_1$, we have

$$\tilde{f}_1 \tilde{e}_2 (\tilde{f}_2)^{-1} \tilde{d}_2 K = \tilde{f}_1 \tilde{e}_1 (\tilde{f}_0)^{-1} \tilde{d}_0 K \in \mathcal{M}X_1,$$

Using (4.2.3), we can suppress $\tilde{f}_1$:

$$\tilde{e}_2 (\tilde{f}_2)^{-1} \tilde{d}_2 K = \tilde{e}_1 (\tilde{f}_0)^{-1} \tilde{d}_0 K \in \mathcal{M}Y_1.$$

We now apply $(g_1)_!$ to both sides of this equality and use (4.2.1), (4.2.2); we get

$$\tilde{\pi}'(g_2)_!(\tilde{f}_2)^{-1}\tilde{d}_2K=\tilde{\pi}(g_0)_!(\tilde{f}_0)^{-1}\tilde{d}_0K,$$

hence (b) is proved.

By assumption, $(g_0)_!\tilde{f}_0\tilde{d}_0K\in\mathscr{M}L$, hence the left-hand side of (b) is in $\mathscr{M}V_1$. By (b), we have $\tilde{\pi}'K'\in\mathscr{M}V_1$ and from (1.8.1) it follows that $K'\in\mathscr{M}V_2$. This completes the proof.

4.3. Let Σ be a subset of L which is the inverse image under $L\to L\mathscr{Z}_L^0$ of an isolated conjugacy class of $L/\mathscr{Z}_L^0$ and let $\mathscr{E}$ be a $\bar{\mathbf{Q}}_l$-local system on Σ which is equivariant for the action of $L\times\mathscr{Z}_L^0$ on Σ defined by $(l,z)\colon\sigma\to z^n lzl^{-1}$; $n\geqslant 1$ is a fixed integer invertible in k. Then $\mathrm{IC}(\bar{\Sigma},\mathscr{E})[d]$, $(d=\dim\Sigma)$ is an L-equivariant perverse sheaf on L. The following result is proved in [4, 4.5]:

(4.3.1) $\mathrm{ind}_P^G(\mathrm{IC}(\bar{\Sigma},\mathscr{E})[d])$ *is a perverse sheaf on* G; *it is a direct sum of irreducible perverse sheaves with support* $\bar{Y}_{(L,\Sigma)}$, (*see* 3.11).

Now, using 3.12, we deduce:

(4.3.2) *If* $A_1\in\hat{L}$ *is cuspidal, then* $\mathrm{ind}_P^G A_1\in\mathscr{M}G$ *and is semisimple.*

We can now state

THEOREM 4.4. (a) *For any* $A\in\hat{G}$, *there exists* $L\subset P$ *as in* 3.1 *and* $A_1\in\hat{L}^{(0)}$ *such that* A *is a direct summand of* $\mathrm{ind}_P^G A_1$.

(b) *If* $L\subset P$ *is as in* 3.1, *and* $A_1\in\hat{L}$, *then* $\mathrm{ind}_P^G A_1\in\mathscr{M}G$.

(c) *If* $L\subset P$ *is as in* 3.1, *and* $A\in\hat{G}$, *then* $\mathrm{res}\,A\in\mathscr{D}L^{\leqslant 0}$.

(d) *If* $L\subset P$ *is as in* 3.1, $A\in\hat{G}$ *and* $A_1\in\hat{L}$, *then*

$$\mathrm{Hom}_{\mathscr{D}L}(\mathrm{res}\,A,A_1)\cong\mathrm{Hom}_{\mathscr{D}G}(A,\mathrm{ind}\,A_1).$$

When G is a maximal torus, the theorem is obvious. Assume now that G is not a torus and that the theorem is already proved for G replaced by L for any $L\subset P$ as in 3.1, with $P\neq G$. We shall prove the theorem for G itself, in 4.5–4.6, using this inductive assumption.

4.5. We first show that

(4.5.1) 4.4(b) holds for G.

Indeed it is enough to check 4.4(b) in the case where $P\neq G$. Then (a) applies to L, hence there exists $M\subset Q$ in L as in 4.2 and $A_2\in\hat{M}^{(0)}$ such that A_1 is a direct summand of $\mathrm{ind}_Q^L(A_2)$ (which is in $\mathscr{M}L$ by (4.3.2)). By

4.2, we have $\mathrm{ind}_P^G(\mathrm{ind}_Q^L(A_2)) = \mathrm{ind}_{\tilde{Q}}^G(A_2)$ $(\tilde{Q} = QU_P)$. Again using (4.3.2), we have $\mathrm{ind}_{\tilde{Q}}^G(A_2) \in \mathscr{M}G$. Hence $\mathrm{ind}_P^G(\mathrm{ind}_Q^L A_2) \in \mathscr{M}G$. Since A_1 is a direct summand of $\mathrm{ind}_Q^L(A_2)$ and ind_P^G takes direct sums to direct sums, it follows that $\mathrm{ind}_P^G(A_1)$ is a direct summand (in $\mathscr{D}G$) of an object in $\mathscr{M}G$. This clearly forces $\mathrm{ind}_P^G(A_1)$ to be in $\mathscr{M}G$, as required.

4.6. Consider the commutative diagram

$$\begin{tikzcd}
V_2 \arrow[r, "f_1"] \arrow[d, "f_2"'] & G & \\
D & G \times P \arrow[u, "\zeta"'] \arrow[ul, "\rho"'] \arrow[r, "\zeta'"] \arrow[d, "\theta"] \arrow[dl, "\phi"'] \arrow[dr, "\theta'"] & G \\
D' \arrow[u, "\beta"'] \arrow[r, "\gamma"] & L & P \arrow[l, "\pi_P"'] \arrow[u, "i"']
\end{tikzcd}$$

where

$$V_2 = \{(g, xP) \mid x^{-1}gx \in P\},$$

$$D = \{(x, l) \mid \in G \times L\} \qquad \text{modulo the } P\text{-action}$$

$$p\colon (x, l) \sim (xp^{-1}, \pi_p(p)x\pi_P(p)^{-1}),$$

$$D' = \{(x, l) \in G \times L\},$$

$f_1(g, xP) = g,$

$f_2(g, xP) = (x, \pi_P(x^{-1}gx)),$

β: obvious map,

$\gamma(x, l) = l,$

$\rho(x, p) = (xpx^{-1}, xP),$

$\phi(x, p) = (x, \pi_P(p)),$

$\zeta(x, p) = xpx^{-1},$

$\zeta'(x, p) = p,$

$\theta'(x, p) = p,$

$\theta(x, p) = \pi_P(p).$

Let $A_1 \in \hat{L}$. Then $\tilde{\gamma}A_1$ is P-equivariant for a P-action on D' which makes β a locally trivial principal P-bundle. By (1.9.3), we have $\tilde{\gamma}A_1 = \tilde{\beta}A_1'$, $(A_1' \in \mathscr{M}D)$. We have

(4.6.1) $\mathrm{ind}\, A_1 = (f_1)_! \tilde{f}_2 A_1'.$

(Indeed, it is enough to show that $\tilde{\rho}\tilde{f}_2 A'_1 = \tilde{\theta} A_1$. But $\tilde{\rho}\tilde{f}_2 A'_1 = \tilde{\phi}\tilde{\beta} A'_1 = \tilde{\phi}\tilde{\gamma} A_1 = \tilde{\theta} A_1$.)

Let $A \in \hat{G}$. Define $\operatorname{Res} A = (f_2)_! f_1^* A[\alpha](\alpha) \in \mathscr{D}D$, $(\alpha = \dim U_P)$. We show that

(4.6.2) $\quad \tilde{\gamma}(\operatorname{res} A) = \tilde{\beta}(\operatorname{Res} A)$.

Indeed, we have

$$\tilde{\gamma}(\operatorname{res} A)[-\dim G](-\alpha) = \gamma^*(\pi_P)_! i^* A = \phi_! \theta'^* i^* A = \phi_! \zeta'^* A,$$
$$\tilde{\beta}(\operatorname{Res} A)[-\dim G](-\alpha) = \beta^*(f_2)_! f_1^* A = \phi_! \rho^* f_1^* A = \phi_! \zeta^* A$$

(we have used $\beta^*(f_2)_! = \phi_! \rho^*$, $\gamma^*(\pi_P)_! = \phi_! \theta'^*$ which follow from (1.7.5)). But ζ, $\zeta' : G \times P \to G$ are compositions of $G \times P \hookrightarrow G \times G$ with the maps $G \times G \to G$ given, respectively, by $(g_1, g_2) \to g_1 g_2 g_1^{-1}$, $(g_1, g_2) \to g_2$. Hence, the G-equivariance of A implies $\zeta^* A = \zeta'^* A$, so that $\phi_! \zeta'^* A = \phi_! \zeta^* A$ and (4.6.2) follows.

Next, we show that, for any integer i, we have

(4.6.3) $\quad \operatorname{Hom}_{\mathscr{D}D}(\operatorname{Res} A, A'_1[i]) \cong \operatorname{Hom}_{\mathscr{D}G}(A, \operatorname{ind} A_1[i])$

Indeed, the left-hand side is

$$\begin{aligned}
&\operatorname{Hom}((f_2)_! f_1^* A[\alpha](\alpha), A'_1[i]) \\
&\quad = \operatorname{Hom}(f_1^* A[\alpha](\alpha), f_2^! A'_1[i]) \qquad \text{(by (1.7.2))} \\
&\quad = \operatorname{Hom}(f_1^* A[\alpha](\alpha), f_2^* A'_1[2\alpha + i](\alpha)) \qquad \text{(by (1.7.4))} \\
&\quad = \operatorname{Hom}(f_1^* A, f_2^* A'_1[\alpha + i]).
\end{aligned}$$

The right-hand side in (4.6.3) is

$$\begin{aligned}
\operatorname{Hom}(A, (f_1)_! \tilde{f}_2 A'_1[i]) &= \operatorname{Hom}(A, (f_1)_* \tilde{f}_2 A'_1[i]) \qquad \text{(by (1.7.3))} \\
&= \operatorname{Hom}(f_1^* A, \tilde{f}_2 A'_1[i]) \qquad \text{(by (1.7.1))} \\
&= \operatorname{Hom}(f_1^* A, f_2^* A'_1[\alpha + i]) \quad \text{(by (1.7.4))};
\end{aligned}$$

and (4.6.3) follows.

If $i < 0$, we have $\operatorname{Hom}_{\mathscr{D}G}(A, \operatorname{ind} A_1[i]) = 0$, since A, $\operatorname{ind} A_1 \in \mathscr{M}G$, (see (4.5.1)). From (4.6.3), it now follows that

(4.6.4) $\quad \operatorname{Hom}_{\mathscr{D}D}(\operatorname{Res} A, A'_1[i]) = 0$ for $i < 0$.

According to 3.9, there exists a sequence $C_1, C_2, \ldots, C_t$ in $\hat{L}$ and a sequence of integers $n_1, n_2, \ldots, n_t$ such that

$$\operatorname{res} A = \bigoplus_{j=1}^{t} C_j[n_j] \qquad \text{in } \mathscr{D}L.$$

Attach $C_j' \in \mathscr{D}D$ to C_j in the same way as A_1' was attached to A_1. We have

$$\begin{aligned}\tilde{\beta}({}^pH^i \operatorname{Res} A) &= {}^pH^i(\tilde{\beta} \operatorname{Res} A) && \text{(see (1.8.1))}\\ &= {}^pH^i(\tilde{\gamma} \operatorname{res} A) && \text{(see (4.6.2))}\\ &= {}^pH^i\left(\tilde{\gamma}\left(\bigoplus_{j=1}^{t} C_j[n_j]\right)\right)\\ &= \bigoplus_{\substack{1 \leqslant j \leqslant t\\ n_j=-i}} \tilde{\gamma} C_j && \text{(see (1.8.1))}\\ &= \bigoplus_{\substack{1 \leqslant j \leqslant t\\ n_j=-i}} \tilde{\beta} C_j'.\end{aligned}$$

Since $\tilde{\beta}$ is fully faithful (1.8.3), we have

(4.6.5) $\quad {}^pH^i \operatorname{Res} A = \bigoplus_{1 \leqslant j \leqslant t, n_j=-i} C_j'$.

Now we show that

(4.6.6) $\quad {}^pH^i \operatorname{Res} A = 0$ for all $i > 0$.

Assume that this is not so; let i be the largest integer such that ${}^pH^i \operatorname{Res} A \neq 0$; then $i > 0$ and there exists a nonzero morphism $\operatorname{Res} A \to {}^pH^i \operatorname{Res} A[-i]$. Now using (4.6.5), we see that there exists a nonzero morphism $\operatorname{Res} A \to C_j'[-i]$ for some $j \in [1, t]$. Since $-i < 0$, this contradicts (4.6.4) with A_1' replaced by C_j'. Thus, (4.6.6) is proved. We can also formulate it as stating that

(4.6.7) $\quad \operatorname{Res} A \in \mathscr{D}D^{\leqslant 0}$.

Applying (1.8.1) to $\tilde{\beta}$, we deduce that $\tilde{\beta}(\operatorname{Res} A) \in \mathscr{D}D'^{\leqslant 0}$. Using (4.6.2), we have then $\tilde{\alpha}(\operatorname{res} A) \in \mathscr{D}D'^{\leqslant 0}$. Applying (1.8.1) to $\tilde{\gamma}$, we deduce that $\operatorname{res} A \in \mathscr{D}L^{\leqslant 0}$. Hence

(4.6.8) $\quad$ 4.4(c) holds for G.

We have

$$\begin{aligned}\operatorname{Hom}(A, \operatorname{ind} A_1) &= \operatorname{Hom}(\operatorname{Res} A, A_1') && \text{(by (4.6.3) with } i=0)\\ &= \operatorname{Hom}(\tilde{\beta} \operatorname{Res} A, \tilde{\beta} A_1') && \text{(by (1.8.2) and (4.6.7))}\\ &= \operatorname{Hom}(\tilde{\gamma} \operatorname{res} A, \tilde{\gamma} A_1) && \text{(by (4.6.2))}\\ &= \operatorname{Hom}(\operatorname{res} A, A_1) && \text{(by (1.8.2) and (4.6.8)).}\end{aligned}$$

(4.6.9) $\quad$ Hence 4.4(d) holds for G.

Finally, we show that 4.4(a) holds for any $A \in \hat{G}$. If A is cuspidal, there is nothing to prove. Thus, we may assume that there exists $L \subset P \neq G$ as in 3.1 such that $\operatorname{res} A[-1] \notin \mathscr{D}L^{\leqslant 0}$, or equivalently, such that ${}^pH^i \operatorname{res} A \neq 0$ for some $i \geqslant 0$. By (4.6.8) we have ${}^pH^i \operatorname{res} A = 0$ for $i > 0$. It follows that ${}^pH^0 \operatorname{res} A \neq 0$ and that there exists a nonzero morphism $\operatorname{res} A \to {}^pH^0 \operatorname{res} A$ (in $\mathscr{D}L$). Since ${}^pH^0 \operatorname{res} A$ is a direct sum of objects in $\hat{L}$ (see 3.9), it follows that there exists $A_1 \in \hat{L}$ and a nonzero morphism $\operatorname{res} A \to A_1$ (in $\mathscr{D}L$). Using (4.6.9), it follows that there exists a nonzero morphism $A \to \operatorname{ind}_P^G A_1$, in $\mathscr{D}G$ (or $\mathscr{M}G$). This must be injective, since A is irreducible. By our inductive assumption, A_1 is a direct summand in $\operatorname{ind}_Q^L(A_2)$ for some $M \subset Q$ as in 4.2 and some $A_2 \in \hat{M}^0$. By transitivity of induction (4.2), $\operatorname{ind}_P^G A_1$ is a direct summand in $\operatorname{ind}_{\tilde{Q}}^G A_2$, where $\tilde{Q} = QU_P$. Hence A is isomorphic to a subobject of $\operatorname{ind}_{\tilde{Q}}^G(A_2)$. By (4.3.2) $\operatorname{ind}_{\tilde{Q}}^G(A_2)$ is a semisimple object of $\mathscr{M}G$, hence A is a direct summand of it. Thus, 4.4(a) holds for G. This completes the proof of Theorem 4.4.

4.7. Let $L \subset P$, U_P, W^*, $W^*_{\mathscr{L}}$, $\mathscr{S}^*$ be as in 3.1, ($\mathscr{L} \in \mathscr{S}(T)$). Let $\mathbf{s} = (s_1, s_2, \ldots, s_r)$ be a sequence in $S^* \cup \{e\}$. We can consider $\mathbf{s}$ also as a sequence in $S \cup \{e\}$. Let δ: $\bar{Y}_{\mathbf{s},L} \to L$, $\bar{\mathscr{L}}_L$, $\delta_!(\bar{\mathscr{L}}_L) = \bar{K}^{\mathscr{L}}_{\mathbf{s},L}$ be defined in terms of $\mathbf{s}$, $\mathscr{L}$, L in the same way as $\bar{\pi}_{\mathbf{s}}$: $\bar{Y}_{\mathbf{s}} \to G$, $\bar{\mathscr{L}}$, $\bar{K}^{\mathscr{L}}_{\mathbf{s}}$ is defined in 2.8 in terms of $\mathbf{s}$, $\mathscr{L}$, G. We shall prove the following result.

Proposition 4.8. (a) *For any i, we have* $\operatorname{ind}_P^G({}^pH^i \bar{K}^{\mathscr{L}}_{\mathbf{s},L}) = {}^pH^{i+d_G-d_L} \bar{K}^{\mathscr{L}}_{\mathbf{s}}$, *where* $d_G = \dim G$, $d_L = \dim L$.

(b) *For any* $A_1 \in \hat{L}_{\mathscr{L}}$, $\operatorname{ind}_P^G(A_1)$ *is semisimple in* $\mathscr{M}G$, *and its irreducible components are in* $\hat{G}_{\mathscr{L}}$.

Proof. We first show that (a) implies (b). If $A_1 \in \hat{L}_{\mathscr{L}}$, we may assume that A_1 is a direct summand of ${}^pH^i(\bar{K}^{\mathscr{L}}_{\mathbf{s},L})$, (2.17(a)). From (a), it follows that $\operatorname{ind}_P^G(A_1)$ is a direct summand of ${}^pH^{i+d_G-d_L}(\bar{K}^{\mathscr{L}}_{\mathbf{s}})$ which is semisimple by 2.17(a); (b) follows.

We now prove (a). Consider the commutative diagram

$$\begin{tikzcd} \bar{Y}_{\mathbf{s},L} \arrow[d, "\delta"'] & V_1 \times_{\mathscr{L}} \bar{Y}_{\mathbf{s},L} \arrow[l, "\mathrm{pr}_2"'] \arrow[r, "\tilde{\pi}'"] \arrow[d, "\mathrm{pr}_1"] & \bar{Y}_{\mathbf{s}} \arrow[d, "\zeta"] & \\ L & V_1 \arrow[l, "\pi"] \arrow[r, "\pi'"'] & V_2 \arrow[r, "\pi''"'] & G \end{tikzcd}$$

whose bottom row is defined in 4.1; ζ is defined by

$$\zeta(g, B_0, B_1, \ldots, B_r) = (g, x_0 P),$$

where $x_0 \in G$ is any element such that $B_0 = x_0 B x_0^{-1}$. The map $\hat{\pi}'$ is defined as follows: for $(g, h) \in V_1$ and $(l, B_0^*, B_1^*, ..., B_r^*) \in \bar{Y}_{\mathbf{s},L}$ such that $\pi_P(h^{-1}gh) = l$, we set

$$\hat{\pi}'((g, h), (l, B_0^*, B_1^*, ..., B_r^*)) = (g, hB_0^* U_P h^{-1}, hB_1^* U_P h^{-1}, ..., hB_r^* U_P h^{-1}).$$

Both squares in the diagram are cartesian and the maps π, π', pr_2, $\hat{\pi}'$ are smooth with connected fibres.

Using (1.7.5) we see that

$$\text{(4.8.1)} \quad \tilde{\pi}\zeta_! = (\mathrm{pr}_1)_! \tilde{\hat{\pi}}', \ \tilde{\pi}\delta_! = (\mathrm{pr}_1)_! \widetilde{\mathrm{pr}}_2.$$

Let $K' \in \mathscr{D}V_2$ be defined by $K' = \zeta_! \bar{\mathscr{L}}$. By the decomposition theorem [1, 6.2.5], K' is semisimple.

From the definitions on $\bar{\mathscr{L}}$, $\bar{\mathscr{L}}_L$ we can check easily that $\mathrm{pr}_2^* \bar{\mathscr{L}}_L = \hat{\pi}'^* \bar{\mathscr{L}}$. It follows that $(\mathrm{pr}_1)_! \mathrm{pr}_2^* \bar{\mathscr{L}}_L = (\mathrm{pr}_1)_! \hat{\pi}'^* \bar{\mathscr{L}}$; using (4.8.1) we obtain $\pi^* \delta_! \bar{\mathscr{L}}_L = \pi'^* \zeta_! \bar{\mathscr{L}}$, hence $\pi^* \bar{K}_{\mathbf{s},L}^{\mathscr{L}} = \pi'^* K'$. We have $\tilde{\pi} = \pi[\frac{3}{2}d_G - \frac{1}{2}d_L]$, $\tilde{\pi}' = \pi'[\frac{1}{2}d_G + \frac{1}{2}d_L]$. It follows that $\tilde{\pi}\bar{K}_{\mathbf{s},L}^{\mathscr{L}} = \tilde{\pi}' K'[d_G - d_L]$.

Applying (1.8.1) to $\tilde{\pi}$ and $\tilde{\pi}'$, we have

$$\begin{aligned}\tilde{\pi}({}^pH^i \bar{K}_{\mathbf{s},L}^{\mathscr{L}}) &= {}^pH^i(\tilde{\pi}\bar{K}_{\mathbf{s},L}^{\mathscr{L}}) = {}^pH^i(\tilde{\pi}' K'[d_G - d_L]) \\ &= \tilde{\pi}'({}^pH^i K'[d_G - d_L]) = \tilde{\pi}'({}^pH^{i + d_G - d_L} K').\end{aligned}$$

By the definition of induction, we have

$$\mathrm{ind}_P^G({}^pH^i \bar{K}_{\mathbf{s},L}^{\mathscr{L}}) = \pi_!''({}^pH^{i + d_G - d_L} K').$$

We have

$$\text{(4.8.2)} \quad \bigoplus_i \mathrm{ind}_P^G({}^pH^i \bar{K}_{\mathbf{s},L}^{\mathscr{L}})[-i] = \bigoplus_i ({}^pH^{i + d_G - d_L} \bar{K}_{\mathbf{s}}^{\mathscr{L}})[-i], \text{ in } \mathscr{D}G.$$

Indeed, the left-hand side is

$$\begin{aligned}\bigoplus_i \pi_!''({}^pH^{i + d_G - d_L} K')[-i] &= \pi_!'' \left(\bigoplus_i ({}^pH^{i + d_G - d_L} K')[-i]\right) \\ &= \pi_!''(K'[d_G - d_L]) \qquad \text{since } K' \text{ is semisimple} \\ &= \pi_!'' \zeta_! \bar{\mathscr{L}}[d_G - d_L] \\ &= \bar{K}_{\mathbf{s}}^{\mathscr{L}}[d_G - d_L],\end{aligned}$$

which is equal to the right-hand side of (4.8.2), since $\bar{K}_{\mathbf{s}}$ is semisimple (2.17(a)). In (4.8.2), we have $\mathrm{ind}_P^G({}^pH^i \bar{K}_{\mathbf{s},L}^{\mathscr{L}}) \in \mathscr{M}G$ for all i; indeed, ${}^pH^i(\bar{K}_{\mathbf{s},L}^{\mathscr{L}})$ is a direct sum of objects of form $A_1 \in \hat{L}$ and for each such A_1, we have $\mathrm{ind}_P^G(A_1) \in \mathscr{M}G$ (see 4.4(b)). Taking ${}^pH^i$ for both sides of (4.8.2), we therefore find $\mathrm{ind}_P^G({}^pH^i\bar{K}_{\mathbf{s},L}^{\mathscr{L}}) = {}^pH^{i+d_G-d_L}\bar{K}_{\mathbf{s}}^{\mathscr{L}}$ and the proposition is proved.

5. Sequences in the Weyl Group

5.1. We fix $\mathscr{L} \in \mathscr{S}(T)$. Besides the notations in 2.3, we shall use the following notation. Let $\Omega_{\mathscr{L}} = \{w \in W'_{\mathscr{L}} \mid w(R^+_{\mathscr{L}}) = R^+_{\mathscr{L}}\}$. Then $W'_{\mathscr{L}}$ is the semidirect product of $\Omega_{\mathscr{L}}$ and $W_{\mathscr{L}}$, with $W_{\mathscr{L}}$ normal.

Let $\tilde{l}$: $W'_{\mathscr{L}} \to \mathbb{N}$ be the function defined by $\tilde{l}(w) = \#\{\alpha \in R^+_{\mathscr{L}} \mid w(\alpha) \in R^-\}$. Then $\tilde{l}$ extends the length function of the Coxeter group $(W_{\mathscr{L}}, S_{\mathscr{L}})$.

5.2. Let $\mathbf{s} = (s_1, s_2, \ldots, s_r)$ be a sequence of elements in $S \cup \{e\}$ such that $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$. When $s_i \neq e$, we shall write α_i for the simple root in R corresponding to s_i. Define

(5.2.1) $\quad I_{\mathbf{s}} = \{i \in [1, r] \mid s_i \neq e, s_r \cdots s_{i+1} s_i s_{i+1} \cdots s_r \in W_{\mathscr{L}}\}$.

We have the following

Lemma 5.3. $|I_{\mathbf{s}}| \geqslant \tilde{l}(s_1 s_2 \cdots s_r)$, *with equality if* $l(s_1 s_2 \cdots s_r) = l(s_1) + \cdots + l(s_r)$.

Proof. Let $X = \{\alpha \in R^+_{\mathscr{L}} \mid (s_1 s_2 \cdots s_r)(\alpha) \in R^-\}$, $X' = \{\alpha \in R^+_{\mathscr{L}} \mid \exists i \in [1, r], s_i \neq e, \alpha = s_r s_{r-1} \cdots s_{i+1}(\alpha_i)\}$. It is clear that $X \subset X'$. We have $|X| = \tilde{l}(s_1 s_2 \cdots s_r)$ hence $\tilde{l}(s_1 s_2 \cdots s_r) \leqslant |X'|$. Let $\phi: I_{\mathbf{s}} \to R_{\mathscr{L}}$ be defined by $\phi(i) = s_r s_{r-1} \cdots s_{i+1}(\alpha_i)$; then $X' = \phi(I_{\mathbf{s}}) \cap R^+_{\mathscr{L}}$. Hence $|X'| \leqslant |\phi(I_{\mathbf{s}})| \leqslant |I_{\mathbf{s}}|$ so that $\tilde{l}(s_1 s_2 \cdots s_r) \leqslant |I_s|$, as required. Assume now that $l(s_1 s_2 \cdots s_r) = l(s_1) + \cdots + l(s_r)$. Then the roots $s_r s_{r-1} \cdots s_{i+1}(\alpha_i)$ $(1 \leqslant i \leqslant r, s_i \neq e)$ are distinct and positive. Hence, for $i \in I_{\mathbf{s}}$, the roots $s_r s_{r-1} \cdots s_{i+1}(\alpha_i)$ are distinct elements of X, so that $|I_{\mathbf{s}}| \leqslant |X|$. It follows that $|I_{\mathbf{s}}| = |X|$.

Lemma 5.4. *Let $J \subset I_{\mathbf{s}}$; we define $\mathbf{s}_J$ to be the sequence $(s'_1, s'_2, \ldots, s'_r)$ with $s'_i = s_i$ for $i \notin J$, $s'_i = e$ for $i \in J$. We have $I_{\mathbf{s}_J} = I_{\mathbf{s}} - J$.*

Proof. Let $h \in I_{\mathbf{s}} - J$. We have $s_r s_{r-1} \cdots s_h \cdots s_{r-1} s_h \in W_{\mathscr{L}}$. Hence if $a_1 > a_2 > \cdots > a_p$ are the indices in $J \cap [h+1, r]$, we have

$$\begin{aligned} &s_r s_{r-1} \cdots \hat{s}_{a_p} \cdots s_h \cdots \hat{s}_{a_p} \cdots s_r \\ &\quad = (s_r s_{r-1} \cdots s_{a_p} \cdots s_{r-1} s_r)(s_r s_{r-1} \cdots s_h \cdots s_{r-1} s_r) \\ &\quad \quad \times (s_r s_{r-1} \cdots s_{a_p} \cdots s_{r-1} s_r) \in W_{\mathscr{L}} \end{aligned}$$

$$\begin{aligned}
&s_r s_{r-1} \cdots \hat{s}_{a_{p-1}} \cdots \hat{s}_{a_p} \cdots s_h \cdots \hat{s}_{a_p} \cdots \hat{s}_{a_{p-1}} \cdots s_{r-1} s_r \\
&\qquad = (s_r s_{r-1} \cdots s_{a_{p-1}} \cdots s_{r-1} s_r)(s_r s_{r-1} \cdots \hat{s}_{a_p} \cdots s_h \cdots \hat{s}_{a_p} \cdots s_{r-1} s_r) \\
&\qquad\quad \times (s_r s_{r-1} \cdots s_{a_{p-1}} \cdots s_{r-1} s_r) \in W_{\mathscr{L}} \\
&\qquad \vdots \\
&s_r s_{r-1} \cdots \hat{s}_{a_1} \cdots \hat{s}_{a_2} \cdots \hat{s}_{a_p} \cdots s_h \cdots \hat{s}_{a_p} \cdots \hat{s}_{a_2} \cdots \hat{s}_{a_1} \cdots s_{r-1} s_r \in W_{\mathscr{L}}.
\end{aligned}$$

(Here, $\hat{}$ stands for an omitted symbol.)

This shows that $h \in I_{\mathbf{s}_J}$. The same computation (in the opposite direction) shows that if $h' \in I_{\mathbf{s}_J}$, then $h' \in I_{\mathbf{s}} - J$.

5.5. We write the elements of $I_{\mathbf{s}}$ in ascending order: $i_1 < i_2 < \cdots < i_a$. Define

$$\begin{aligned}
\sigma_a &= s_r s_{r-1} \cdots s_{i_a} \cdots s_{r-1} s_r, \\
\sigma_{a-1} &= s_r s_{r-1} \cdots \hat{s}_{i_a} \cdots s_{i_{a-1}} \cdots \hat{s}_{i_a} \cdots s_{r-1} s_r, \\
&\vdots \\
\sigma_1 &= s_r s_{r-1} \cdots \hat{s}_{i_a} \cdots \hat{s}_{i_{a-1}} \cdots \hat{s}_{i_2} \cdots s_{i_1} \cdots \hat{s}_{i_2} \\
&\quad \cdots \hat{s}_{i_{a-1}} \cdots \hat{s}_{i_a} \cdots s_{r-1} s_r, \\
\omega &= s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_2} \cdots \hat{s}_{i_a} \cdots s_r.
\end{aligned}$$

PROPOSITION 5.6. (a) *$\sigma_1, \sigma_2, \ldots, \sigma_a \in S_{\mathscr{L}}$ (see 2.3) and $\omega \in \Omega_{\mathscr{L}}$, (see* 5.1).

(b) $s_1 s_2 \cdots s_r = w\sigma_1 \sigma_2 \cdots \sigma_a$.

(c) *More generally, if J is a subset of $I_{\mathbf{s}}$, then*

$$\prod_{\substack{1 \leqslant i \leqslant r \\ i \notin J}} s_i = \omega \prod_{\substack{1 \leqslant j \leqslant a \\ i_j \notin J}} \sigma_j$$

(in both products, the factors are written in ascending order of indices).

(d) *If $l(s_1 s_2 \cdots s_r) = l(s_1) + \cdots + l(s_r)$, then $\sigma_1 \sigma_2 \cdots \sigma_a$ is a reduced expression in $W_{\mathscr{L}}$.*

Proof. We set $h = i_a \in I_{\mathbf{s}}$. Let $\mathbf{s}'$ be the sequence $(s_r, s_{r-1}, \ldots, s_h, \ldots, s_{r-1}, s_r)$. We show that $I_{\mathbf{s}'}$ has a single element. We have $s_r s_{r-1} \cdots s_i \cdots s_{r-1} s_r \in W_{\mathscr{L}}$. But $s_r s_{r-1} \cdots s_{h+1} \cdots s_{r-1} s_r \notin W_{\mathscr{L}}$, $s_r s_{r-1} \cdots s_{h+2} \cdots s_{r-1} s_r \notin W_{\mathscr{L}}, \ldots$, since h is the largest index in $I_{\mathbf{s}}$. Hence the middle term in $\mathbf{s}'$ has an index in $I_{\mathbf{s}'}$ but all the terms following it have an index outside $I_{\mathbf{s}'}$. We now show that the term in $\mathbf{s}'$ immediately preceding the middle term has an index outside $I_{\mathbf{s}'}$. If this is not so, we would have $s_r s_{r-1} \cdots$

$s_{h+1}s_hs_{h+1}s_hs_{h+1}\cdots s_{r-1}s_r \in W_{\mathscr{L}}$. Multiplying on the left and right by $s_rs_{r-1}\cdots s_h \cdots s_{r-1}s_r$, we find that $s_rs_{r-1}\cdots s_{h+1}\cdots s_{r-1}s_r \in W_{\mathscr{L}}$, a contradiction. Similarly, we see that all terms in $\mathbf{s}'$, preceding the middle term have an index outside $I_{\mathbf{s}'}$. Thus, $I_{\mathbf{s}'}$ has a single element. By 5.3, we have $\tilde{l}(\sigma_a)=\tilde{l}(s_rs_{r-1}\cdots s_h\cdots s_{r-1}s_r)\leqslant 1$. Since σ_a has odd length in W, it must be $\neq e$ hence $\tilde{l}(\sigma_a)=1$. We have $\sigma_a \in W_{\mathscr{L}}$, hence $\sigma_a \in S_{\mathscr{L}}$.

We now prove (a) and (b) by induction on a. Assume first that $a=0$. By 5.3, we have $\tilde{l}(s_1s_2\cdots s_r)=0$, hence $s_1s_2\cdots s_r \in \Omega_{\mathscr{L}}$ and (a), (b) are clear. Assume now that $a \geqslant 1$ and that (a), (b) are proved for $a-1$. Consider the sequence $\mathbf{s}_J$, where $J=\{i_a\}$, (see 5.4). By 5.4, we have $I_{\mathbf{s}_J}=I_{\mathbf{s}}-\{i_a\}$. The induction hypothesis applies to $\mathbf{s}_J$. It follows that $\sigma_{a-1},\dots,\sigma_1 \in S_{\mathscr{L}}$ and $\omega \in \Omega_{\mathscr{L}}$. We have checked already that $\sigma_a \in S_{\mathscr{L}}$. Hence (a) for $\mathbf{s}$ follows. The induction hypothesis shows that $s_1\cdots \hat{s}_{i_a}\cdots s_r=\omega\sigma_1\sigma_2\cdots\sigma_{a-1}$. It follows that $s_1s_2\cdots s_r=(s_1s_2\cdots\hat{s}_{i_a}\cdots s_r)(s_r\cdots s_{i_a}\cdots s_r)=\omega\sigma_1\sigma_2\cdots\sigma_{a-1}\sigma_a$ hence (b) for $\mathbf{s}$ follows. The more general statement (c) follows from (b) using 5.4. Statement (d) follows from 5.3.

5.7. Let $\mathbf{s}=(s_1,s_2,\dots,s_r)$, $\mathbf{s}'=(s_1',s_2',\dots,s_{r'}')$ be two sequences in $S\cup\{e\}$ such that $s_1s_2\cdots s_r\in W_{\mathscr{L}}'$, $s_1's_2'\cdots s_{r'}'\in W_{\mathscr{L}}'$. Let ω, $\sigma_{i_1},\sigma_{i_2}\cdots\sigma_{i_a}$ be the elements attached to $\mathbf{s}$ in 5.5, let ω', $\sigma_{j_1},\sigma_{j_2},\dots,\sigma_{j_{a'}}$ be the elements defined in the same way for $\mathbf{s}'$ instead of $\mathbf{s}$, and let ω'', $\sigma_{h_1},\sigma_{h_2},\dots,\sigma_{h_{a''}}$ be the elements defined in the same way for the sequence

$$\mathbf{s}\mathbf{s}'=(s_1,s_2,\dots,s_r,s_1',s_2',\dots,s_{r'}').$$

Then, we have

$$a''=a+a',\qquad \omega''=\omega\omega',\qquad \sigma_{h_1}=\omega'^{-1}\sigma_{i_1}\omega',\dots,\sigma_{h_a}=\omega'^{-1}\sigma_{i_a}\omega',$$

$$\sigma_{h_{a+1}}=\sigma_{j_1},\dots,\sigma_{h_{a+a'}}=\sigma_{j_{a'}}.$$

(This follows easily from the definitions.)

5.8. We let $S^*\subset S$, $W^*\subset W$, $R^*\subset R$, be as in 3.1. The statements (5.8.1), (5.8.2) below are well known.

(5.8.1) *Any coset $W^*y\subset W$ contains a unique element of minimal length y_0; it is characterized by the property $y_0^{-1}(R^*\cap R^+)\subset R^+$.*

(5.8.2) *Any coset $zW_{\mathscr{L}}\subset W$ contains a unique element of minimal length z_0; it is characterized by the property $z_0(R^+_{\mathscr{L}})\subset R^+$.*

(5.8.3) *Let $w\in W$, $s\in S$ be such that w has minimal length in $wW_{\mathscr{L}}$. Then either* (a) *sw has minimal length in $swW_{\mathscr{L}}$ or* (b) *$w^{-1}sw\in W_{\mathscr{L}}$.*

Indeed, assume that (a) does not hold, so that there exist $\alpha\in R^+_{\mathscr{L}}$ such that $sw(\alpha)<0$. By our assumption we have $w(\alpha)>0$. Hence $w(\alpha)$ must be the simple root α_s corresponding to s. Thus, $w^{-1}(\alpha_s)=\alpha\in R_{\mathscr{L}}$, hence $w^{-1}sw\in W_{\mathscr{L}}$, hence (b) holds. This proves (5.8.3).

(5.8.4) *Let $w, w'\in W$ be such that $W^*wW_{\mathscr{L}}=W^*w'W_{\mathscr{L}}$. Assume that*

(a) *w has minimal length in $wW_{\mathscr{L}}$* and also in W^*w,

(b) *w' has minimal length in $w'W_{\mathscr{L}}$* and also in W^*w'.

Then $w=w'$.

Indeed, our assumption implies that there exist $s_1, s_2,..., s_t\in S^*$, such that $s_1s_2\cdots s_tw\in w'W_{\mathscr{L}}$. Assume that there exists $i\in[2,t]$, such that $s_i\cdots s_tw$ has minimal length in $s_i\cdots s_twW_{\mathscr{L}}$ and $s_{i-1}s_i\cdots s_tw$ does not have minimal length in $s_{i-1}s_i\cdots s_tW_{\mathscr{L}}$. Then by (5.8.3), we have $s_{i-1}s_i\cdots s_tw=s_is_{i+1}\cdots s_tw\sigma$ for some $\sigma\in W_{\mathscr{L}}$. Hence $s_1\cdots s_{i-2}s_i\cdots s_tw=s_1\cdots s_tw\sigma\in w'W_{\mathscr{L}}\sigma=w'W_{\mathscr{L}}$. Iterating this, we are reduced to the case where for all $i\in[1,t]$, $s_i\cdots s_tw$ has minimal length in $s_i\cdots s_twW_{\mathscr{L}}$. In particular, $s_1s_2\cdots s_tw$ has minimal length in $s_1s_2\cdots s_twW_{\mathscr{L}}=w'W_{\mathscr{L}}$. Since w' has also minimal length in $w'W_{\mathscr{L}}$, we have $s_1s_2\cdots s_tw=w'$, by (5.8.2). Thus, $W^*w=W^*w'$. Since w, w' both have minimal length in $W^*w=W^*w'$, we have $w=w'$, by (5.8.1). This proves (5.8.4). We can now state:

PROPOSITION 5.9. *Any double coset $W^*yW_{\mathscr{L}}$ contains a unique element y_0 of minimal length. It is characterized by the property: $y_0^{-1}(R^*\cap R^+)\subset R^+$ and $y_0(R^+_{\mathscr{L}})\subset R^+$.*

Proof. The existence of an element y_0 of minimal length in $W^*yW_{\mathscr{L}}$ is obvious. It is clear that y_0 must have minimal length in $y_0W_{\mathscr{L}}$ and also in W^*y_0 the proposition follows from (5.8.4), (5.8.1), (5.8.2).

6. HECKE ALGEBRAS

6.1. We fix $\mathscr{L}\in S(T)$. Let $\mathscr{A}=\mathbb{Z}[u^{1/2},u^{-1/2}]$, where u is an indeterminate.

Let $H'_{\mathscr{L}}$ be the Hecke algebra (over $\mathscr{A}$) corresponding to $W'_{\mathscr{L}}$; it is a free $\mathscr{A}$-module with basis T_w ($w\in W'_{\mathscr{L}}$). The multiplication is characterized by

$$T_wT_{w'}=T_{ww'}\qquad\text{if}\quad w, w'\in W'_{\mathscr{L}}\text{ satisfy }\tilde{l}(w)+\tilde{l}(w')=\tilde{l}(ww')$$

$$(T_\sigma+1)(T_\sigma-u)=0\qquad\text{if}\quad \sigma\in S_{\mathscr{L}}.$$

(Recall that $\tilde{l}$ is defined in 5.1.)

Let $\tilde{H}'_{\mathscr{L}}$ be te free $\mathscr{A}$-module with basis $e_{\mathbf{s}}$ indexed by the sequences $\mathbf{s} = (s_1, s_2, ..., s_r)$ in $S \cup \{e\}$ $(r \geqslant 1)$, such that $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$.

Let Γ_G be the abelian group with generators $[A]$ (corresponding to the various isomorphism classes of objects A in $\mathscr{M}G$) and relations $[A] + [A'] = [A \oplus A']$ for any two objects $A, A' \in \mathscr{M}G$.

Define an $\mathscr{A}$-linear map $\gamma: \tilde{H}'_{\mathscr{L}} \to \Gamma_G \otimes_{\mathbb{Z}} \mathscr{A}$ by $\gamma(e_{\mathbf{s}}) = \sum_{j \in \mathbb{Z}} (-1)^j [{}^pH^j(\bar{K}^{\mathscr{L}}_{\mathbf{s}})] \otimes u^{j/2}$.

Define an $\mathscr{A}$-linear map $\delta: \tilde{H}'_{\mathscr{L}} \to H'_{\mathscr{L}}$ by $\delta(e_{\mathbf{s}}) = T_\omega(1 + T_{\sigma_1})(1 + T_{\sigma_2}) \cdots (1 + T_{\sigma_a})\, u^{(m - a + \dim G)/2}$, where $\omega, \sigma_1, \sigma_2, ..., \sigma_a$ are the elements of $W'_{\mathscr{L}}$ associated to $\mathbf{s} = (s_1, s_2, ..., s_r)$ in 5.6, and $m = \#\{i \in [1, r] \mid s_i \neq e\}$. With these definitions, we can state

PROPOSITION 6.2. (a) *There is a unique $\mathscr{A}$-linear map $\varepsilon: H'_{\mathscr{L}} \to \Gamma_G \otimes_{\mathbb{Z}} \mathscr{A}$ such that the diagram*

$$\begin{tikzcd} \tilde{H}'_{\mathscr{L}} \arrow[rr, "\delta"] \arrow[dr, "\gamma"'] & & H'_{\mathscr{L}} \arrow[dl, "\varepsilon"] \\ & \Gamma_G \otimes_{\mathbb{Z}} \mathscr{A} & \end{tikzcd}$$

is commutative.

(b) *We have $\varepsilon(h_1 h_2) = \varepsilon(h_2 h_1)$ for all $h_1, h_2 \in H'_{\mathscr{L}}$.*

(c) *Let $\bar{\ }: H'_{\mathscr{L}} \to H'_{\mathscr{L}}$ be the ring involution defined by $T_w \to T^{-1}_{w^{-1}}$, $(w \in W'_{\mathscr{L}})$ and $\overline{u^{j/2}} = u^{-j/2}$. Let $\bar{\ }: \underline{\Gamma_G \otimes_{\mathbb{Z}} \mathscr{A}}$ be the group involution defined by $[A] \otimes u^{j/2} \to [A] \otimes u^{-j/2}$. Then $\varepsilon(\bar{h}) = \overline{\varepsilon(h)}$ for all $h \in H'_{\mathscr{L}}$.*

First, note that δ is surjective. Indeed, given $\omega \in \Omega_{\mathscr{L}}$, and a sequence $\sigma_1, \sigma_2, ..., \sigma_a$ in $S_{\mathscr{L}}$, we consider reduced expressions $\omega = t_1 t_2 \cdots t_p$, $\sigma_j = \tau_{j1} \tau_{j2} \cdots \tau_{jr_j} \cdots \tau_{j2} \tau_{j1}$ $(1 \leqslant j \leqslant a)$ in S, and let $\mathbf{s}$ be the sequence

$$(t_1, t_2, ..., t_p, \tau_{11}, \tau_{12}, ..., \tau_{1r_1}, ..., \tau_{11}, \tau_{21}, \tau_{22}, ..., \\ \tau_{2r_2}, ..., \tau_{22}, ..., \tau_{a1}, \tau_{a2}, ..., \tau_{ar_a}, ..., \tau_{a2}, \tau_{a1})$$

in S. It is easy to see that $\delta(\mathbf{s}) = T_\omega(1 + T_{\sigma_1}) \cdots (1 + T_{\sigma_a})$; these elements clearly generate $H'_{\mathscr{L}}$ as an $\mathscr{A}$-module, so that δ is surjective.

It follows that ε is unique (if it exists). Assume that (a) is already proved. To prove (c), it is enough in view of surjectivity of δ to show that $\varepsilon(\overline{\delta e_{\mathbf{s}}}) = \overline{\varepsilon(\delta e_{\mathbf{s}})}$ for all basis elements $e_{\mathbf{s}}$ of $\tilde{H}'_{\mathscr{L}}$. We have $\overline{\delta e_{\mathbf{s}}} = u^{-(m + \dim G)} \delta e_{\mathbf{s}}$ since $\overline{1 + T_{\sigma_i}} = u^{-1}(1 + T_{\sigma_i})$, $\bar{T}_\omega = T_\omega$. Hence, we must check that $u^{-m'} \varepsilon(\delta e_{\mathbf{s}}) = \overline{\varepsilon(\delta e_{\mathbf{s}})}$, $(m' = m + \dim G)$ or, equivalently, that $u^{-m'} \gamma(e_{\mathbf{s}}) = \overline{\gamma(e_{\mathbf{s}})}$. This is equivalent to the statement 2.17(b). Thus, (c) follows from (a).

It remains to prove (a) and (b). The statement (a) is a consequence of the following statement:

(6.2.1) Let $(e_{\mathbf{s}_1}, e_{\mathbf{s}_2},..., e_{\mathbf{s}_t})$, $(e_{\mathbf{s}'_1}, e_{\mathbf{s}'_2},..., e_{\mathbf{s}'_{t'}})$ be two sequences of basis elements of $\tilde{H}'_{\mathscr{L}}$ and let $(n_1, n_2,..., n_t)$, $(n'_1, n'_t,..., n'_{t'})$ be two sequences of integers. Assume that

$$\sum_{i=1}^{t} e_{\mathbf{s}_i} \otimes u^{n_i/2} - \sum_{i=1}^{t'} e_{\mathbf{s}'_i} \otimes u^{n'_i/2}$$

is in the kernel of $\delta: \tilde{H}'_{\mathscr{L}} \to H'_{\mathscr{L}}$. Then for any integer j, the perverse sheaves $\bigoplus_{i=1}^{t} {}^pH^{j-n_i}(\bar{K}^{\mathscr{L}}_{\mathbf{s}_i})$ and $\bigoplus_{i=1}^{t'} {}^pH^{j-n'_i}(\bar{K}^{\mathscr{L}}_{\mathbf{s}'_i})$ are isomorphic in $\mathscr{M}G$.

By general principles [1, Sect. 6], the statement (6.2.1) for general k is a consequence of the statement (6.2.1) for k an algebraic closure of a finite field. The same applies to (b). Thus, it is enough to prove (6.2.1) and (b) in the special case where k is an algebraic closure of a finite field.

6.3. We now prove (6.2.1) under the assumption that k is an algebraic closure of a finite field. Since the two perverse sheaves in (6.2.1) are semisimple (2.17(a)), they are isomorphic if and only if they define the same element of the Grothendieck group $\mathscr{K}G$ of the abelian category $\mathscr{M}G$. Hence, if $\rho: \Gamma_G \to \mathscr{K}G$ is the natural homomorphism, it is enough to prove that there exists an $\mathscr{A}$-linear map $\varepsilon': \mathscr{H}'_{\mathscr{L}} \to \mathscr{K}(G) \otimes_{\mathbf{Z}} \mathscr{A}$ such that

(6.3.1) $\varepsilon'\delta = (\rho \otimes 1)\gamma.$

We may regard $K^{\mathscr{L}}_w$, $K^{\mathscr{L}}_{\mathbf{w}}$, $\bar{K}^{\mathscr{L}}_{\mathbf{s}}$ as well as the complexes and morphisms appearing in 2.13–2.16 (for fixed $\mathscr{L}$) as being in the derived category of mixed complexes over G_0 (a split F_q-form of G) with B, T defined over F_q, for a sufficiently large $F_q \subset k$. Then the ${}^pH^i$ of these complexes will have natural weight filtrations (see [1, 5.3.5]) whose subquotients (denoted ${}^pH^i_j$) are pure perverse sheaves of weight j. For any mixed complex K on G_0, we define

$$\chi_u(K) = \sum_{i,j} (-1)^i \{{}^pH^i_j(K)\} \otimes u^{j/2} \in \mathscr{K}(G) \otimes_{\mathbf{Z}} \mathscr{A}.$$

Here $\{{}^pH^i_j(K)\}$ denotes the image of ${}^pH^i_j(K)$ in the Grothendieck group $\mathscr{K}G$.

We define an $\mathscr{A}$-linear map $\varepsilon': H'_{\mathscr{L}} \to \mathscr{K}(G) \otimes_{\mathbf{Z}} \mathscr{A}$ by

(6.3.2) $\varepsilon'(T_w) = \chi_u(K^{\mathscr{L}}_w)\, u^{(-l(w)+\bar{l}(w)-\dim G)/2}.$

Let $\mathbf{s} = (s_1,..., s_r)$ be a sequence in $S \cup \{e\}$ such that $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$, and let $\omega, \sigma_1, \sigma_2,..., \sigma_a$ be the elements of $W'_{\mathscr{L}}$ associated to $\mathbf{s}$ in 5.5. We shall prove by induction on $m = \#\{i \in [1, r] \mid s_i \neq e\}$ that

(6.3.3) $\chi_u(K^{\mathscr{L}}_{\mathbf{s}}) = u^{(m-a+\dim G)/2} \varepsilon'(T_\omega T_{\sigma_1} T_{\sigma_2} \cdots T_{\sigma_a}).$

We can assume that all s_i are in S by dropping the ones which are e. Then $m=r$.

When $m=0$, we have $K_{\mathbf{s}}^{\mathscr{L}}=K_e^{\mathscr{L}}$, $a=0$, and (6.3.3) follows from (6.3.2).

Assume now that $m \geqslant 1$ and that (6.3.3) is already known for sequences m replaced by $m' < m$.

Assume first that $l(s_1 s_2 \cdots s_r)=r$ so that $K_{\mathbf{s}}^{\mathscr{L}}=K_w^{\mathscr{L}}$, where $w=s_1 s_2 \cdots s_r$ (see 2.11, (2.5.1)). By 5.6, we have $\tilde{l}(\omega)=0$, $\tilde{l}(\sigma_1)=\cdots=\tilde{l}(\sigma_a)=1$, $\tilde{l}(\omega\sigma_1\sigma_2\cdots\sigma_a)=a$, hence $T_\omega T_{\sigma_1} T_{\sigma_2} \cdots T_{\sigma_a}=T_{\omega\sigma_1\sigma_2\cdots\sigma_a}=T_{s_1 s_2\cdots s_r}=T_w$ so that

$$u^{(m-a+\dim G)/2}\varepsilon'(T_\omega T_{\sigma_1} T_{\sigma_2}\cdots T_{\sigma_a})=u^{(l(w)-\tilde{l}(w)+\dim G)/2}\varepsilon'(T_w)$$
$$=\chi_u(K_w^{\mathscr{L}})$$
$$=\chi_u(K_{\mathbf{s}}^{\mathscr{L}})$$

as required.

Assume next that $l(s_1 s_2 \cdots s_r)<r$. Then we can find h $(2 \leqslant h \leqslant r)$ such that $s_h \cdots s_{r-1} s_r$ is a reduced expression and $s_{h-1} s_h \cdots s_r$ is not a reduced expression. We can find $s'_h,\ldots,s'_{r-1},s'_r$ in S such that $s'_h \cdots s'_{r-1} s'_r = s_h \cdots s_{r-1} s_r$ and $s'_h=s_{h-1}$.

Let $\boldsymbol{\sigma}=(s_1,s_2,\ldots,s_{h-1},s'_h,\ldots,s'_{r-1},s'_r)$. As shown in 2.16, we have $K_{\mathbf{s}}^{\mathscr{L}}=K_{\boldsymbol{\sigma}}^{\mathscr{L}}$; hence $\chi_u(K_{\mathbf{s}}^{\mathscr{L}})=\chi_u(K_{\boldsymbol{\sigma}}^{\mathscr{L}})$. The definition 5.5 of ω, $\sigma_1,\ldots,\sigma_a$ attached to $\mathbf{s}$ can be also applied to $\boldsymbol{\sigma}$ instead of $\mathbf{s}$, and it leads to the same sequence $\omega,\sigma_1,\ldots,\sigma_a$. Hence to prove (6.3.3) for $\mathbf{s}$ it is enough to prove it for $\boldsymbol{\sigma}$. Thus, we are reduced to the case where $\mathbf{s}$ satisfies $s_{h-1}=s_h$. In this case. we shall use the notations in 2.15.

If $h \notin I_{\mathbf{s}}$, then from (2.15.4) we have $\chi_u(K_{\mathbf{s}}^{\mathscr{L}})=u\cdot\chi_u(K_{\mathbf{s}''}^{\mathscr{L}})$. By the induction hypothesis, we have $\chi_u(K_{\mathbf{s}''}^{\mathscr{L}})=u^{(r-2-a+\dim G)/2}\varepsilon'(T_\omega T_{\sigma_1}\cdots T_{\sigma_a})$ hence $\chi_u(K_{\mathbf{s}}^{\mathscr{L}})=u^{(r-a+\dim G)/2}\varepsilon'(T_\omega T_{\sigma_1}\cdots T_{\sigma_a})$.

If $h \in I_{\mathbf{s}}$, then from (2.15.2) we have

$$\chi_u(K_{\mathbf{s}}^{\mathscr{L}})=\chi_u((\pi'_{\mathbf{s}})_!\tilde{\mathscr{L}})+u\chi_u(K_{\mathbf{s}''}^{\mathscr{L}})$$

and from (2.15.3) we have

$$u\chi_u(K_{\mathbf{s}'}^{\mathscr{L}})=\chi_u((\pi'_{\mathbf{s}})_!\tilde{\mathscr{L}})+\chi_u(K_{\mathbf{s}'}^{\mathscr{L}}).$$

(Indeed, since weight filtrations are strictly compatible with morphisms [1, 5.3.5] the exact sequences (2.15.2), (2.15.3) remain exact when each ${}^pH^i$ is replaced by ${}^pH^i_j$ for fixed j.) It follows that

$$\chi_u(K_{\mathbf{s}}^{\mathscr{L}})=u\chi_u(K_{\mathbf{s}''}^{\mathscr{L}})+(u-1)\chi_u(K_{\mathbf{s}'}^{\mathscr{L}}).$$

The induction hypothesis is applicable to $K_{\mathbf{s}'}^{\mathscr{L}}, K_{\mathbf{s}''}^{\mathscr{L}}$:

$$\chi_u(K_{\mathbf{s}'}^{\mathscr{L}}) = u^{((r-1)-(a-1)+\dim G)/2}\varepsilon'(T_\omega T_{\sigma_1} \cdots T_{\sigma_{h-1}} \hat{T}_{\sigma_h} \cdots T_{\sigma_a}),$$
$$\chi_u(K_{\mathbf{s}''}^{\mathscr{L}}) = u^{((r-2)-(a-2)+\dim G)/2}\varepsilon'(T_\omega T_{\sigma_1} \cdots \hat{T}_{\sigma_{h-1}} \hat{T}_{\sigma_h} \cdots T_{\sigma_a}).$$

Moreover, we have $\sigma_{h-1} = \sigma_h$ so that $uT_{\sigma_{h-1}} + (u-1)1 = T_{\sigma_{h-1}} T_{\sigma_h}$. Hence

$$\chi_u(K_{\mathbf{s}}^{\mathscr{L}}) = u^{(r-a+\dim G)/2}\varepsilon'(T_\omega T_{\sigma_1} \cdots T_{\sigma_a}),$$

as required. Thus, (6.3.3) is proved.

We now prove that with the notation in (6.3.3), we have

$$(6.3.4) \quad \chi_u(\bar{K}_{\mathbf{s}}^{\mathscr{L}}) = u^{(m-a+\dim G)/2}\varepsilon'(T_\omega(1+T_{\sigma_1})(1+T_{\sigma_2}) \cdots (1+T_{\sigma_a})).$$

We shall use the notation in 2.13. From (2.13.1) (or rather, from the corresponding exact sequences obtained by considering the subquotients of fixed weight of the weight filtrations), we get

$$\chi_u((\bar{\pi}_{\mathbf{s}})_! \, \phi_!^{(i)}(\phi^{(i)})^* \bar{\mathscr{L}})$$
$$= \chi_u((\bar{\pi}_{\mathbf{s}})_! \, \phi_!^{(i+1)}(\sigma^{(i+1)})^* \bar{\mathscr{L}}) + \sum_{\substack{J \subset I_{\mathbf{s}} \\ |J| = i}} \chi_u(K_{\mathbf{s}_J}^{\mathscr{L}})$$

for any i.

Summing these equalities over all i, $0 \leqslant i \leqslant |I_s|$ and taking into account (2.13.2), we find

$$\chi_u(\bar{K}_{\mathbf{s}}^{\mathscr{L}}) = \sum_{J \subset I_{\mathbf{s}}} \chi_u(K_{\mathbf{s}_J}^{\mathscr{L}}).$$

We now use (6.3.3) for each $\mathbf{s}_J$ is the last sum, and 5.6(c); (6.3.4) follows.

The mixed complex $\bar{K}_s^{\mathscr{L}}$ is pure of weight 0 (see the proof of 3.7) hence

$${}^pH_j^i(\bar{K}_{\mathbf{s}}^{\mathscr{L}}) = \begin{cases} {}^pH^i(\bar{K}_s^{\mathscr{L}}) & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

It follows that $\chi_u(\bar{K}_s^{\mathscr{L}}) = (\rho \otimes 1)\gamma(e_{\mathbf{s}})$. On the other hand, the right-hand side of (6.3.4) is equal to $\varepsilon'(\delta e_{\mathbf{s}})$. Hence (6.3.4) implies (6.3.1). This completes the proof of 6.2(a).

6.4. We shall now prove 6.2(b) assuming again (without loss of generality) that k is an algebraic closure of a finite field. We again place ourselves in the setup of 6.3. It is enough to prove the following statement:

(6.4.1) Let $\mathbf{s} = (s_1, s_2, ..., s_r)$, $\mathbf{s}' = (s_1', s_2', ..., s_{r'}')$ be two sequences in S as in 5.7 and let $(\omega, \sigma_1, \sigma_2, ..., \sigma_a)$, $(\omega', \sigma_1', \sigma_2', ..., \sigma_{a'}')$ be the sequence in $W_{\mathscr{L}}'$

attached to them in 5.5 Then $\varepsilon'((T_\omega T_{\sigma_1}\cdots T_{\sigma_a})(T_{\omega'}T_{\sigma'_1}\cdots T_{\sigma'_a}))=\varepsilon'((T_{\omega'}T_{\sigma'_1}\cdots T_{\sigma'_a})(T_\omega T_{\sigma_1}\cdots T_{\sigma_a}))$.

Let $\mathbf{ss}'$, $\mathbf{s}'\mathbf{s}$ be defined as in (2.19.1). Using 5.7 and (6.3.3) we see that the equality (6.4.1) is equivalent to the equality

$$\chi_u(K^{\mathscr{L}}_{\mathbf{ss}'})=\chi_u(K^{\mathscr{L}}_{\mathbf{s}'\mathbf{s}}).$$

But this follows from (2.19.1).

This completes the proof of Proposition 6.2.

6.5. Let us define for any $K\in\mathscr{D}G$,

$$\chi(K)=\sum_i(-1)^i\{{}^pH^iK\}\in\mathscr{K}G.$$

The proofs in 6.3 and 6.4 (specialized for $u=1$) give the following result:

Let $\varepsilon'_1:\mathbb{Z}[W'_{\mathscr{L}}]\to\mathscr{K}(G)$ be the homomorphism defined by $\varepsilon'_1(w)=\chi(K^{\mathscr{L}}_w)$. Then ε'_1 is constant on conjugacy classes in $W'_{\mathscr{L}}$. With the notations in (6.3.3), we have

$$\chi(K^{\mathscr{L}}_{\mathbf{s}})=\varepsilon'_1(s_1s_2\cdots s_r)=\varepsilon'_1(\omega\sigma_1\sigma_2\cdots\sigma_a),$$
$$\chi(\bar K^{\mathscr{L}}_{\mathbf{s}})=\varepsilon'_1(\omega(1+\sigma_1)(1+\sigma_2)\cdots(1+\sigma_a)).$$

6.6. We now return to the setting in 3.1. Let $\mathbf{s}=(s_1,s_2,\dots,s_r)$ be a sequence in S such that $s_1s_2\cdots s_r\in W'_{\mathscr{L}}$, $(\mathscr{L}\in\mathscr{S}(T))$. We apply the functor $\operatorname{res}:\mathscr{D}G\to\mathscr{D}L$ to $\bar K_{\mathbf{s}}$. We wish to describe ${}^pH^i(\operatorname{res}\bar K_{\mathbf{s}})$ in terms of the function $\varepsilon_{\mathscr{L},L}:H'_{\mathscr{L},L}\to\Gamma_L\otimes_{\mathbb{Z}}\mathscr{A}$ (defined as ε in 6.2, for L instead of G); here $H'_{\mathscr{L},L}$ is $H'_{\mathscr{L}}$ defined with respect to L instead of G. We shall denote by $H_{\mathscr{L}}$ (resp. $H_{\mathscr{L},L}$) the $\mathscr{A}$-submodule of $H'_{\mathscr{L}}$ (resp. $H'_{\mathscr{L},L}$) spanned by the elements T_w, $w\in W_{\mathscr{L}}$, (resp. by the elements T_w, $w\in W^*_{\mathscr{L}}$).

We shall denote by $\mathscr{T}$ the set of elements y_0 in W which have minimal length in their $W^*-W_{\mathscr{L}}$ double coset.

Let $\omega,\sigma_1,\sigma_2,\dots,\sigma_a$ be the sequence in $W'_{\mathscr{L}}$ attached to $\mathbf{s}$ in 5.5. Thus, $\omega\in\Omega_{\mathscr{L}}$, $\sigma_i\in S_{\mathscr{L}}$. If $y_0\in\mathscr{T}$, then $\omega'=y_0\omega y_0^{-1}$ is in $\Omega_{\mathscr{L}'}$, $\sigma'_i=y_0\sigma_iy_0^{-1}$ are in $S_{\mathscr{L}'}$ where $\mathscr{L}'=(y^{-1})^*\mathscr{L}$. If we assume that $\omega'\in W^*$, then conjugation by ω' is an automorphism of the Coxeter group $(W_{\mathscr{L}'},S_{\mathscr{L}'})$ leaving stable its length function $\tilde l'$ and its parabolic subgroup $W_{\mathscr{L}'}\cap W^*$. Hence it also leaves stable the set

$$\mathscr{S}=\{z\in W_{\mathscr{L}'}\mid z\text{ has minimal }\tilde l'\text{-length in the coset }(W^*\cap W_{\mathscr{L}'})z\}.$$

For any $h\in H_{\mathscr{L}'}$ and any $z\in\mathscr{S}$, there are well-defined elements $x_{z,z'}(h)\in H_{\mathscr{L}',L}$ $(z'\in\mathscr{S})$, such that

(6.6.1) $\quad T_{w'^{-1}z\omega'}\cdot h=\sum_{z'\in\mathscr{S}}x_{z,z'}(h)\,T_{z'}$ in $H_{\mathscr{L}'}$.

(Indeed, $H_{\mathscr{L}'}$ is free as a left $H_{\mathscr{L}',L}$ module with basis $T_{z'}$, $z'\in\mathscr{S}$.)

With these notations, we set

(6.6.2) $\gamma(y_0) = \varepsilon_{\mathscr{L}',L}(T_{\omega'} \sum_{z\in\mathscr{S}} x_{z,z}((1 + T_{\sigma'_1})(1 + T_{\sigma'_2}) \cdots (1 + T_{\sigma'_a}))) \in \Gamma_L \otimes_{\mathbb{Z}} \mathscr{A}$.

We can now state:

PROPOSITION 6.7. *The following identity holds in* $\Gamma_L \otimes_{\mathbb{Z}} \mathscr{A}$:

(6.7.1) $\sum_j (-1)^j [{}^pH^j(\operatorname{res} \bar{K}_{\mathbf{s}}^{\mathscr{L}})]\, u^{(j-m')/2} = \sum_{y_0} \gamma(y_0)\, u^{-a/2}$.

where y_0 *runs over all elements of* $\mathscr{F}$ *such that* $y_0 \omega y_0^{-1} \in W^*$, *and* $m' = r + \dim G$.

Proof. We shall give the proof in the case where $\mathscr{L}$ is the constant sheaf $\bar{\mathbf{Q}}_l$. In this case we have $W = W_{\mathscr{L}} = W'_{\mathscr{L}}$; we denote $H = H_{\mathscr{L}} = H'_{\mathscr{L}}$, $H_L = H_{\mathscr{L},L} = H'_{\mathscr{L},L}$, $\varepsilon_L = \varepsilon_{\mathscr{L},L}$. We have also $\omega = e$, $a = r$, $\sigma_i = s_i$ $(1 \leqslant i \leqslant r)$. Using 3.7(c), 3.5, and (3.8.1) we see that the left-hand side of (6.7.1) is equal to

(6.7.2) $\sum_{\mathbf{v}} \sum_j (-1)^j [{}^pH^{j-d(\mathbf{v})} \bar{K}_{\boldsymbol{\tau}}]\, u^{(j-m')/2}$;

here, $\mathbf{v}$ runs over all sequences $\mathbf{v} = (v_0, v_1,..., v_r)$ of P-orbits on $\mathscr{B}$ satisfying (3.3.1), $d(\mathbf{v})$ is defined by (3.4.1), $\mathbf{t}$ is the sequence in $S^* \cup \{e\}$ defined in (3.3.4), and $\boldsymbol{\tau}$ is the sequence in S^* obtained from $\mathbf{t}$ by dropping all t_i which are equal to e. (Thus, $\boldsymbol{\tau}$ is completely determined by $\mathbf{v}$.) We denote by $\bar{K}_{\boldsymbol{\tau}}$ the complex in $\mathscr{D}L$ defined in 2.8 in terms of $\boldsymbol{\tau}$, $\mathscr{L} = \bar{\mathbf{Q}}_l$, L.

Now using 6.2(a), we can rewrite (6.7.2) as

(6.7.3) $\varepsilon_L(\sum_{\mathbf{v}} (1 + T_{\tau_1})(1 + T_{\tau_2}) \cdots (1 + T_{\tau_h})\, u^{d(\mathbf{v}) - \dim U_P}) u^{-r/2}$,

where $\mathbf{v}$ and $\boldsymbol{\tau} = (\tau_1, \tau_2,..., \tau_h)$ are as in (6.7.2).

For any $y \in W_*$, $s \in S$, we have

$$T_y(1 + T_s) = \begin{cases} T_y + T_{ys}, & \text{if } ys \in W_*,\ ys > y, \\ u(T_y + T_{ys}), & \text{if } ys \in W_*,\ ys < y, \\ (1 + T_{ysy^{-1}})\, T_y, & \text{if } ys \in W^*y \quad (\text{so that } ysy^{-1} \in S^*). \end{cases}$$

Applying this repeatedly, we see that for any $y \in W_*$, we have

(6.7.4) $T_y(1 + T_{s_1})(1 + T_{s_2}) \cdots (1 + T_{s_r})$
$= \sum_{\mathbf{y}} q^{\delta(\mathbf{y})}(1 + T_{\tau_1})(1 + T_{\tau_2}) \cdots (1 + T_{\tau_h})\, T_{y_r}$;

here, the sum is over all sequences $\mathbf{y} = (y_0, y_1,..., y_r)$ in W_* such that $y = y_0$, $W^*y_i = W^*y_{i-1}$ or $W^*y_{i-1}s_i$ $(1 \leqslant i \leqslant r)$, $\delta(\mathbf{y})$ is defined to be $\#\{i \in [1, r] \mid y_i > y_i s_i,\ y_i s_i \in W_*\}$ and $\boldsymbol{\tau} = (\tau_1, \tau_2,..., \tau_h)$ is the sequence in S^* consisting of those terms in $(y_0 s_1 y_0^{-1}, y_1 s_2 y_1^{-1},..., y_{r-1} s_r y_{r-1}^{-1})$ which are in S^*.

Now (6.7.1) (in the case $\mathscr{L} = \bar{\mathbf{Q}}_l$) follows directly from (6.7.3), (6.7.4), and the definition (6.6.2) of the function γ. The proof in the general case is similar, but the notation is more complicated; we shall omit it.

COROLLARY 6.8. *For any j, we have ${}^pH^j(\operatorname{res} \bar{K}_s^{\mathscr{L}}) \cong {}^pH^{2m'-j}(\operatorname{res} \bar{K}_s^{\mathscr{L}})$ in $\mathscr{D}L$, where $m' = r + \dim G$.*

Proof. Since ${}^pH^j(\operatorname{res} \bar{K}_s^{\mathscr{L}})$ are semisimple objects of $\mathscr{M}L$ for all j, it is enough to show that $[{}^pH^j \operatorname{res} \bar{K}_s^{\mathscr{L}}] = [{}^pH^{2m'-j} \operatorname{res} \bar{K}_s^{\mathscr{L}}]$ (equality in Γ_L). By (6.7.1), it is then enough to show that for any y_0 in the sum (6.7.1), the expression $\gamma(y_0)\, u^{-a/2} \in \Gamma_L \otimes \mathscr{A}$ is fixed by the involution $^-$ of $\Gamma_L \otimes \mathscr{A}$ defined in 6.2(c). Since $\varepsilon_{\mathscr{L}',L}$ commutes with the involutions $^-$ (see 6.2(c)) and $(1 + T_{\sigma_1}) \cdots (1 + T_{\sigma_a})\, u^{-a/2}$ is fixed by the bar involution, we see that it is enough to prove the following statement:

(6.8.1) $\varepsilon_{\mathscr{L}',L}(T_{\omega'} \sum_{z \in \mathscr{S}} x_{z,z}(h)) = \varepsilon_{\mathscr{L}',L}(\overline{T_{\omega'} \sum_{z \in \mathscr{S}} x_{z,z}(\bar{h})})$ for all $h \in H_{\mathscr{L}',L}$ (notation as in 6.6).

Applying the involution $^-: H_{\mathscr{L}'} \to H_{\mathscr{L}'}$ to the identity (6.6.1), we get

$$(6.8.2) \quad \bar{T}_{\omega'^{-1} z \omega'} \cdot \bar{h} = \sum_{z' \in \mathscr{S}} \overline{x_{z,z'}(h)}\, \bar{T}_{z'} \ (z \in \mathscr{S}).$$

Since (T_z),$(\bar{T}_z)$, $(z \in \mathscr{S})$, form two bases of $H_{\mathscr{L}'}$, as a free left $H_{L,\mathscr{L}'}$-module, we have

$$(6.8.3) \quad \bar{T}_z = \sum_{z' \in \mathscr{S}} r_{z,z'} T_{z'},\ T_z = \sum_{z' \in \mathscr{S}} q_{z,z'} \bar{T}_{z'} \ (z \in \mathscr{S}),$$

where $r_{z,z'} \in H_{L,\mathscr{L}'}$, $q_{z,z'} \in H_{L,\mathscr{L}'}$.

Introducing in (6.8.2) we get

$$\begin{aligned}
\sum_{z' \in \mathscr{S}} \overline{x_{z,z'}(h)}\, \bar{T}_{z'}
&= \sum_{z'' \in \mathscr{S}} r_{\omega'^{-1} z \omega', z''} T_{z''} \bar{h} \\
&= \sum_{z'', z''' \in \mathscr{S}} r_{\omega'^{-1} z \omega', z''} x_{\omega' z'' \omega'^{-1}, z'''}(\bar{h})\, \bar{T}_{z'''} \\
&= \sum_{z', z'', z''' \in \mathscr{S}} r_{\omega'^{-1} z \omega', z''} x_{\omega' z'' \omega'^{-1}, z'''}(\bar{h})\, q_{z''', z'} \bar{T}_{z'}.
\end{aligned}$$

From this, we deduce

$$\sum_{z \in \mathscr{S}} x_{z,z}(h) = \sum_{z, z'', z''' \in \mathscr{S}} \overline{r_{\omega'^{-1} z \omega', z''}}\ \overline{x_{\omega' z'' \omega'^{-1}, z'''}(\bar{h})}\ \overline{q_{z''', z}}.$$

Multiply both sides with $T_{\omega'}$ and apply $\varepsilon_{\mathscr{L}',L}$:

$$\varepsilon_{\mathscr{L}',L}\left(T_{\omega'}\sum_{z\in\mathscr{S}} x_{z,z}(h)\right)$$

$$=\varepsilon_{\mathscr{L}',L}\left(T_{\omega'}\sum_{z,z'',z'''\in\mathscr{S}} \overline{r_{\omega'^{-1}z\omega',z''}}\,\overline{x_{\omega'z''\omega'^{-1},z'''}(\bar{h})}\,\overline{q_{z''',z}}\right)$$

$$=\varepsilon_{\mathscr{L}',L}\left(\sum_{z,z'',z'''\in\mathscr{S}} \overline{q_{z''',z}}\,T_{\omega'}\,\overline{r_{\omega'^{-1}z\omega',z''}}\,\overline{x_{\omega'z''\omega'^{-1},z'''}(\bar{h})}\right) \text{ by 6.2(b)}$$

$$=\varepsilon_{\mathscr{L}',L}\left(T_{\omega'}\sum_{z,z'',z'''\in\mathscr{S}} \overline{q_{\omega'^{-1}z'''\omega',\omega'^{-1}z\omega'}}\,\overline{r_{\omega'^{-1}z\omega',z''}}\,\overline{x_{\omega'z''\omega'^{-1},z'''}(\bar{h})}\right)$$

$$=\varepsilon_{\mathscr{L}',L}\left(T_{\omega'}\sum_{z'',z'''\in\mathscr{S}} \delta_{\omega'^{-1}z'''\omega',z''}\overline{x_{\omega'z''\omega'^{-1},z'''}(\bar{h})}\right)$$

$$=\varepsilon_{\mathscr{L}',L}\left(T_{\omega'}\overline{\sum_{z\in\mathscr{S}} x_{z,z}(\bar{h})}\right).$$

This proves (6.8.1) and hence the corollary.

We can now state

THEOREM 6.9. (a) *If* $L\subset P$ *is as in* 3.1, *and* $A\in\hat{G}$, *then* $\operatorname{res} A\in\mathscr{M}L$; *moreover*, $\operatorname{res} A$ *is semisimple and its irreducible components are in* $\hat{L}$.

(b) $A\in\hat{G}$ *is cuspidal* (*see* 3.10) *if and only if for any* $L\subset P$ *as in* 3.1 *with* $P\neq G$, *we have* $\operatorname{res} A=0$.

Proof. Assuming that (a) holds, the proof of (b) is immediate: if $A\in\hat{G}$ is cuspidal, then ${}^pH^i(\operatorname{res} A)=0$ for all $i\geqslant 0$ and by (a), ${}^pH^i(\operatorname{res} A)=0$ for all $i\neq 0$; hence ${}^pH^i(\operatorname{res} A)=0$ for all i, so that $\operatorname{res} A=0$.

We now prove that in (a), we have $\operatorname{res} A\in\mathscr{M}L$ for $A\in\hat{G}$. (The other statement in (a) follows from 3.9.) Let $\mathbf{s}=(s_1,\dots,s_r)$ be a sequence in S such that $s_1s_2\cdots s_r\in W'_{\mathscr{L}}$, $\mathscr{L}\in\mathscr{S}(T)$.

Let $K=\bar{K}^{\mathscr{L}}_{\mathbf{s}}[m']$, $K'=\operatorname{res} K$, $K_i={}^pH^iK$, $K'_i=\operatorname{res} K_i$, $(m'=r+\dim G)$. It is enough to prove that $K'_i\in\mathscr{M}L$ (since $\operatorname{res} A$ may be assumed to be direct summand of K'_i.)

Fix $A'\in\hat{L}$ and let b_{ij} be the multiplicity of A' in ${}^pH^j(K'_i)$. Then $b_{ij}\geqslant 0$ and it is enough to prove that $b_{ij}=0$ whenever $j\neq 0$. Let b_j be the multiplicity of A' in ${}^pH^j(K')$. From 2.17(a), we have ${}^pH^j(K')={}^pH^j(\bigoplus_i \operatorname{res} K_i[-i])=\bigoplus_i {}^pH^{j-i}(K'_i)$, hence $b_j=\sum_i b_{i,j-i}$. From 6.8, we have $b_j=b_{-j}$ for all j, hence

(6.9.1) $\quad 0=\sum_j jb_j=\sum_{i,j} jb_{i,j-i}=\sum_{i,j}(i+j)\,b_{i,j}.$

From 2.17(b), we have $K_i = K_{-i}$; it follows that $b_{ij} = b_{-i,j}$, so that $\sum_{i,j} ib_{i,j} = 0$. Introducing this into (6.9.1), we find $\sum_{i,j} jb_{ij} = 0$.

From 4.4(c) and 2.17(a) we see that $b_{ij} = 0$ for all $j > 0$. Therefore, we have $\sum_{i,j \leqslant 0} jb_{ij} = 0$. Since $jb_{ij} \leqslant 0$ for all terms in the previous sum, we must have $jb_{ij} = 0$ for all i, j. It follows that $b_{ij} = 0$ for $j \neq 0$ and the theorem is proved.

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References

1. A. A. Beilinson, J. Bernstein, and P. Deligne, Faisceaux pervers, *Astérisque* **100** (1982).
2. P. Deligne, La conjecture de Weil II, *Publ. Math. I.H.E.S.* **52** (1980), 137–252.
3. P. Deligne and G. Lusztig, Representations of reductive groups over finite fields, *Ann. of Math.* **103** (1976), 103–161.
4. G. Lusztig, Intersection cohomology complexes on a reductive group, *Invent. Math.* **75** (1984), 205–272.

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Character Sheaves II

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Contents.

This paper is a continuation of [5]; we preserve its notations. The numbering of chapters, sections and references will continue that of [5]. In [5] we have defined, for any connected reductive algebraic group G over an algebraically closed field k, a class $\hat{G}$ of irreducible perverse sheaves on G, called the character sheaves. In the case where k is an algebraic closure of the finite field F_q and G is defined over F_q, we may consider the subset $\hat{G}(F_q)$ of $\hat{G}$ consisting of the character sheaves which are isomorphic to their inverse image under the Frobenius map F. Any character sheaf K in $\hat{G}(F_q)$ gives rise to a class function χ_K on $G(F_q)$ well defined up to multiplication by a nonzero constant, as follows. We choose a definite isomorphism $\varphi: F^*K \xrightarrow{\sim} K$, and we define $\chi_K(g)$ ($g \in G(F_q)$, to be the alternating sum over i of traces of φ on the stalks of the cohomology sheaves $\mathcal{H}^i_g K$.

One expects [6, 13.7] that the class functions χ_K coincide with the "almost-characters" of $G(F_q)$ (see [6, 13.6]) so that, in particular, they should form an orthogonal basis of the space of class functions on $G(F_q)$. Here, we shall try to prove the orthogonality relations for the class functions $\chi_K(g)$. We shall prove them only under an assumption on cuspidal character sheaves. Thus, here we have only a relative result; it is one step in an inductive proof which will be completed (in good characteristic) in another paper in this series. An important role in our arguments is played by certain class functions on the unipotent set of $G(F_q)$, called generalized Green functions, since they are closely related to the Green functions of [3] and [7].

We shall adhere to the notations in [4, Sect. 0].

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7. Strongly Cuspidal Complexes

7.1. Let A be a perverse sheaf on G

(7.1.1) We say that A is *cuspidal* if it satisfies:

(a) There exists an integer $n \geqslant 1$, invertible in k, such that A is $G \times \mathscr{Z}_G^0$-equivariant for the action of $G \times \mathscr{Z}_G^0$ on G given by (g_0, z): $g \to z^n g_0 g g_0^{-1}$.

(b) For any parabolic subgroup $P \subsetneq G$ with Levi subgroup L, we have $\operatorname{res} A[-1] \in \mathscr{D}L^{\leqslant 0}$ or, equivalently, $\dim \operatorname{supp} \mathscr{H}^i (\operatorname{res} A) < -i$ for all i. (Here $\operatorname{res}: \mathscr{D}G \to \mathscr{D}L$ is defined wit respect to P, L just as in 3.8; the assumptions on P, L made in 3.8 are not necessary for that definition.)

The proof and conclusions of Proposition 3.12 remain valid:

(7.1.2) If A is a irreducible cuspidal perverse sheaf on G, then there is a unique $G \times \mathscr{Z}_G^0$-orbit $\Sigma \subset G$ and a unique irreducible $G \times \mathscr{Z}_G^0$-equivariant local system $\mathscr{E}$ on Σ such that $A = \mathrm{IC}(\bar{\Sigma}, \mathscr{E})[d]$, extended to the whole of G, by 0 on $G - \bar{\Sigma}$ (where $d = \dim \Sigma$). Moreover, the image of Σ in $G/\mathscr{Z}_G^0$ is an isolated conjugacy class of $G/\mathscr{Z}_G^0$ and, for $g \in \Sigma$, the group $\mathscr{Z}_G^0(g)/\mathscr{Z}_G^0$ is unipotent.

It is easy to see that

(7.1.3) In the set up of (7.1.2), $(\Sigma, \mathscr{E})$ is a cuspidal pair for G, in the sense of [4, 2.4].

The converse is also true:

(7.1.4) If $(\Sigma, \mathscr{E})$ is a cuspidal pair for G, in the sense of [4, 2.4] and $d = \dim \Sigma$, then $\mathrm{IC}(\bar{\Sigma}, \mathscr{E})[d]$ extended to the whole of G, by 0 on $G - \bar{\Sigma}$ is an irreducible cuspidal perverse sheaf on G in the sense of (7.1.1).

(The proof is immediate, using [4, 2.2(a)].) We shall need the following variant of definition (7.1.1):

(7.1.5) A perverse sheaf on G is said to be *strongly cuspidal* if it satisfies condition (a) in (7.7.1) and if for any parabolic subgroup $P \subsetneq G$ with Levi subgroup L, we have $\operatorname{res} A = 0 \in \mathscr{D}\mathscr{L}$ (where res is as in (7.1.1)(b)).

It is clear that if A is strongly cuspidal, then it is also cuspidal.

On the other hand, if A is a character sheaf of G, then A is cuspidal in the sense of (7.1.1) if and only if it is cuspidal in the sense of 3.10. Using now 6.9(b), we see that

(7.1.6) A character sheaf is cuspidal if and only if it is strongly cuspidal.

We now consider the following data on G:

(7.1.7) P is a parabolic subgroup of G with Levi subgroup L and unipotent radical U_P, and K_0 is an irreducible cuspidal perverse sheaf on L.

Let π_P be the canonical projection $P \to L$. Consider the diagram

$$L \xleftarrow{\pi} V_1 \xrightarrow{\pi'} V_2 \xrightarrow{\pi''} G, \tag{7.1.8}$$

where $V_1 = \{(g, h) \in G \times G \mid h^{-1}gh \in P\}$,

$$V_2 = \{(g, hP) \in G \times (G/P) \mid h^{-1}gh \in P\}, \qquad \pi(g, h) = \pi_P(h^{-1}gh),$$
$$\pi'(g, h) = (g, hP), \qquad \pi''(g, h) = h.$$

There is a well-defined perverse sheaf $K_1 \in \mathcal{M}V_2$ such that $\tilde{\pi}K_0 = \tilde{\pi}'K_1$ (cf. 4.1); we define

$$K = (\pi'')_! K_1. \tag{7.1.9}$$

Then K is a semisimple object of $\mathcal{M}G$, (see (4.3.1)). Note that K is obtained by inducing (see 4.1) K_0 from P to G; the assumptions on P, L made in 4.1 are not necessary for the definition of induction.

(7.1.10) The irreducible perverse sheaves on G which appear as irreducible components of K (for various P, L, K_0 as in (7.1.7)) are called the "admissible complexes" of G.

This definition is the same (up to shift and extension by 0) as that in [4, 4.1].

(7.1.11) The irreducible perverse sheaves on G which appear as irreducible components of K in (7.1.9) (for various P, L, K_0 as in (7.1.7) and with K_0 assumed to be strongly cuspidal for L) are called the "strongly admissible complexes" of G.

(7.1.12) The class $\mathscr{C}$ of irreducible perverse sheaves on G which appear as irreducible components of K in (7.1.9) (for various P, L, K_0 as in (7.1.7) and with K_0 assumed to be a cuspidal character sheaf of L) coincides with the class of character sheaves of G.

Indeed, the class $\mathscr{C}$ considered in (7.1.12) is contained in the class of character sheaves on G, by 4.8(b). Conversely, let A be a character sheaf of G. If A is cuspidal, then A is clearly in $\mathscr{C}$ (take $P = L = G$, $K_0 = A$). If A is not cuspidal, then there exist $P \supset L$ as in (7.1.7), $P \subsetneq G$, such that res $A \neq 0$, (res defined with respect to P, L, G). By 6.9(a), there exists a character sheaf A' of L and a nonzero morphism res $A \to A'$ in $\mathscr{D}L$. We may assume (by induction) that A' is in the class $\mathscr{C}$ (defined in terms of L, instead of G). From 4.4(d) it follows that there is a nonzero morphism $A \to \operatorname{ind} A'$ in $\mathscr{D}G$ (ind is defined with respect to P, L, G). Hence A is a direct summand of ind A', (see 4.8(b)). But transitivity of induction shows that by inducing a complex in $\mathscr{C}$ (with respect to L) to G one obtains a complex which is direct sum of complexes in $\mathscr{C}$ (with respect to G). Hence A is in $\mathscr{C}$, as required.

In particular, we see that for an irreducible perverse sheaf A on G we have the following implications:

(7.1.13) A character sheaf $\Rightarrow$ A strongly admissible complex $\Rightarrow$ A admissible complex.

(7.1.14) Now let $P, L. K_0$ be as in (7.1.7) and let V_1, V_2, K_1, K be defined in terms of $P, L. K_0$, as in (7.1.8), (7.1.9). We also consider another set of data P', L', K_0' of the same type as P, L, K_0 and define V_1', V_2', K_1', K' in terms of P', L', K_0' in the same way as V_1, V_2, K are defined in terms of P, L, K_0.

With these notations, we have the following result.

PROPOSITION 7.2. *Assume that K_0, K_0' are strongly cuspidal. Assume also that for an isomorphism $f: L \xrightarrow{\sim} L'$ which can be realized by conjugation by an element of G, we have $H_c^i(L, K_0 \otimes f^*K_0') = 0$ for all i. (This condition is automatically satisfied if L, L' are not conjugate in G). Then, for any irreducible components A of K and A' of K' we have*

$$H_c^i(G, A \otimes A') = 0 \qquad \textit{for all } i.$$

Proof. Since K, K' are semisimple objects of $\mathscr{M}G$ and H_c^i commutes with direct sums, it is enough to show that $H_c^i(G, K \otimes K') = 0$ for all i, or equivalently that

$$H_c^i(X, \tilde{K}) = 0 \qquad \text{for all } i; \tag{7.2.1}$$

here $X = V_2 \times V_2' = \{(g, hP, h'P') \in G \times (G/P) \times (G/P') \mid h^{-1}gh \in P, h'^{-1}gh' \in P'\}$, $\tilde{K} = K_1 \boxtimes K_1'$, where V_2, K_1, V_2', K_1' are as at the end of 7.1. Each G-orbit $\mathcal{O}$ on $(G/P) \times (G/P')$ we define $X_{\mathcal{O}} = \{(g, hP, h'P') \in X \mid (hP, h'P') \in \mathcal{O}\}$. The $X_{\mathcal{O}}$ form a finite partition of X into locally closed subvarieties. Hence (7.2.1) is a consequence of

$$H_c^i(X_{\mathcal{O}}, \tilde{K}) = 0 \qquad \text{for all } i \text{ and all } \mathcal{O}. \tag{7.2.2}$$

Consider the morphism $\phi_{\mathcal{O}}: X_{\mathcal{O}} \to \mathcal{O}$ given by $(g, hP, h'P') \to (hP, h'P')$. The Leray spectral sequence of $\phi_{\mathcal{O}}$ shows that to prove (7.2.2) it is enough to show that $H_c^i(\varphi_{\mathcal{O}}^{-1}(\xi), \tilde{K}) = 0$ for all i, all $\mathcal{O}$ and all $\xi \in \mathcal{O}$. By G-homogeneity of $\mathcal{O}$, it is moreover enough to check this for a single ξ in each orbit $\mathcal{O}$. Thus, it is enough to check that for any element $n \in G$ such that $L, nL'n^{-1}$ contain a common maximal torus we have

$$H_c^i(X(n), \tilde{K}) = 0 \qquad \text{for all } i, \tag{7.2.3}$$

where $X(n)$ is the subvariety of X defined by the conditions $hP = P$, $h'P' = nP'$. Let $P'' = nP'n^{-1}$, $L'' = nL'n^{-1}$, $U_{P''} = nU_{P'}n^{-1}$ and let $\pi_{P''}$ be the canonical projection $P'' \to L''$. Let $f: L'' \xrightarrow{\sim} L'$ be defined by $f(x'') = n^{-1}x''n$. Then $g \to (g, P, nP')$ is an isomorphism

$$P \cap P' \xrightarrow{\sim} X(n). \tag{7.2.4}$$

Consider the morphism

$$\rho: P \cap P' \to E = \{(x, x') \in (P'' \cap L) \times (P \cap L'') \mid \pi_{P''}(x) = \pi_P(x'')\}$$

defined by $g \mapsto (\pi_P(g), \pi_{P''}(g))$. (See [4, 1.2].) This is a locally trivial fibration with fibres $\approx U_P \cap U_{P''}$, which is an affine space. The restriction of $\tilde{K}$ to $X(n)$ becomes under (7.2.4), the complex $\tilde{\rho}(\hat{K})$, where $\hat{K}$ is the following complex on E:

$$\hat{K} = K_0 \boxtimes f^*(K_0')[j]$$

with a suitable shift $[j]$. Thus (7.2.3) is a consequence of

$$H_c^i(E, K_0 \boxtimes f^*(K_0')) = 0 \qquad \text{for all } i. \tag{7.2.5}$$

Note that $P'' \cap L$ is a parabolic subgroup of L with Levi subgroup $L \cap L''$ and that $P \cap L''$ is a parabolic subgroup of L'' with Levi subgroup $L \cap L''$.

Assume first that $P'' \cap L \neq L$. The orbits of the unipotent radical of $P'' \cap L$ acting on E by left multiplication on the x coordinate are precisely the fibres of the map $pr_2: E \to P \cap L''$. Since K_0 is strongly cuspidal, the H_c^i of any such orbit with coefficients in $K_0 \boxtimes f^*K_0'$ are zero. From the Leray spectral sequence of pr_2, it then follows that (7.2.5) holds.

Similarly, if $P \cap L'' \neq L''$, then using the fact that f^*K_0' is strongly cuspidal we see that (7.2.5) olds.

Thus, we are reduced to the case where $P'' \cap L = L$ and $P \cap L'' = L''$. Then $L = L''$ and E is the diagonal in $L \times L$. In this case (7.2.5) follows immediately from the assmptions in the proposition. This completes the proof.

7.3

We shall state a variant of Proposition 7.2. Let $\mathscr{A}$ be the affine variety which parametrizes the semisimple classes of G and let $\sigma\colon G\to\mathscr{A}$ be the Steinberg morphism which attaches to each $g\in G$ the conjugacy class of the semisimple part of g. Let us fix $a\in\mathscr{A}$. We then have the following result

(7.3.1) *We preserve the assumptions of* 7.2 *except that we replace* "$H_c^i(L, K_0\otimes f^*K_0')=0$ *for all* i" *by* "$H_c^i(L\cap\sigma^{-1}(a), K_0\otimes f^*K_0')=0$ *for all* i." *Then for any irreducible components* A *of* K *and* A' *of* K' *we have* $H_c^i(\sigma^{-1}(a), A\otimes A')=0$ *for all* i.

As in the proof of 7.2 we see that it is enough to prove that $H_c^i(X^a, \tilde{K})=0$, where $X^a=\{(g, hP, h'P')\in X\mid \sigma(g)=a\}$ and $X, \tilde{K}$ are as in (7.2.1). As in that proof, this can be reduced to the following statement

$$H_c^i(E^a, K_0\boxtimes f^*K_0')=0 \qquad \text{for all } i, \tag{7.3.2}$$

where E, $K_0\boxtimes f^*K_0'$ are as in (7.2.5) and E^a is the subspace of E defined by $E^a=\{(x,x')\in E\mid \sigma(x)=\sigma(x')=a\}$. Just as in the proof of (7.2.5), we consider the map $pr_2\colon E^a\to(P\cap L'')\cap\sigma^{-1}(a)$; if $P\cap L''\neq L''$, we use 6.9(b) to deduce that (7.3.2) holds. Similarly, we see that (7.3.2) olds if $P''\cap L\neq L$. If we have $P\cap L''=L''$ and $P''\cap L=L$ then $L=L''$ and then (7.3.2) follows from the assumption in (7.3.1). Thus, (7.3.1) is proved.

7.4

Let A, A' be two perverse sheaves on a variety Z. Then

$$H_c^i(Z, A\otimes A')=0 \quad \text{for} \quad i>0. \tag{7.4.1}$$

Moreover if A, A' are irreducible, then

$$H_c^0(Z, A\otimes A')=0 \Leftrightarrow A' \text{ is not isomorphic to } DA. \tag{7.4.2}$$

This is proved as follows (cf. [4, 6.7].) From the inequalities $\dim\operatorname{supp}\mathscr{H}^iA\leqslant -i$ and the analogous inequalities for A', it follows that $\dim\operatorname{supp}(\mathscr{H}^iA\otimes\mathscr{H}^jA')\leqslant\min(-i,-j)\leqslant -\frac{1}{2}(i+j)$, so that $\dim\operatorname{supp}\mathscr{H}^i(A\otimes A')\leqslant -i/2$. It follows that $H_c^a(Z, \mathscr{H}^i(A\otimes A'))=0$ for $a>-i$, i.e., for $i+a>0$. This is the E_2-term of a spectral sequence converging to $H_c^*(Z, A\otimes A')$, hence (7.4.1) follows. We now prove (7.4.2). Assume that A, A' are irreducible. For $i<-d$, we have $\mathscr{H}^i(A)=0$ and for $i>-d=-\dim\operatorname{supp}A$, we have $\dim\operatorname{supp}\mathscr{H}^i(A)<-i$; an analogous result holds for A'. Hence $\dim\operatorname{supp}(\mathscr{H}^i(A)\otimes\mathscr{H}^jA)<-\frac{1}{2}(i+j)$ for $i\neq -d$, $j\neq -d'=-\dim\operatorname{supp}A'$, and $H_c^a(Z, \mathscr{H}^i(A\otimes A'))$ is 0 for $i+a\geqslant 0$ except possibly in the following case: $\operatorname{supp}A=\operatorname{supp}A'$, $-a=i=-d-d'=-2d$, in which case it is $H_c^{2d}(Z, \mathscr{H}^{-d}(A) \otimes \mathscr{H}^{-d}(A'))$. Let Z_1 be an open dense subset of $\operatorname{supp} A = \operatorname{supp} A'$ on which $\mathscr{H}^{-d}(A)$, $\mathscr{H}^{-d}(A')$ are irreducible local systems $\mathscr{E}$, $\mathscr{E}'$. We have $H_c^{2d}(Z, \mathscr{H}^{-d}(A) \otimes \mathscr{H}^{-d}(A')) = H_c^{2d}(Z_1, \mathscr{E} \otimes \mathscr{E}')$ and this is zero if $\mathscr{E}'$ is not isomorphic to the dual of $\mathscr{E}$, and is 1-dimensional otherwise. From this, (7.4.2) follows as in the proof of (7.4.1).

We shall need the following variant of Proposition 7.2.

PROPOSITION 7.5. *With the notations in* (7.1.14) *assume that the following condition is satisfied: for any isomorphism $f: L \xrightarrow{\sim} L'$ which can be realized by conjugation by an element of G, f^*K_0' is not isomorphic to DK_0. Then, for any irreducible components A of K and A' of K', A' is not isomorphic to DA.*

Proof. The proof will follow closely the proof of 7.2. We shall use the notations in that proof. Using (7.4.2), we see that it is enough to show that $H_c^0(X, \tilde{K}) = 0$, (see (7.2.1)). This, in turn, is a consequence of the following statement:

> $H_c^0(X_{\mathcal{O}}, \tilde{K}) = 0$, for all $\mathcal{O}$ (as in (7.2.2)), which follows from the equality $H_c^i(\varphi_{\mathcal{O}}^{-1}(\xi), \tilde{K}) = 0$, for all $i \geqslant -2\delta$ ($\delta = \dim \mathcal{O}$), all $\mathcal{O}$, and all $\xi \in \mathcal{O}$.

Thus, it is enough to prove the following variant of (7.2.3): $H_c^i(X(n), \tilde{K}) = 0$ for all $i \geqslant -2\delta$, where $(P, nP') \in \mathcal{O}$ and L, $nL'n^{-1}$ have a common maximal torus. In the case where $P'' \cap L \neq L$ or $P \cap L'' \neq L''$, this is proved as in the proof of 7.2, using the fact that K_0, K_0' are cuspidal. In the case where $P'' \cap L = L$, $P \cap L'' = L''$, we have $L = L''$ and we are reduced just as in that proof to showing that $H_c^i(L, K_0 \otimes f^*(K_0')) = 0$ for all $i \geqslant 0$, (where $f: L \xrightarrow{\sim} L'$ is defined by $f(x) = n^{-1}xn$). This follows from our assumptions and from (7.4.1), (7.4.2). (Alternatively, a proof of the proposition could be extracted from the proof of Theorem 5.5 in [4].)

COROLLARY 7.6. *With the notations in* (7.1.14), *assume that A is an admissible complex on G which is a component of both K and K'. Then there exists $g \in G$ such that $gLg^{-1} = L'$, $K_0 = \operatorname{ad}(g)^*K_0'$ ($\operatorname{ad}(g): L \to L'$, $\operatorname{ad}(g)x = gxg^{-1}$).*

Proof. Note that DK_0' is again a cuspidal complex on L'. This follows from the fact that if $(\Sigma, \mathscr{E})$ is a cuspidal pair for L, then $(\Sigma, \mathscr{E}^*)$, (where $\mathscr{E}^*$ is the dual of $\mathscr{E}$) is again a cuspidal pair for L. (See [4, 2.5].) Now DK' is obtained from P', L', DK_0' in the same way as K is obtained from P, L, K_0; (see (7.1.9)), since induction commutes with D. Clearly, DA is an irreducible component of DK'. It remains to apply 7.5 to K and DK' and to $A' = DA$.

DEFINITION 7.7. Let A be a cuspidal perverse sheaf on G. We say that A is clean if there exists $\Sigma \subset G$ (the inverse image under $G \to G/\mathscr{Z}_G^0$ of a conjugacy class in $G/\mathscr{Z}_G^0$) such that $\operatorname{supp} A = \bar{\Sigma}$ and the restriction of A to $\bar{\Sigma} - \Sigma$ is zero.

PROPOSITION 7.8. *Let A, A' be two clean irreducible cuspidal perverse sheaves on G such that A' is not isomorphic to DA. Then $H_c^i(G, A \otimes A') = 0$ for all i.*

Proof. Let Σ be defined in terms of A as in (7.1.2). Define in the same way Σ' in terms of A'. We may clearly assume that $\Sigma = \Sigma'$. Let $\mathscr{E}$ (resp. $\mathscr{E}'$) be the local system on Σ such that $A = \mathscr{E}[d]$ (resp. $A' = \mathscr{E}'[d]$) on Σ_0, where $d = \dim \Sigma$. We must show that $H_c^i(\Sigma, \mathscr{E} \otimes \mathscr{E}') = 0$ for all i. The local system $\mathscr{E} \otimes \mathscr{E}'$ on Σ is equivariant for the action $(g_0, z)\colon g \to z^n g_0 g g_0^{-1}$ of $G \times \mathscr{Z}_G^0$ on Σ (for some $n \geqslant 1$, invertible in k); moreover, it is semisimple and contains no summand $\approx \bar{\mathbb{Q}}_l$ (since $\mathscr{E}'$ is not isomorphic to $\mathscr{E}^*$). It is enough to show that $H_c^i(\Sigma, \mathscr{F}) = 0$ for all irreducible local systems $\mathscr{F} \not\approx \bar{\mathbb{Q}}_l$, which are equivariant for the $G \times \mathscr{Z}_G^0$ action above. Let $G_1 \subset G \times \mathscr{Z}_G^0$ be the stabilizer of some base point $y \in \Sigma$. Let $\tilde{\Sigma} = (G \times \mathscr{Z}_G^0)/G_1^0$ and let $f\colon \tilde{\Sigma} \to \Sigma$ be the map $(g_0, z) \mapsto z^n g_0 y g_0^{-1}$. Then f is a principal G_1/G_1^0-covering (G_1/G_1^0 acts on $\tilde{\Sigma}$ by right multiplication) and every local system $\mathscr{F}$ as above is a direct summand of $f_* \bar{\mathbb{Q}}_l$. It is then enough to show that $H_c^i(\Sigma, f_* \bar{\mathbb{Q}}_l) = H_c^i(\Sigma, \bar{\mathbb{Q}}_l)$, or equivalently, that G_1/G_1^0 acts trivially on $H_c^i(\tilde{\Sigma}, \bar{\mathbb{Q}}_l)$.

Consider the map $f'\colon (G/\mathscr{Z}_G^0) \times \mathscr{Z}_G^0 \to \tilde{\Sigma}$ defined by $(g\mathscr{Z}_G^0, z) \to$ class of (g, z). It is clear that f' is surjective. Moreover, the fibres of f' are the orbits of the group $(H_0/\mathscr{Z}_G^0)$ acting on $(G/\mathscr{Z}_G^0) \times \mathscr{Z}_G^0$ by right multiplication, where H is the centralizer of y in G. (Note that $G_1^0 = H^0 \times \{e\}$.) By (7.1.2), $H^0/\mathscr{Z}_G^0$ is a (connected) unipotent group. Hence f' is an affine space bundle, so that $H_c^i(\tilde{\Sigma}, \bar{\mathbb{Q}}_l) = H_c^{i+2a}((G/\mathscr{Z}_G^0) \times \mathscr{Z}_G^0, \bar{\mathbb{Q}}_l)$ with $a = \dim H^0/\mathscr{Z}_G^0$. Moreover, the action of G_1/G_1^0 on $\tilde{\Sigma}$ and the action of $G_1/\mathscr{Z}_G^0$ on $(G/\mathscr{Z}_G^0) \times \mathscr{Z}_G^0$ (by right multiplication on the first factor) are compatible with the map f'. Hence to prove that G_1/G_1^0 acts trivially on $H_c^i(\tilde{\Sigma}, \bar{\mathbb{Q}}_l)$ it is enough to show that $G_1/\mathscr{Z}_G^0$ acts trivially on $H_c^{i+2a}((G/\mathscr{Z}_G^0) \times \mathscr{Z}_G^0, \bar{\mathbb{Q}}_l)$. But this is clear since the action of $G_1/\mathscr{Z}_G^0$ is the restriction of the action of the connected group $G/\mathscr{Z}_G^0$ and a connected group must act trivially on cohomology. The proposition is proved.

PROPOSITION 7.9. *Let A be a strongly cuspidal irreducible perverse sheaf on G and let $\Sigma, \mathscr{E}$ be as in* (7.1.2). *Assume that G is semisimple and that any Levi subgroup L of a proper parabolic subgroup of G has the following property: any irreducible cuspidal perverse sheaf on L whose support contains some unipotent element is strongly cuspidal. Assume that Σ is a unipotent* *class and that for any cuspidal pair $(\Sigma', \mathscr{E}')$ in G with $\Sigma' \subset \bar{\Sigma} - \Sigma$, the character by which the centre of G acts (in the conjugation action) on any stalk of $\mathscr{E}$ differs from that by which it acts on any stalk of $\mathscr{E}'$. Then A is clean.*

Proof. Assume that A is not clean. Let $C \subset \bar{\Sigma} - \Sigma$ be a unipotent class of minimal possible dimension such that $\oplus_i \mathscr{H}^i(A)|_C \neq 0$. Let i_0 be the largest i such that $\mathscr{H}^i(A)|_C \neq 0$. Let $\mathscr{L}$ be a direct summand of the local system $\mathscr{H}^{i_0}(A)|_C$. The center of G acts on each stalk of $\mathscr{H}^{i_0}(A)$ by the same character by which it acts on each stalk of $\mathscr{E}$. Therefore, from our assumption it follows that $(C, \mathscr{L})$ cannot be a cuspidal pair in G. By [4, 2.5], $(C, \mathscr{L}^*)$ is also not a cuspidal pair in G ($\mathscr{L}^* =$ dual of $\mathscr{L}$). According to [4, 6.5] there exist P, L, K_0 as in (7.1.7) with $P \subsetneq G$ such that the support of K_0 contains unipotent elements and such that if K is the corresponding induced complex on G, then for some direct summand A' of K the following property holds: The restriction of A' to the unipotent variety of G is (up to shift) $\mathrm{IC}(\bar{C}, \mathscr{L}^*)$ extended to the unipotent variety by zero outside $\bar{C}$. By our assumption, K_0 is strongly cuspidal. We may apply Proposition 7.2 to A, A'. (In our case, the two Levi subgroups appearing in 7.2 are L and G, hence are not conjugate.) It follows that $H^i_c(G, A \otimes A') = 0$ for all i. As $\operatorname{supp} A = \bar{\Sigma}$, we have also $\operatorname{supp}(A \otimes A') \subset \bar{\Sigma}$ hence $H^i_c(G, A \otimes A') = H^i_c(\bar{\Sigma}, A \otimes A')$. As $\operatorname{supp}(A') \cap \bar{\Sigma} \subset \bar{C}$, we have $H^i_c(\bar{\Sigma}, A \otimes A') = H^i_c(\bar{C}, A \otimes A')$. As A is zero on $\bar{C} - C$ (by minimality of C) we have $H^i_c(\bar{C}, A \otimes A') = H^i_c(C, A \otimes A')$. Comparing the last four equalities, we see that $H^i_c(C, A \otimes A') = 0$ for all i. Since $A' \mid C$ is equal to $\mathscr{L}^*$ up to a shift, it follows that $H^i_c(C, A \otimes \mathscr{L}^*) = 0$ for all i. In particular, we have $H^{2d+i_0}_c(C, A \otimes \mathscr{L}^*) = 0$, where $d = \dim C$. Consider the spectral sequence $E_2^{p,q} = H^p_c(C, \mathscr{H}^q(A) \otimes \mathscr{L}^*) \Rightarrow H^{p+q}_c(C, A \otimes \mathscr{L}^*)$. Then $E_2^{p,q} = 0$ if $q > i_0$ (by our choice of i_0) or if $p > 2d$. It follows that $E_2^{2d,i_0} = E_3^{2d,i_0} = \cdots = E_\infty^{2d,i_0}$. But E_∞^{2d,i_0} is a subquotient of $H^{2d+i_0}_c(C, A \otimes \mathscr{L}^*)$, hence it is zero. It follows that $0 = E_2^{2d,i_0} = H^{2d}_c(C, \mathscr{H}^{i_0}(A) \otimes \mathscr{L}^*)$. Since $\mathscr{L}$ is a direct summand $\mathscr{H}^{i_0}(A)|_C$, it follows that $H^{2d}_c(C, \mathscr{L} \otimes \mathscr{L}^*) = 0$. This is clearly a contradiction. The proposition is proved.

7.10

Let $(\Sigma, \mathscr{E})$ be a cuspidal pair for G and let $K = \mathrm{IC}(\bar{\Sigma}, \mathscr{E})[d]$, extended to the whole of G, by 0 on $G - \bar{\Sigma}$ ($d = \dim \Sigma$). Let Σ_1 be the set of semisimple parts of elements in Σ (or, equivalently, in its closure $\bar{\Sigma}$). Let s be an element of Σ_1. We denote $G' = Z^0(s)$ and $C = \{u \in G' \mid u \text{ unipotent}, su \in \Sigma\}$. This is a single orbit under the conjugation action of $\{g \in G \mid gsg^{-1} \in s\mathscr{Z}^0_G\}$ which contains G' as a subgroup of finite index. Hence C is a union of finitely many unipotent conjugacy classes of G'. Let $\mathscr{E}'$ be the local system on $\mathscr{Z}^0_G \cdot C$ defined as the inverse image of $\mathscr{E}$ under the map $\mathscr{Z}^0_G \cdot C \to \Sigma$,

$g \to sg$. We define $K' = \mathrm{IC}(\mathscr{Z}^0_G \cdot \bar{C}, \mathscr{E}')[d_1]$, extended to the whole of G', by 0 on $G' - \mathscr{Z}^0_G \cdot \bar{C}$ ($d_1 = \dim \mathscr{Z}^0_{G'} \cdot C$). Note that $\mathscr{Z}^0_G \cdot \bar{C}$ has all its irreducible components of the same dimension d_1 and $\mathscr{Z}^0_G \cdot C$ is open, dense, smooth in $\mathscr{Z}^0_G \cdot \bar{C}$ so that the intersection cohomology complex K' is well defined.

With these notations, we can state

PROPOSITION 7.11. (a) *K' is a cuspidal perverse sheaf on G'.*

(b) *If K is strongly cuspidal, then K' is a strongly cuspidal perverse sheaf on G'.*

(c) *C is a single unipotent class of G'.*

(d) *if and only if K is clean, K' is clean.*

Proof. We first prove (b). Let P' be a proper parabolic subgroup of G' with unipotent radical $U_{P'}$. We must show that for any $z \in \mathscr{Z}^0_{G'}$, $u \in \bar{C} \cap P'$, we have $H^i_c(zuU_{P'}, K') = 0$ for all i. We may assume that $z = e$. Hence we must show that $H^i_c(uU_{P'} \cap \bar{C}, K') = 0$ for all i. The restriction of K' to $\bar{C}$ coincides (up to a shift) with the inverse image of K under the map $\bar{C} \to \bar{\Sigma}$, $u \mapsto su$, (because the map $\pi: \bar{\Sigma} \to \Sigma_1$ defined by taking semisimple parts is a locally trivial fibration and Σ_1 is smooth). Hence we must show that $H^i_c(suU_{P'} \cap c\bar{C}, K) = 0$ for all i, or equivalently, that $H^i_c(suU_{P'} \cap \bar{\Sigma}, K) = 0$. Let P be a parabolic subgroup of G such that $P' = P \cap G'$. Define $\rho: suU_P \cap \bar{\Sigma} \to sU_P \cap \Sigma_1$ to be he restriction of $\pi: \bar{\Sigma} \to \Sigma_1$. The group U_P acts by conjugation on both the source and the target of ρ and the action is compatible with ρ; moreover, this action is transitive on $sU_P \cap \Sigma_1$. We have $\rho^{-1}(s) = suU_{P'} \cap \bar{\Sigma}$, hence we must only show that $H^i_c(\rho^{-1}(s), K) = 0$ for all i. Consider the Leray spectral sequence for ρ:

$$E_2^{p,q} = H^p_c(sU_P \cap \Sigma_1, \mathscr{H}^q \rho_! K) \Rightarrow H^{p+q}_c(suU_P \cap \bar{\Sigma}, K).$$

The last vector space is zero since K is strongly cuspidal. Thus, $E_\infty^{p,q} = 0$ for all p, q. Now $\mathscr{H}^q \rho_! K$ is a U_P-equivariant local system on $sU_P \cap \Sigma_1$ and $sU_P \cap \Sigma_1 \cong U_P/U_{P'} \cong$ affine space. Hence $E_2^{p,q} = 0$ for $p \neq 2 \dim U_P/U_{P'}$. This implies that $E_2^{p,q} = E_\infty^{p,q}$ for all p, q; it follows that $E_2^{p,q} = 0$, for all p, q, so that $\mathscr{H}^q \rho_! K = 0$ for all q. Taking the stalk at s, we see that $H^q_c(\rho^{-1}(s), K) = 0$ for all q, and (b) is proved.

The proof of (a) is similar; it will be omitted. It is clear hat the previous argument, together with (c) also proves (d).

We now prove (c). It is easy to show that our statement for G follows from that for $G/\mathscr{Z}^0_G$; hence we may assume that G is semisimple. Assume that C contains at lest two unipotent classes $C_1 \neq C_2$ of $Z^0(s)$. As we have seen in 7.10, there exists $g \in Z_G(s)$ such that $gC_1 g^{-1} = C_2$. Moreover, from (a) we see that there exist irreducible local systems $\mathscr{E}_1$ on C_1 and $\mathscr{E}_2$ on C_2 such that $(C_1, \mathscr{E}_1)$, $(C_2, \mathscr{E}_2)$ are cuspidal pairs for G. Let $\pi: \tilde{G} \to G$ be the simply connected covering of G. Let $\tilde{C}_1, \tilde{C}_2$ be the sets of unipotent elements in $\pi^{-1}(C_1)$, $\pi^{-1}(C_2)$; let $\tilde{\mathscr{E}}_1 = \pi^*(\mathscr{E}_1)$, $\tilde{\mathscr{E}}_2 = \pi^*\mathscr{E}_2$, $\tilde{g} \in \pi^{-1}(g)$, $\tilde{s} \in \pi^{-1}(s)$. Then $(\tilde{C}_1, \tilde{\mathscr{E}}_1)$, $(\tilde{C}_2, \tilde{\mathscr{E}}_2)$ are cuspidal pairs for $Z_G(\tilde{s})$, $\tilde{C}_1 \neq \tilde{C}_2$ and we have $\tilde{g}\tilde{C}_1\tilde{g}^{-1} = \tilde{C}_2$, $\tilde{g}\tilde{s}\tilde{g}^{-1} = z\tilde{s}$, $z \in \ker \pi$. Thus it is enough to prove

LEMMA 7.12. *Let G be simply connected, let $s \in G$ be an isolated semisimple element. Assume that C_1, C_2 are two unipotent classes in $Z_G(s)$, and $g \in G$ is such that $gC_1g^{-1} = C_2$, $gsg^{-1} = zs$, $z \in$ centre of G. Assume that there exist irreducible local systems $\mathscr{E}_1$ on C_1, $\mathscr{E}_2$ on C_2 such that $(C_1, \mathscr{E}_1)$, $(C_2, \mathscr{E}_2)$ are cuspidal pairs for $Z_G(s)$. Then $C_1 = C_2$.*

By decomposing G into a product of almost simple groups we are reduced to the case where G is almost simple and simply connected. We may assume that $z \neq e$, for otherwise the result is obvious. In this case, by results of [4], $Z_G(s)$ has at most one unipotent class C which can carry a cuspidal pair (except possibly when G is a spin-group in odd characteristic); hence $C_1 = C_2$. It remains to consider the case where G is a spin-group in odd characteristic. Then $Z_G(s)$ is of type $D_n \times D_m$. If conjugation by g preserves both D factors of $Z_G(s)$, then it leaves stable each unipotent class of $Z_G(s)$ except possibly for some classes of unipotent elements contained in a proper Levi subgroup; such classes cannot carry cuspidal pairs, by [4, 2.8], hence $C_1 = C_2$.

We may therefore assume that conjugation by g switches the two D-factors of $Z_G(s)$. Then $Z_G(s) = (\mathrm{Spin}_{2n} \times \mathrm{Spin}_{2n})/(\varepsilon, \varepsilon)$, where ε is the generator of the kernel of $\mathrm{Spin}_{2n} \to SO_{2n}$. In this case, any unipotent class of $Z_G(s)$ which can carry cuspidal pairs for $Z_G(s)$ is of the form $C' \times C'$, where C' is a unipotent class in Spin_{2n}, (strictly speaking, $C' \times C'$ is a unipotent class of $\mathrm{Spin}_{2n} \times \mathrm{Spin}_{2n}$; we identify it with its image in $Z_G(s)$). This follows from the following result [4, §13, 14]: given a character χ of the group $\{1, \varepsilon\}$, there is at most one unipotent class C'_χ of Spin_{2n} which can carry a cuspidal pair $(C'_\chi, \mathscr{E}')$ for Spin_{2n} such that ε acts on each stalk of $\mathscr{E}'$ as multiplication by $\chi(\varepsilon)$. This completes the proof of the lemma and also the proof of 7.11.

(Note that in the proof of the lemma we have made use of the results in [4] on classification of cuspidal pairs carried by unipotent classes only in the case of classical groups.)

8. GENERALIZED GREEN FUNCTIONS

8.0

In this chapter, k denotes an algebraic closure of a finite field F_q with q elements and G (see 2.1) has a fixed F_q-rational structure compatible with the group structure. We denote by $F: G \to G$ the corresponding Frobenius map.

8.1

We consider the following data in G:

(8.1.1) L is the centralizer of a torus in G, Σ is a subset of L which is the inverse image under $L \to L/\mathscr{Z}_L^0$ of an isolated conjugacy class of $L/\mathscr{Z}_L^0$, $\mathscr{E}$ is a local system on L which is isomorphic to a direct sum of irreducible local systems $\mathscr{E}_i$ on Σ such that each $(\Sigma, \mathscr{E}_i)$ is a cuspidal pair for G, (see (7.1.2)).

To (L, Σ) we associate the open set Σ_{reg} of Σ and the locally closed smooth irreducible subvariety $Y = Y_{(L,\Sigma)} = \bigcup_{x \in G} x\Sigma_{\text{reg}} x^{-1}$ of G, as in 3.11. Consider the diagram

$$\Sigma \xleftarrow{\alpha} \hat{Y} \xrightarrow{\beta} \tilde{Y} \xrightarrow{\pi} Y,$$

where

$$\tilde{Y} = \{(g, xL) \in G \times (G/L) \mid x^{-1}gx \in \Sigma_{\text{reg}}\},$$

$$\hat{Y} = \{(g, x) \in G \times G \mid x^{-1}gx \in \Sigma_{\text{reg}}\},$$

$$\alpha(g, 0) = x^{-1}gx, \qquad \beta(g, x) = (g, xL), \qquad \pi(g, xL) = g.$$

The local system $\alpha^*\mathscr{E}$ on $\hat{Y}$ is L-equivariant for the action of L on $\hat{Y}$ given by $l: (g, x) \mapsto (g, xl^{-1})$ hence it is equal to $\beta^*\tilde{\mathscr{E}}$ for a well-defined local system $\tilde{\mathscr{E}}$ on $\tilde{Y}$. (We take $\tilde{\mathscr{E}} = R^0\beta_*(\alpha^*\mathscr{E})$.) The map π is a finite principal covering map, hence $\pi_*\tilde{\mathscr{E}}$ is a semisimple local system on Y. Consider the closure $\bar{Y}$ of Y in G and let

(8.1.2) $K = \mathrm{IC}(\bar{Y}, \pi_*\tilde{\mathscr{E}})[\dim Y]$, extended to the whole of G (by 0 on $G - \bar{Y}$).

(8.1.3) In the case where $FL = L$, $F\Sigma = \Sigma$, and there exists an isomorphism $\varphi_0: F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ of local systems over Σ, we can define an isomorphism $\varphi: F^*K \xrightarrow{\sim} K$ as follows. The varieties Y, $\tilde{Y}$, $\hat{Y}$ have natural F_q-structures and φ_0 gives rise to an isomorphism $F^*\tilde{\mathscr{E}} \xrightarrow{\sim} \tilde{\mathscr{E}}$ of loal systems over $\tilde{Y}$, to an isomorphism $F^*\pi_*\tilde{\mathscr{E}} \xrightarrow{\sim} \pi_*\tilde{\mathscr{E}}$ of local systems over Y and hence to an isomorphism $\varphi: F^*K \xrightarrow{\sim} K$ (in $\mathscr{M}G$).

8.2

Another construction of K is given in [4, 4.5]. We shall recall it briefly. Let P be a parabolic subgroup of G having L as a Levi subgroup. Let $\pi_P: P \to L$ be the canonical projection. Consider the diagram

$$\bar{\Sigma} \xleftarrow{\hat{\alpha}} \hat{X} \xrightarrow{\beta} X \xrightarrow{\psi} \bar{Y},$$

where

$$X=\{(g, xP)\in G\times (G/P) \mid x^{-1}gx\in \bar{\Sigma}\cdot U_P\},$$
$$\hat{X}=\{(g, x)\in G\times G \mid x^{-1}gx\in \bar{\Sigma}\cdot U_P\},$$
$$\bar{Y}=\text{closure of } Y \text{ in } G \qquad (\text{see } 8.1),$$
$$\hat{\alpha}(g, x)=\pi_P(x^{-1}gx), \quad \hat{\beta}(g, x)=(g, xP), \qquad \psi(g, xP)=g.$$

(According to [4, 4.3] we have $\bar{Y}=\bigcup_{x\in G} x\bar{\Sigma}U_P x^{-1}$.)

There is a canonical perverse sheaf $\tilde{K}$ on X such that $(\bar{\hat{\beta}})\tilde{K}=(\bar{\hat{\alpha}})(\mathrm{IC}(\bar{\Sigma}, \mathscr{E})[\dim \Sigma])$.

The following results are proved in [4, 4.3, 3.1, 4.5].

(8.2.1) The map $(g, xL)\to (g, xP)$ gives an isomorphism of $\tilde{Y}$ onto the open dense subset $\psi^{-1}(Y)$ of X. The map ψ is a proper map of X on $\bar{Y}$.

(8.2.2) Y is locally closed in G, smooth irreducible of dimension equal to $\dim G-\dim L+\dim \Sigma$.

(8.2.3) There is a canonical isomorphism $K\mid \bar{Y}\simeq \psi_!\tilde{K}$.

The model $\psi_!\tilde{K}$ of K has the disadvantage that in the situation of (8.1.3), there is no direct way to define an isomorphism $F^*\psi_!\tilde{K}\simeq\psi_!\tilde{K}$. This is due to the fact that, although $FL=L$, we do not necessarily have $FP=P$. On the other hand, if we denote $P'=FP$, then we may define $\psi'\colon X'\to\bar{Y}$, $\tilde{K}'\in\mathscr{M}(X')$ in terms of $G, L, P', \Sigma, \mathscr{E}$ in the same way as $\psi\colon X\to\bar{Y}$, $\tilde{K}\in\mathscr{M}(X)$ were defined above in terms of G, L, P, Σ, $\mathscr{E}$. Then (8.2.3) applies again and gives an isomorphism $K\simeq\psi'_!K'$. Moreover, the map $\varphi_0\colon F^*\mathscr{E}\simeq\mathscr{E}$ in (8.1.3) gives rise in a natural way to an isomorphism $F^*\tilde{K}'\simeq\tilde{K}$ in $\mathscr{M}X$ (note that F maps naturally X onto X'); hence it gives rise to an isomorphism $\tilde{\varphi}\colon F^*(\psi'_!\tilde{K}')\simeq\psi_!\tilde{K}$ in $\mathscr{M}\bar{Y}$, such that the following diagram is commutative

$$\begin{array}{ccc} F^*(K\mid \bar{Y}) & \xrightarrow{\varphi} & K\mid \bar{Y} \\ \big\downarrow\wr & & \big\downarrow\wr \\ F^*(\psi'_!\tilde{K}') & \xrightarrow{\tilde{\varphi}} & \psi_!(\tilde{K}) \end{array} \tag{8.2.4}$$

(The vertical maps are defined by (8.2.3).)

8.3

Let L, Σ be as in (8.1.1); assume that $\Sigma=\mathscr{Z}_L^0\cdot C$, where C is a unipotent class of L. Assume also that $FL=L$, $F\Sigma=\Sigma$. Let $\mathscr{F}$ be a local system on C

and let $\varphi_1: F^*\mathscr{F} \xrightarrow{\sim} \mathscr{F}$ be an isomorphism. We assume that there exists a local system $\mathscr{E}$ on Σ, as in (8.1.1), such that $\mathscr{F} = \mathscr{E} \mid C$, and that there exists an isomorphism $\varphi_0: F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ extending the given isomorphism $\varphi_1: F^*\mathscr{F} \xrightarrow{\sim} \mathscr{F}$.

We define K, $\varphi: F^*K \xrightarrow{\sim} K$ in terms of $\mathscr{E}$ as in 8.1. We define a function

$$Q_{L,G,C,\mathscr{F},\varphi_1}: \{\text{unipotent elements in } G^F\} \to \bar{\mathbb{Q}}_l$$

by

$$Q_{L,G,C,\mathscr{F},\phi_1}(u) = \sum_i (-1)^i \operatorname{Tr}(\varphi, \mathscr{H}^i_u K). \tag{8.3.1}$$

(The map induced by $\varphi: F^*K \xrightarrow{\sim} K$ on the stalk $\mathscr{H}^i_u K$ ($u \in G^F$), is denoted again by φ; the trace is taken over $\bar{\mathbb{Q}}_l$.)

This definition makes sense in view of the following result:

(8.3.2) *The function $Q_{L,G,C,\mathscr{F},\varphi_1}$ is independent of the choice of $\mathscr{E}$ and $\varphi_0: F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ extending $\mathscr{F}$ and $\varphi_1: F^*\mathscr{F} \xrightarrow{\sim} \mathscr{F}$ to Σ.*

(In particular, to compute this function we may take $\mathscr{E}$, φ_0 to be the inverse image of $\mathscr{F}$, φ_1 under the canonical map $\Sigma \to C$.) We consider two local systems $\mathscr{E}_1, \mathscr{E}_2$ on Σ as in (8.1.1) and isomorphisms $\varphi_{01}: F^*\mathscr{E}_1 \xrightarrow{\sim} \mathscr{E}_1$, $\varphi_{02}: F^*\mathscr{E}_2 \xrightarrow{\sim} \mathscr{E}_2$ such that over C we have a commutative diagram

$$\begin{tikzcd} F^*\mathscr{E}_1 \mid C \arrow[r, "F^*\gamma", "\sim"'] \arrow[d, "\varphi_{01}"', "\wr"] & F^*\mathscr{E}_2 \mid C \arrow[d, "\varphi_{02}", "\wr"'] \\ \mathscr{E}_1 \mid C \arrow[r, "\gamma", "\sim"'] & \mathscr{E}_2 \mid C \end{tikzcd}$$

Let K_1, K_2 be defined in terms of $\mathscr{E}_1, \mathscr{E}_2$ in the same way as K was defined in terms of $\mathscr{E}$ and let $\varphi_1: F^*K_1 \xrightarrow{\sim} K_1$, $\varphi_2: F^*K_2 \xrightarrow{\sim} K_2$ be the isomorphisms induced by $\varphi_{01}, \varphi_{02}$. It is enough to show that $\operatorname{Tr}(\varphi_1, \mathscr{H}^i_u K_1) = \operatorname{Tr}(\varphi_2, \mathscr{H}^i_u K_2)$ for any unipotent element $u \in G^F$. We may assume that $u \in \bar{Y} = \operatorname{supp} K_1 = \operatorname{supp} K_2$.

Let P, P' be as in 8.2. We shall use the notations in 8.2 except that we shall write $\tilde{K}_1$ (resp. $\tilde{K}_2$) for $\tilde{K}$ defined in terms of $\mathscr{E}_1$ (resp. $\mathscr{E}_2$) and we shall write $\tilde{K}'_1$ (resp. $\tilde{K}'_2$) for $\tilde{K}'$ defined in terms of $\mathscr{E}_1$ (resp. $\mathscr{E}_2$).

From (8.2.4) we get commutative diagrams

$$\begin{tikzcd} \mathscr{H}^i_u(K_1) \arrow[r, "\varphi_1", "\sim"'] \arrow[d, "\wr"] & \mathscr{H}^i_u(K_1) \arrow[d, "\wr"'] \\ \mathscr{H}^i_u(\psi'_! \tilde{K}'_1) \arrow[r, "\tilde{\varphi}_1", "\sim"'] & \mathscr{H}^i_u(\psi_! \tilde{K}_1) \end{tikzcd}$$

$$\begin{array}{ccc}\mathscr{H}^i_u(K_2) & \xrightarrow[\sim]{\varphi_2} & \mathscr{H}^i_u(K_2)\\ \wr\downarrow & & \wr\downarrow\\ \mathscr{H}^i_u(\psi'_!\tilde{K}'_2) & \xrightarrow[\sim]{\tilde{\varphi}_2} & \mathscr{H}^i_u(\psi_!\tilde{K}_2)\end{array}$$

where the vertical maps are given by (8.2.3). It is therefore enough to construct isomorphisms δ', δ such that the diagram

$$\begin{array}{ccc}\mathscr{H}^i_u(\psi'_!\tilde{K}'_1) & \xrightarrow[\sim]{\tilde{\varphi}_1} & \mathscr{H}^i_u(\psi_!\tilde{K}_1)\\ \delta'\downarrow\wr & & \wr\downarrow\delta\\ \mathscr{H}^i_u(\psi'_!\tilde{K}'_2) & \xrightarrow[\sim]{\tilde{\varphi}_2} & \mathscr{H}^i_u(\psi_!\tilde{K}_2)\end{array}$$

is commutative. This would follow from the existence of two isomorphisms $\varepsilon'\colon \tilde{K}'_1|_{\text{uni}} \xrightarrow{\sim} \tilde{K}'_2|_{\text{uni}}$, $\varepsilon\colon \tilde{K}_1|_{\text{uni}} \xrightarrow{\sim} \tilde{K}_2|_{\text{uni}}$ which are compatible with the isomorphisms $F^*\tilde{K}'_1 \xrightarrow{\sim} \tilde{K}_1$, $F^*\tilde{K}'_2 \xrightarrow{\sim} \tilde{K}_2$ induced by $\varphi_{01}, \varphi_{02}$. (Here $|_{\text{uni}}$ denotes restriction to the subvariety of X or X' which is inverse image under ψ or ψ' of the set of unipotent elements of $\bar{Y}$.)
But from the definition of $\tilde{K}_1$ it is clear that $\tilde{K}_1|_{\text{uni}}$ can be defined purely in terms of the restriction of $\mathscr{E}_1$ to C (without using any information on $\mathscr{E}$ on $\Sigma - C$). A similar statement holds for $\tilde{K}_2, \tilde{K}'_1, \tilde{K}'_2$. It follows that $\gamma\colon \mathscr{E} \mid C \xrightarrow{\sim} \mathscr{E}' \mid C$ induces the required isomorphisms $\varepsilon, \varepsilon'$; their compatibility with $\varphi_{01}, \varphi_{02}$ follows from the compatibility of γ with $\varphi_{01}, \varphi_{02}$.

This completes the proof of (8.3.2). The function (8.3.1) is called a *generalized Green function.*

8.4

Given an algebraic variety Z defined over F_q (with Frobenius map $F\colon G \to G$), an object $A \in \mathscr{D}X$ and an isomorphism $\varphi\colon F^*A \xrightarrow{\sim} A$, in $\mathscr{D}X$, we define the *characteristic function* $\chi_{A,\varphi}\colon Z^F - \bar{\mathbb{Q}}_l$ by

$$\chi_{A,\varphi}(z) = \sum_i (-1)^i \operatorname{Tr}(\varphi, \mathscr{H}^i_z A) \qquad (z \in Z^F). \tag{8.4.1}$$

(cf. (8.3.1)).

We wish to state a result expressing the characteristic function $\chi_{K,\varphi}$ of K, φ (defined from $L, \Sigma, \mathscr{E}, \varphi_0$ as in 8.1) in terms of generalized Green functions. We shall assume that $\mathscr{E}$ is irreducible.

Let s be a semisimple element of G^F and let u be a unipotent element of G^F, commuting with s. (In the rest of this chapter s and u will be fixed.)

Let Σ_1 be the set of semisimple parts of elements of Σ. Assume that $x \in G^F$ is an element such that $x^{-1}sx \in \Sigma_1$. Then $s \in xLx^{-1}$ so that the group L_x defined by $L_x = xLx^{-1} \cap Z^0_G(s)$ is a Levi subgroup of some parabolic subgroup of $Z_G^0(s)$. Let C_x be the set of unipotent elements in $Z_G^0(s)$ such that $sv \in x\Sigma x^{-1}$. Let $\mathscr{F}_x$ be the local system on C_x defined as the inverse image of $\mathscr{E}$ under the map $C_x \to \Sigma$, $v \mapsto x^{-1}svx$. Since this map is defined over F_q, $\varphi_1: F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ induces an isomorphism $\varphi_x: F^*\mathscr{F}_x \xrightarrow{\sim} \mathscr{F}_x$ of local systems over C_x. By 7.11(a) and (c), the set C_x is a single unipotent class of $Z_G^0(s)$ and $\mathscr{F}_x$, $\varphi_x: F^*\mathscr{F}_x \to \mathscr{F}_x$ are as in 8.3. We can now state:

THEOREM 8.5. *With the notations and assumptions of* 8.4, *we have*

$$\chi_{K,\varphi}(su) = \sum_{\substack{x \in G^F \\ x^{-1}sx \in \Sigma_1}} (|L_x^F|/|Z_G^0(s)^F|\,|L^F|)\, Q_{L_x, Z_G^0(s), C_x, \mathscr{F}_x, \varphi_x}(u).$$

(In the case where $L = \Sigma = \Sigma_1$ is a maximal torus of G, this formula should be compared with the character formula in [3, 4.2].)

The proof of the theorem will make use of the following result.

LEMMA 8.6. *Let P be a parabolic subgroup of G having L as a Levi subgroup. There exists an open set $\mathscr{U}$ in $Z_G^0(s)$ containing e and satisfying properties* (a)–(e) *below*:

(a) *$g\mathscr{U}g^{-1} = \mathscr{U}$ for all $g \in Z_G^0(s)$.*

(b) *$x \in \mathscr{U} \Leftrightarrow x_s \in \mathscr{U}$, ($x_s$ = semisimple part of x).*

(c) *$F\mathscr{U} = \mathscr{U}$.*

(d) *If $g \in \mathscr{U}$, $x \in G$, $x^{-1}sgx \in \bar{\Sigma}U_P$, then $x^{-1}g_sx \in \mathscr{Z}_L^0 U_P$ and $x^{-1}sx \in \Sigma_1 U_P$.*

(e) *If $g \in \mathscr{U}$, $x \in G$, $x^{-1}sgx \in \Sigma$, then $x^{-1}g_sx \in \mathscr{Z}_L^0$ and $x^{-1}sx \in \Sigma_1$.*

8.7

In this section we fix P and $\mathscr{U}$ as in 8.6. Let

$$X_{\mathscr{U}} = \{(g, xP) \in X \mid g \in s\mathscr{U}\}, \tag{8.7.1}$$

where X is as in 8.2. Let

$$\mathscr{M} = \{x \in G \mid x^{-1}sx \in \Sigma_1\}, \qquad \Gamma = Z^0(s)\backslash\mathscr{M}/L, \tag{8.7.2}$$

$$\hat{\mathscr{M}} = \{x \in G \mid x^{-1}sx \in \Sigma_1 U_P\}, \qquad \hat{\Gamma} = Z^0(s)\backslash\hat{\mathscr{M}}/P. \tag{8.7.3}$$

We shall assume that $\mathscr{M}$ is nonempty. It is easy to see that Γ is finite and that the natural map $\Gamma \to \hat{\Gamma}$ is bijective. We shall regard elements of Γ (resp. $\hat{\Gamma}$) as subsets of G: double cosets with respect to $Z^0(s)$, L (resp. $Z^0(s)$, P). We define

$$X_{\mathscr{U},\hat{\mathscr{O}}} = \{(sg, xP) \in X_{\mathscr{U}} \mid x \in \hat{\mathscr{O}}\}, \qquad \hat{\mathscr{O}} \in \hat{\Gamma}. \tag{8.7.4}$$

The orbits of $Z^0(s)$ on $\{xP \in G/P \mid x^{-1}sx \in \Sigma_1 U_P\}$ are clearly complete varieties, hence they are closed. The number of these orbits being finite (in 1–1 correspondence with $\hat{\Gamma}$) they are also open. From 8.6(d) it follows that the sets $X_{\mathscr{U},\hat{\mathcal{O}}}$ ($\hat{\mathcal{O}} \in \hat{\Gamma}$) cover $X_{\mathscr{U}}$. Thus

(8.7.5) *The sets* $X_{\mathscr{U},\hat{\mathcal{O}}}$, ($\hat{\mathcal{O}} \in \hat{\Gamma}$), *are open and closed in* $X_{\mathscr{U}}$; *they form a finite partition of* $X_{\mathscr{U}}$.

For $\mathcal{O} \in \Gamma$ we denote by $\hat{\mathcal{O}}$ the corresponding element of $\hat{\Gamma}$. We choose a base point $x_{\mathcal{O}} \in \mathcal{O}$ for each $\mathcal{O}$. We may assume that this choice is such that $F(x_{\mathcal{O}}) = x_{F(\mathcal{O})}$ for all $\mathcal{O}$. (Note that F acts naturally on Γ.) Let $P_{\mathcal{O}} = x_{\mathcal{O}} P x_{\mathcal{O}}^{-1} \cap Z_G^0(s)$; it is a parabolic subgroup of $Z_G^0(s)$ with Levi subgroup $L_{\mathcal{O}} = x_{\mathcal{O}} L x_{\mathcal{O}}^{-1} \cap Z_G^0(s)$, since $s \in x_{\mathcal{O}} L x_{\mathcal{O}}^{-1}$. Let $C_{\mathcal{O}} = \{v \in Z_G^0(s) \mid v \text{ unipotent}, x_{\mathcal{O}}^{-1} svx_{\mathcal{O}} \in \Sigma\}$; as we have seen in 8.4, $C_{\mathcal{O}}$ is a unipotent class in $Z_G^0(s)$. Let $\Sigma_{\mathcal{O}} = \mathscr{Z}_{L_{\mathcal{O}}}^0 \cdot C_{\mathcal{O}}$. Let $\mathscr{E}_{\mathcal{O}}$ be the local system on $\Sigma_{\mathcal{O}}$ obtained as the inverse image of $\mathscr{E}$ under the map $\Sigma_{\mathcal{O}} \to \Sigma$, $g \to x_{\mathcal{O}}^{-1} sgx_{\mathcal{O}}$.

(8.7.6) Let $\pi'_{\mathcal{O}}\colon \tilde{Y}'_{\mathcal{O}} \to Y'_{\mathcal{O}}$, $\tilde{\mathscr{E}}_{\mathcal{O}}$, $K_{\mathcal{O}}$, $\psi_{\mathcal{O}}\colon X'_{\mathcal{O}} \to \bar{Y}'_{\mathcal{O}}$, $\tilde{K}_{\mathcal{O}}$ be defined in terms of $Z^0(s)$, $L_{\mathcal{O}}$, $\Sigma_{\mathcal{O}}$, $\mathscr{E}_{\mathcal{O}}$ in the same way as $\pi\colon \tilde{Y} \to Y$, $\tilde{\mathscr{E}}$, K, $\psi\colon X \to \bar{Y}$, $\tilde{K}$ were defined in 8.1, 8.2 in terms of G, L, Σ, $\mathscr{E}$.

Let

$$X'_{\mathscr{U},\hat{\mathcal{O}}} = \psi_{\mathcal{O}}^{-1}(\mathscr{U}) \subset X'_{\hat{\mathcal{O}}}. \tag{8.7.7}$$

We now show

(8.7.8) *The map* $(g, zP_{\mathcal{O}}) \to (sg, zx_{\mathcal{O}}P)$ *defines an isomorphism* $X'_{\mathscr{U},\hat{\mathcal{O}}} \cong X_{\mathscr{U},\hat{\mathcal{O}}}$.

Assume that $(g, zP_{\mathcal{O}}) \in X'_{\mathscr{U},\hat{\mathcal{O}}}$. Then $z^{-1}gz \in \mathscr{Z}_{L_{\mathcal{O}}}^0 \bar{C}_{\mathcal{O}} U_{P_{\mathcal{O}}}$ hence $(zx_{\mathcal{O}})^{-1} sg(zx_{\mathcal{O}}) \in x_{\mathcal{O}}^{-1} s \mathscr{Z}_{L_{\mathcal{O}}}^0 \bar{C}_{\mathcal{O}} U_{P_{\mathcal{O}}} x_{\mathcal{O}} \subset x_{\mathcal{O}}^{-1} s \mathscr{Z}_{x_{\mathcal{O}} L x_{\mathcal{O}}^{-1}}^0 \bar{C}_{\mathcal{O}} x_{\mathcal{O}} U_P \subset \mathscr{Z}_{L_{\mathcal{O}}}^0 \bar{\Sigma} U_P = \bar{\Sigma} U_P$.

(We have $\mathscr{Z}_{L_{\mathcal{O}}}^0 = \mathscr{Z}_{x_{\mathcal{O}} L x_{\mathcal{O}}^{-1}}^0$, since s is isolated in $x_{\mathcal{O}} L x_{\mathcal{O}}^{-1}$, and $U_{P_{\mathcal{O}}} \subset x_{\mathcal{O}} U_P x_{\mathcal{O}}^{-1}$.) Hence $(sg, zx_{\mathcal{O}}P) \in X_{\mathscr{U},\hat{\mathcal{O}}}$ so that the map (8.7.8) is well defined. Now let $(sg, zx_{\mathcal{O}}P) \in X_{\mathscr{U},\hat{\mathcal{O}}}$ $(z \in Z_G^0(s))$. Then $x_{\mathcal{O}}^{-1} z^{-1} sgzx_{\mathcal{O}} \in \bar{\Sigma} U_P$; hence, by 8.6, we have $x_{\mathcal{O}}^{-1} z^{-1} g_s x_{\mathcal{O}} \in \mathscr{Z}_L^0 U_P$. Thus, we have $z^{-1} g_s z \in (\mathscr{Z}_{x_{\mathcal{O}} L x_{\mathcal{O}}^{-1}}^0 \cdot U_{x_{\mathcal{O}} P x_{\mathcal{O}}^{-1}}) \cap Z_G^0(s) = (\mathscr{Z}_{x_{\mathcal{O}} L x_{\mathcal{O}}^{-1}}^0 \cap Z_G^0(s))\, U_{P_{\mathcal{O}}}$. We have

$$z^{-1}gz \in (s^{-1} x_{\mathcal{O}} \bar{\Sigma} x_{\mathcal{O}}^{-1} U_{x_{\mathcal{O}} P x_{\mathcal{O}}^{-1}}) \cap Z_G^0(s) = ((s^{-1} x_{\mathcal{O}} \bar{\Sigma} x_{\mathcal{O}}^{-1}) \cap Z^0(s))\, U_{P_{\mathcal{O}}}.$$

Thus, $z^{-1}gz \in \xi U_{P_{\mathcal{O}}}$, where $\xi \in (s^{-1} x_{\mathcal{O}} \bar{\Sigma} x_{\mathcal{O}}^{-1}) \cap Z_G^0(s)$. Let $\xi = \xi_s \xi_u$ be the Jordan decomposition of ξ. Then $\xi_s \in \mathscr{Z}_{x_{\mathcal{O}} L x_{\mathcal{O}}^{-1}}^0 \cap Z_G^0(s) \subset \mathscr{Z}_{x_{\mathcal{O}} L x_{\mathcal{O}}^{-1}}^0 = \mathscr{Z}_{L_{\mathcal{O}}}^0$. We have $\xi \in Z_G^0(s)$ hence $\xi_u \in Z_G^0(s)$. We have $s\xi_u \xi_s = s\xi \in x_{\mathcal{O}} \bar{\Sigma} x_{\mathcal{O}}^{-1}$. Hence $s\xi_u \in x_{\mathcal{O}} \bar{\Sigma} x_{\mathcal{O}}^{-1} \xi_s^{-1} \subset x_{\mathcal{O}} \bar{\Sigma} \mathscr{Z}_L^0 x_{\mathcal{O}}^{-1} = x_{\mathcal{O}} \bar{\Sigma} x_{\mathcal{O}}^{-1}$. Thus, $\xi_s \xi_u \in \mathscr{Z}_{L_{\mathcal{O}}}^0 \bar{C}$. Hence the map (8.7.8) is onto. It is injective: if $zP_{\mathcal{O}}, z'P_{\mathcal{O}}$, $(z, z' \in Z_G^0(s))$ satisfy $zx_{\mathcal{O}}P =$

$z'x_{\mathcal{C}}P$, then $z^{-1}z' \in x_{\mathcal{C}}Px_{\mathcal{C}}^{-1} \cap Z_G^0(s) = P_{\mathcal{C}}$ hence $zP_{\mathcal{C}} = z'P_{\mathcal{C}}$. Thus, the map (8.7.8) is bijective. The proof of the fact that its inverse is a morphism is standard and will be omitted.

We now define $\tilde{Y}_{\mathcal{U}} = \pi^{-1}(Y \cap s\mathcal{U}) \subset \tilde{Y}$,

$$\tilde{Y}_{\mathcal{U},\mathcal{C}} = \{(g, xL) \in Y_{\mathcal{U}} \mid x \in \mathcal{C}\}, \qquad \mathcal{C} \in \Gamma.$$

$$\tilde{Y}'_{\mathcal{U},\mathcal{C}} = \pi_{\mathcal{C}}'^{-1}(\mathcal{U}) \subset \tilde{Y}'_{\mathcal{C}}, \qquad \mathcal{C} \in \Gamma.$$

From (8.2.1), (8.2.2), we deduce:

(8.7.9) The map $(g, xL) \mapsto (g, xP)$ gives an isomorphism of $\tilde{Y}_{\mathcal{U}}$ onto the open subset $\psi^{-1}(Y \cap s\mathcal{U})$ of $\chi_{\mathcal{U}}$. The map ψ is a proper map of $X_{\mathcal{U}}$ onto $\bar{Y} \cap s\mathcal{U}$ and $Y \cap s\mathcal{U}$ is open, dense in $\bar{Y} \cap s\mathcal{U}$.

From (8.2.1), (8.2.2) applies to $Z^0(s)$, $L_{\mathcal{C}}$, $P_{\mathcal{C}}$, $\Sigma_{\mathcal{C}}$ (instead of G, L, P, Σ) we deduce:

(8.7.10) The map $(g, zL_{\mathcal{C}}) \mapsto (g, zP_{\mathcal{C}})$ gives an isomorphism of $\tilde{Y}'_{\mathcal{U},\mathcal{C}}$ onto the open dense subset $\psi_{\mathcal{C}}^{-1}(Y'_{\mathcal{C}} \cap \mathcal{U})$ of $X'_{\mathcal{U},\mathcal{C}}$. The map $\psi_{\mathcal{C}}$ is a proper map of $X'_{\mathcal{U},\mathcal{C}}$ onto $\bar{Y}'_{\mathcal{C}} \cap \mathcal{U}$ and $Y'_{\mathcal{C}} \cap \mathcal{U}$ is open, dense in $\bar{Y}'_{\mathcal{C}} \cap \mathcal{U}$.

We now prove:

(8.7.11) *$\tilde{Y}_{\mathcal{U},\mathcal{C}}$ is a nonempty, open and closed subset of $\tilde{Y}_{\mathcal{U}}$. The subsets $\tilde{Y}_{\mathcal{U},\mathcal{C}}$ $(\mathcal{C} \in \Gamma)$, form a finite partition of $\tilde{Y}_{\mathcal{U}}$.*

From 8.6(e) we see that the sets $\tilde{Y}_{\mathcal{U},\mathcal{C}}$ $(\mathcal{C} \in \Gamma)$ cover $\tilde{Y}_{\mathcal{U}}$. Note that (8.7.9) identifies $\tilde{Y}_{\mathcal{U}}$ with an open subset of $X_{\mathcal{U}}$. It also identifies $\tilde{Y}_{\mathcal{U},\mathcal{C}}$ with the intersection of $\tilde{Y}_{\mathcal{U}}$ with $X_{\mathcal{U},\mathcal{C}}$. Since $X_{\mathcal{U},\mathcal{C}}$ is open and closed in $X_{\mathcal{U}}$ (see (8.7.5)) it follows that $\tilde{Y}_{\mathcal{U},\mathcal{C}}$ is open and closed in $\tilde{Y}_{\mathcal{U}}$. We now show that $\tilde{Y}_{\mathcal{U},\mathcal{C}}$ is nonempty. Let v be an element of $C_{\mathcal{C}}$. The intersection $\mathscr{Z}^0_{L_{\mathcal{C}}} \cap \mathcal{U}v^{-1}$ is an open dense subset of $\mathscr{Z}^0_{L_{\mathcal{C}}}$ (it contains e, since $v \in \mathcal{U}$ by 8.6(b), and is open since $\mathcal{U}$ is open in $Z_G^0(s)$). Clearly, the intersection $\mathscr{Z}^0_{L_{\mathcal{C}}} \cap s^{-1}(x_{\mathcal{C}}\Sigma_{\mathrm{reg}}x_{\mathcal{C}}^{-1})v^{-1}$ is also an open dense subset of $\mathscr{Z}^0_{L_{\mathcal{C}}}$. Hence $\mathscr{Z}^0_{L_{\mathcal{C}}} \cap \mathcal{U}v^{-1} \cap s^{-1}(x_{\mathcal{C}}\Sigma_{\mathrm{reg}}x_{\mathcal{C}}^{-1})v^{-1}$ is an open dense subset of $\mathscr{Z}^0_{L_{\mathcal{C}}}$; in particular, it is nonempty. Hence there exists $\zeta \in \mathscr{Z}^0_{L_{\mathcal{C}}}$ such that $(s\zeta v, x_0L) \in \tilde{Y}_{\mathcal{U},\mathcal{C}}$, proving that $\tilde{Y}_{\mathcal{U},\mathcal{C}}$ is nonempty.

(8.7.12) *The map $\psi: X_{\mathcal{U},\mathcal{C}} \to \bar{Y} \cap s\mathcal{U}$ is proper, with image equal to $s(\bar{Y}'_{\mathcal{C}} \cap \mathcal{U})$, and $X_{\mathcal{U},\mathcal{C}}$ is irreducible. We have $\psi(\tilde{Y}_{\mathcal{U},\mathcal{C}}) = Y_{\mathcal{U},\mathcal{C}}$, $\tilde{Y}_{\mathcal{U},\mathcal{C}} = \psi^{-1}(Y_{\mathcal{U},\mathcal{C}}) \cap X_{\mathcal{U},\mathcal{C}}$ and $Y_{\mathcal{U},\mathcal{C}}$ is open, dense in $s(\bar{Y}'_{\mathcal{C}} \cap \mathcal{U})$.*

The fact that $\psi: X_{\mathcal{U},\mathcal{C}} \to \bar{Y} \cap s\mathcal{U}$ is proper follows from (8.7.9) and (8.7.5); the statement about its image follows from (8.7.8) and (8.7.10). The irreducibility of $X_{\mathcal{U},\mathcal{C}}$ follows from (8.7.8) and the fact that $X'_{\mathcal{C}}$ and $X'_{\mathcal{U},\mathcal{C}}$ are irreducible. As $\tilde{Y}_{\mathscr{U},\mathcal{O}}$ is nonempty and open in $X_{\mathscr{U},\hat{\mathcal{O}}}$, (8.7.11), and $X_{\mathscr{U},\hat{\mathcal{O}}}$ is irreducible, it follows that $\tilde{Y}_{\mathscr{U},\mathcal{O}}$ is dense in $Y_{\mathscr{U},\hat{\mathcal{O}}}$. It is clear that $\tilde{Y}_{\mathscr{U},\mathcal{O}}$ is a union of fibres of $\psi: X_{\mathscr{U},\hat{\mathcal{O}}} \to s(\bar{Y}'_{\mathcal{O}} \cap \mathscr{U})$. Since this map is proper, the image of $\tilde{Y}_{\mathscr{U},\mathcal{O}}$ must be open and dense in $s(\bar{Y}'_{\mathcal{O}} \cap \mathscr{U})$.

We now describe the irreducible components of the locally closed subset $Y \cap s\mathscr{U}$ of G. The map $\pi: \tilde{Y} \to Y$ is proper hence its restriction $\pi: \tilde{Y}_{\mathscr{U}} \to Y \cap s\mathscr{U}$ is also proper. Using (8.7.11), it follows that the sets $Y_{\mathscr{U},\mathcal{O}}$ (images of $\tilde{Y}_{\mathscr{U},\mathcal{O}}$ under π) cover $Y \cap s\mathscr{U}$ and are closed in $Y \cap s\mathscr{U}$. It is easy to see that for $\mathcal{O}, \mathcal{O}' \in \Gamma$, the sets $Y_{\mathscr{U},\mathcal{O}}, Y_{\mathscr{U},\mathcal{O}'}$ are either disjoint or coincide; more precisely, they coincide if and only if $\mathcal{O}, \mathcal{O}'$ are in the same orbit of the group $N(L, \Sigma)/L = \{n \in G \mid n^{-1}Ln = L,\ n^{-1}\Sigma n = \Sigma\}/L$ which acts on Γ by right multiplication. Since $Y_{\mathscr{U},\mathcal{O}}$ are irreducible, (see (8.7.12)) it follows that

(8.7.13) *The irreducible components of $Y \cap s\mathscr{U}$ are disjoint. They are in 1–1 correspondence with the orbits of $N(L, \Sigma)/L$ on Γ. The irreducible component corresponding to the orbit of $\mathcal{O} \in \Gamma$ is $Y_{\mathscr{U},\mathcal{O}}$.*

We now define for each $N(L, \Sigma)/L$-orbit Z in Γ an open subset $\mathscr{V}_Z$ of $Y \cap s\mathscr{U}$:

$$\mathscr{V}_Z = \bigcap_{\mathcal{O} \in Z} (s(Y'_{\mathcal{O}} \cap \mathscr{U}) \cap Y_{\mathscr{U},\mathcal{O}}). \tag{8.7.14}$$

Then $\mathscr{V}_Z$ is an open dense subset of $Y_{\mathscr{U},\mathcal{O}}$ $(\mathcal{O} \in Z)$, as we can see from (8.7.10), (8.7.12). Note that $\mathscr{V}_Z$ is smooth, since it is an open suset of $s(Y'_{\mathcal{O}} \cap \mathscr{U})$ hence it is isomorphic to an open subset of $Y'_{\mathcal{O}}$ which is known to be smooth. This shows also that $\dim \mathscr{V}_Z = \dim Y'_{\mathcal{O}} = \dim Z^0_G(s) - \dim L_{\mathcal{O}} + \dim \mathscr{Z}^0_{L_{\mathcal{O}}} C_{\mathcal{O}}$ (see (8.2.2)). Note that $L_{\mathcal{O}}$ is the connected centralizer in L of an element in Σ_1 hence $\dim L_{\mathcal{O}} = \dim L - \dim(\Sigma_1/\mathscr{Z}^0_L)$. Moreover, $\dim \mathscr{Z}^0_{L_{\mathcal{O}}} = \dim \mathscr{Z}^0_L$ and $\dim C_{\mathcal{O}} = \dim \Sigma - \dim \Sigma_1$. Thus,

(8.7.15) $\dim \mathscr{V}_Z = \dim Z^0_G(s) - \dim L + \dim \Sigma$ *is independent of Z.*

It is easy to check that $F(\mathscr{V}_Z) = \mathscr{V}_{FZ}$. Let

(8.7.16) $\mathscr{V} = \bigcup_Z \mathscr{V}_Z$ (union over all $N(L, \Sigma)/L$-orbits Z in Γ).

(8.7.17) *$\mathscr{V}$ is an open dense smooth equidimensional subset of $Y \cap s\mathscr{U}$ and $F\mathscr{V} = \mathscr{V}$; the subsets V_Z are its irreducible components.*

8.8

We now prove Theorem 8.5, assuming Lemma 8.6. We may assume that $su \in \bar{Y}$; otherwise, su is not in the support of K and the identity in the theorem is trivial. This implies in particular that $\mathscr{M}$ in (8.7.2) is nonempty.

From (8.7.8), (8.7.9), (8.7.10), (8.7.11) we see that we have a commutative diagram

$$\begin{tikzcd} \tilde{Y} \mid \mathscr{V} \arrow[d] & \coprod_{\mathcal{O} \in \Gamma} (\tilde{Y}'_{\mathcal{O}} \mid s^{-1}\mathscr{V}) \arrow[l, "\approx"', "\alpha"] \arrow[d] \\ \mathscr{V} & s^{-1} \cdot \mathscr{V} \arrow[l, "\approx"', "\varepsilon"] \end{tikzcd} \tag{8.8.1}$$

where $\tilde{Y} \mid \mathscr{V} = \{(g, xL) \in \tilde{Y} \mid g \in \mathscr{V}\}$,

$\tilde{Y}'_{\mathcal{O}} \mid s^{-1}\mathscr{V} = \{(g, zL_{\mathcal{O}}) \in Y'_{\mathcal{O}} \mid g \in s^{-1}\mathscr{V}_Z\} \qquad (\mathcal{O} \in Z) \qquad$ see (8.7.14)),

$\alpha(g, zL_{\mathcal{O}}) = (sg, zx_{\mathcal{O}} L)$,

$\varepsilon(g) = sg$,

and the vertical maps are projections to the first component.

All maps in the diagram (8.8.1) are clearly defined over F_q. (The parabolic group P which is in general not defined over F_q, does not enter in (8.8.1).) It follows that we have a canonical isomorphism of local systems over $s^{-1}\mathscr{V}$:

$$\varepsilon^*(\pi_*(\bar{\mathscr{E}}) \mid \mathscr{V}) \cong \bigoplus_{\mathcal{O} \in \Gamma} ((\pi'_{\mathcal{O}})_*(\bar{\mathscr{E}}_{\mathcal{O}}) \mid s^{-1}\mathscr{V}) \tag{8.8.2}$$

and this isomorphism is compatible with the liftings of the Frobenius map, given by φ. (Here, $(\pi'_{\mathcal{O}})_*(\bar{\mathscr{E}}_{\mathcal{O}}) \mid s^{-1}\mathscr{V}$ has the following meaning. For each $\mathcal{O} \in \Gamma$, $(\pi'_{\mathcal{O}})_*\bar{\mathscr{E}}_{\mathcal{O}}$ may be restricted to only one irreducible component $s^{-1}\mathscr{V}_Z$ of $s^{-1}\mathscr{V}$; we regard it as zero on the other irreducible components of $s^{-1}\mathscr{V}$.) By the definition of K, $K_{\mathcal{O}}$, (8.8.2) can be also regarded as an isomorphism

$$\varepsilon^*(K \mid \mathscr{V})[-\delta] \cong \bigoplus_{\mathcal{O} \in \Gamma} (K_{\mathcal{O}} \mid s^{-1}\mathscr{V}), \tag{8.8.3}$$

where $\delta = \dim Y - \dim Y'_{\mathcal{O}} = \dim G - \dim Z^0_G(s)$ (see (8.2.2)). Assume that we can show that the isomorphism (8.8.3) is the restriction to $s^{-1}\mathscr{V}$ of an isomorphism

$$\varepsilon^*(K \mid \bar{Y} \cap s\mathscr{U})[-\delta] \cong \bigoplus_{\mathcal{O} \in \Gamma} (K_{\mathcal{O}} \mid s^{-1}\bar{Y} \cap \mathscr{U}). \tag{8.8.4}$$

(Here, ε is regarded as an isomorphism $s^{-1}\bar{Y} \cap \mathscr{U} \to \bar{Y} \cap s\mathscr{U}$, $g \mapsto sg$.) The isomorphism (8.8.4) extending (8.8.3) is unique (if it exists) and is automatically compatible with the liftings φ of the Frobenius maps since (8.8.3) is. This follows from properties of intersection cohomology complexes. (We use the following fact: The right-hand side of (8.8.4) is (up to shift) an intersection cohomology complex on $s^{-1}\bar{Y}\cap\mathscr{U}$ associated to a local system on the open dense smooth equidimensional subset $s^{-1}\mathscr{V}$.) The isomorphism (8.8.4) gives rise to an isomorphism of stalks:

$$\mathscr{H}^{i-\delta}_{su}K\cong\bigoplus_{\mathscr{O}\in\Gamma}\mathscr{H}^{i}_{u}K_{\mathscr{O}} \tag{8.8.5}$$

(for all i) which is automatically compatible with the action of φ and φ_0, where φ_0: $F^*F_{F_{\mathscr{O}}}\xrightarrow{\sim}K_{\mathscr{O}}$ is the isomorphism induced by φ. Taking now alternating sums of traces of φ in (8.8.5), we find

$$\chi_{K,\varphi}(su)=\sum_{\substack{\mathscr{O}\in\Gamma\\ F\mathscr{O}=\mathscr{O}}}\chi_{K_{\mathscr{O}},\varphi_{\mathscr{O}}}(u).$$

(Note that δ in (8.8.3) is even since it is the dimension of the conjugacy class of s in G.) This implies the theorem, in view of the definition (8.3.1) of generalized Green functions and the identity $|\mathscr{O}^F|=|Z^0_G(s)^F|\cdot|L^F|\cdot|L^F_{\mathscr{O}}|^{-1}$. It remains to construct the isomorphism (8.8.4). Using (8.2.3) we find isomorphisms

$$\begin{aligned} K\mid\bar{Y}\cap s\mathscr{U}&\xrightarrow{\sim}\psi_!(\tilde{K}\mid X_{\mathscr{U}}),\\ K_{\mathscr{O}}\mid\bar{Y}'_{\mathscr{O}}\cap\mathscr{U}&\xrightarrow{\sim}(\psi_{\mathscr{O}})_!(\tilde{K}_{\mathscr{O}}\mid X_{\mathscr{U},\hat{\mathscr{O}}}).\end{aligned} \tag{8.8.6}$$

From (8.7.5), (8.7.8) we get an isomorphism

$$\varepsilon^*(\psi_!(\tilde{K}\mid X_{\mathscr{U}}))[-\delta]\xrightarrow{\sim}\bigoplus_{\mathscr{O}\in\Gamma}(\psi_{\mathscr{O}})_!(\tilde{K}_{\mathscr{O}}\mid X'_{\mathscr{U},\hat{\mathscr{O}}}). \tag{8.8.7}$$

(We regard $(\psi_{\mathscr{O}})_!(\tilde{K}_{\mathscr{O}}\mid X'_{\mathscr{U},\hat{\mathscr{O}}})$ as a perverse sheaf on $s^{-1}\bar{Y}\cap\mathscr{U}$, equal to zero outside $\bar{Y}'_{\mathscr{O}}\cap\mathscr{U}$.) The shift $[-\delta]$ in (8.8.7) comes from the fact that under (8.7.8), the restriction of $\tilde{K}[-\delta]$ to $X_{\mathscr{U},\hat{\mathscr{O}}}$ corresponds to the restriction of $\tilde{K}_{\mathscr{O}}$ to $X_{\mathscr{U},\hat{\mathscr{O}}}$.

Combining the isomorphisms (8.8.6), (8.8.7) we find an isomorphism as in (8.8.4); from the definitions it follows that it extends the isomorphism (8.8.3). This completes the proof of Theorem 8.5 assuming Lemma 8.6.

8.9. *Proof of Lemma* 8.6

A subset $\mathscr{U}$ of $Z^0_G(s)$ is said to be stable if it has the properties (b) and (c) in 8.8.

(8.9.1) *There exists a stable open subset $\mathscr{U}_1\subset Z^0_G(s)$ containing e such that $F\mathscr{U}_1=\mathscr{U}_1$ and such that $x\in\mathscr{U}_1\Rightarrow Z_G(sx_s)\subset Z_G(s)$.*

We imbed G into $\hat{G}=GL_n(k)$ as a closed subgroup defined over F_q. Let

$\mathscr{U}'_1 = \{ g \in \hat{G} \mid Z_{\hat{G}}(g_s) \subset Z_{\hat{G}}(s) \}$, and let $\mathscr{U}_1 = s^{-1}\mathscr{U}'_1 \cap Z^0_G(s)$. It is clear that $\mathscr{U}_1$ has the required properties.

Let $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$ be the inverse images under $L \to L/\mathscr{Z}^0_L$ of the various isolated semisimple classes in $L/\mathscr{Z}^0_L$. (We number them in such a way that the first one of these sets is the set Σ_1 introduced in 8.4.)

Let $\mathscr{M}_j = \{ x \in G \mid x^{-1}sx \in \Sigma_j \}$ $(1 \leqslant j \leqslant m)$. Let $\mathscr{C}_{j,i}$ $(1 \leqslant i \leqslant r(j))$, be the orbits of $Z^0_G(s) \times L$ acting on $\mathscr{M}_j$ by $x \mapsto zxl^{-1}$ $(z \in Z^0_G(s),\ l \in L)$.

Let

$$\mathscr{S} = Z^0_G(s) \cap \left(\bigcup_{i,j} \left(\bigcup_{x \in \mathscr{C}_{j,i}} s^{-1}x \left(\bigcup_{k \neq j} \Sigma_k \right) x^{-1} \right) \right).$$

Let $\mathscr{S}' = \{ g \in Z^0_G(s) \mid g_s \in \mathscr{S} \}$. Then $\mathscr{S}'$ is a closed stable subset of $Z^0_G(s)$ not containing e.

(8.9.2) *The set $\mathscr{U}_2 = \mathscr{U}_1 - \mathscr{S}'$ is a stable open subset of $Z^0_G(s)$ containing e and we have $g \in \mathscr{U}_2$, $x \in G$, $x^{-1}sgz \in \Sigma \Rightarrow x^{-1}sx \in \Sigma_1$.*

Assume that $g \in \mathscr{U}_2$, $x \in G$, $x^{-1}sgx \in \Sigma_1$. Then $x^{-1}sg_sx \in \Sigma_1$. By (8.9.1) we have $Z_G(x^{-1}sg_sx) \subset Z_G\,(x^{-1}sx)$. It follows that $x^{-1}sx$ is isolated in L, i.e., $x^{-1}sx \in \Sigma_j$ for some j. If $j \neq 1$, then by the definition of $\mathscr{S}$, we have $g_s \in \mathscr{S}$, hence $g \in \mathscr{S}'$. Thus, if $g \in \mathscr{U}_1 - \mathscr{S}'$, we have $j = 1$, i.e., $x^{-1}sx \in \Sigma_1$, as required.

Let $\mathscr{T} = Z^0_G(s) \cap \bigcup_{x \in \mathscr{M}_1}((s^{-1}x\Sigma_1 x^{-1} \cap L_x) - \mathscr{Z}^0_{L_x})$, where L_x is as in 8.4. Let $\mathscr{T}' = \{ g \in Z^0_G(s) \mid g_s \in \mathscr{T} \}$. Then $\mathscr{T}'$ is a closed, stable subset of $Z^0_G(s)$ not containing e.

(8.9.3) *The set $\mathscr{U}_3 = \mathscr{U}_2 - \mathscr{T}'$ is a stable open subset of $Z^0_G(s)$ containing e and we have $g \in \mathscr{U}_3$, $x \in G$, $x^{-1}sgx \in \Sigma \Rightarrow x^{-1}g_sx \in \mathscr{Z}^0_L$.*

Let $g \in \mathscr{U}_3$, $x \in G$ be such that $x^{-1}sgx \in \Sigma$. Then $x^{-1}sg_s \in \Sigma_1$. From (8.9.3) we know that we must have $x^{-1}sx \in \Sigma_1$, i.e., $x \in \mathscr{M}_1$. Assume that $x^{-1}g_sx \notin \mathscr{Z}^0_L$. Then $g_s \notin \mathscr{Z}^0_{xLx^{-1}} = \mathscr{Z}^0_{L_x}$, hence $g_s \in \mathscr{T}$, hence $g \in \mathscr{T}'$, contradicting $g \in \mathscr{U}_3$. Thus, we have $x^{-1}g_sx \in \mathscr{Z}^0_L$, as required.

(8.9.4) *If $g \in \mathscr{U}_3$, $x \in G$, $x^{-1}sgx \in \bar{\Sigma}U_P$, then $x^{-1}sx \in \Sigma_1 U_P$ and $x^{-1}g_sx \in \mathscr{Z}^0_L U_P$.*

We have $x^{-1}sg_sx \in \Sigma_1 U_P$. Replacing x by xp for some $p \in P$, we can assume that $x^{-1}sg_sx \in \Sigma_1$. But then $x^{-1}sx \in \Sigma_1$, $x^{-1}g_sx \in \mathscr{Z}^0_L$ by (8.9.2), (8.9.3). Thus (8.9.4) follows.

We may take $\mathscr{U} = \mathscr{U}_3$; Lemma 8.6 is proved.

9. Orthogonality for the Generalized Green Functions

9.1

In this chapter we preserve the assumptions of 8.0. Let L, Σ, $\mathscr{E}$, φ_0: $F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$, K, $\varphi: F^*K \xrightarrow{\sim} K$ be as in 8.1. Let L', Σ', $\mathscr{E}'$, φ'_0: $F^*\mathscr{E}' \xrightarrow{\sim} \mathscr{E}'$, K', $\varphi': F^*K' \xrightarrow{\sim} K'$ be another set of data of the same kind in G. Let Σ_1 be as in 8.4 and let Σ'_1 be defined similarly, in terms of Σ'.

Let θ be the set of all $n \in G^F$ such that $nLn^{-1} = L'$, $n\Sigma n^{-1} = \Sigma'$.

We shall make the following assumption.

(9.1.1) Assume that $K_0 = \mathrm{IC}(\bar{\Sigma}, \mathscr{E})[\dim \Sigma]$, $K'_0 = \mathrm{IC}(\bar{\Sigma}', \mathscr{E}')[\dim \Sigma']$ are strongly cuspidal complexes for L, L', respectively. (We regard these as complexes on L, L' equal to 0 on $L - \bar{\Sigma}$, $L' - \bar{\Sigma}'$.) Assume also that either L, L' are not conjugate in G or L, L' are conjugate in G and both K_0, K'_0 are clean.

With these assumptions, we can state the following two results, which will be prove in 9.4–9.6.

Theorem 9.2. *If $\mathscr{E}, \mathscr{E}'$ are irreducible, then*

$$|G^F|^{-1} \sum_{g \in G^F} \chi_{K,\varphi}(g)\, \chi_{K',\varphi'}(g)$$
$$= |L^F|^{-1}\, |L'^F|^{-1} \sum_{n \in \theta} \sum_{\xi \in \Sigma^F} \chi_{\mathscr{E},\varphi_0}(\xi)\, \chi_{\mathscr{E}',\varphi'_0}(n\xi n^{-1}). \tag{9.2.1}$$

Theorem 9.3. *If $\Sigma = \mathscr{Z}^0_L C$, $\Sigma' = \mathscr{Z}^0_{L'} C'$ (C, C' are unipotent classes in L, L') and if $\mathscr{F}$, φ_1 (resp. $\mathscr{F}'$, φ'_1) is the restriction of $\mathscr{E}$, φ_0 (resp. $\mathscr{E}'$, φ'_0) to C (resp. C'), then*

$$|G^F|^{-1} \sum_{\substack{u \in G^F \\ \text{unipotent}}} Q_{L,G,C,\mathscr{F},\varphi_1}(u)\, Q_{L',G,C',\mathscr{F}',\varphi'_1}(u)$$
$$= |L^F|^{-1}\, |L'^F|^{-1} \sum_{n \in \theta} \sum_{\xi \in C^F} \chi_{F,\varphi_1}(\xi)\, \xi_{\mathscr{F}',\varphi'_1}(n\xi n^{-1}). \tag{9.3.1}$$

9.4

First we note that

(9.4.1) *Theorem 9.2 holds if L, L' are not conjugate under an element of G.*

Indeed, by the trace formula for Frobenius maps, the left-hand side of (9.2.1) is equal to $\Sigma(-1)^i \operatorname{Tr}(\mathrm{Fr}, H^i_c((G, K \otimes K')))$. (The map Fr is defined as the composition $H^i_c(G, K \otimes K') \rightarrow^{F^*} H^i_c(G, F^*K \otimes F^*K') \rightarrow^{\varphi \otimes \varphi'}$

$H_c^i(G, K \otimes K')$). By 7.2 we have $H_c^i(G, K \otimes K') = 0$ for all i, hence the left-hand side of (9.2.1) is zero. The right-hand side is also zero since, under our assumption, θ is empty.

We now prove that

(9.4.2) *Theorem* 9.2 *holds under the following assumptions*: $\Sigma = \mathscr{Z}_L^0 C$, $\Sigma' = \mathscr{Z}_{L'}^0 C'$ (*as in* 9.3), $\mathscr{E}' \mid \mathscr{Z}_{L'}^0$ *is a constant sheaf*, $\mathscr{E} \mid \mathscr{Z}_L^0$ *is a nonconstant sheaf*.

We first show that $H_c^i(G, K \otimes K') = 0$ for all i. We may assume that L, L' are conjugate in G (see (9.4.1)). Then K_0, K_0' are clean, see (9.1.1), and by 7.2 it is enough to show that $H_c^i(L, K_0 \otimes f^* K_0') = 0$ for any i and any isomorphism $f: L \xrightarrow{\sim} L'$ given by conjugation by an element in G; the last equality follows from 7.7 and our assumption on $\mathscr{E}$, $\mathscr{E}'$. From the vanishing of $H_c^i(G, K \otimes K')$ we deduce, as in the proof of (9.4.1), that the left-hand side of (9.2.1) is zero. We now show that the right-hand side of (9.2.1) is zero. It is enough to show that for any $n \in \theta$, the sum $\Sigma_{\xi \in \Sigma^F} \chi_{\mathscr{E}, \varphi_0}(\xi) \chi_{\mathscr{E}', \varphi_0}(n \xi n^{-1})$ is zero. By the trace formula for Frobenius maps, this sum is equal to the alternating sum of traces of the Frobenius map on $H_c^i(\Sigma, \mathscr{E} \otimes \mathrm{ad}(n)^* \mathscr{E}')$. It is enough to show that the last space is zero for all i. This follows from 7.7 and our assumptions on $\mathscr{E}$, $\mathscr{E}'$.

Next we show:

(9.4.3) *Theorem* 9.3 *follows from its special case in which* $\mathscr{Z}_L^0 \cap \mathscr{Z}_{L'}^0 \cap \mathscr{Z}_G = \{e\}$.

Let $\Gamma = \mathscr{Z}_L^0 \cap \mathscr{Z}_{L'}^0 \cap \mathscr{Z}_G$ and let $\bar{G} = G/\Gamma$. Let $\bar{L}$, $\bar{L}'$, $\bar{C}$, $\bar{C}'$ be the images of L, L', C, C' under the canonical map $\rho: G \to \bar{G}$. Note that $\bar{G}$, $\bar{L}$, $\bar{L}'$, $\bar{C}$, $\bar{C}'$ have natural F_q-structures. Let $\bar{\mathscr{F}}$ (resp. $\bar{\mathscr{F}}'$) be the local system on $\bar{C}$ (resp. $\bar{C}'$) defined by $\mathscr{F}$ (resp. $\mathscr{F}'$) by the isomorphism $C \xrightarrow{\sim} \bar{C}$ (resp. $C' \xrightarrow{\sim} \bar{C}'$) induced by ρ. Let $\bar{\varphi}_1: F^* \bar{\mathscr{F}} \xrightarrow{\sim} \bar{\mathscr{F}}$, $\bar{\varphi}_1': F^* \bar{\mathscr{F}}' \xrightarrow{\sim} \bar{\mathscr{F}}'$ be induced by φ_1, φ_1'. Then $\bar{\mathscr{F}}$ is $\bar{L}$-equivariant (since $\Gamma \subset \mathscr{Z}_L^0$) and similarly $\bar{\mathscr{F}}'$ is $\bar{L}'$-equivariant. We can define a complex $\bar{K}$ on $\bar{G}$ in terms of $\bar{G}$, $\bar{L}$, $1 \boxtimes \bar{\mathscr{F}}$,..., in the same way as K was defined in 8.1 in terms of G, L, $\mathscr{E}$,.... We define similarly $\bar{K}'$ on $\bar{G}$, in terms of $\bar{G}$, $\bar{L}$, $1 \boxtimes \bar{\mathscr{F}}'$,.... We have $K = \rho^* \bar{K}[\dim \Gamma]$, $K' = \rho^* \bar{K}'[\dim \Gamma]$, $Q_{L,G,C,\mathscr{F},\varphi_1}(u) = (-1)^{\dim \Gamma} Q_{\bar{L},\bar{G},\bar{C},\bar{\mathscr{F}},\bar{\varphi}_1}(\rho(u))$, and $Q_{L',G,C,\mathscr{F}',\varphi_1'}(u) = (-1)^{\dim \Gamma} Q_{\bar{L}',\bar{G},\bar{C}',\bar{\mathscr{F}}',\bar{\varphi}_1'}(\rho(u))$.

It follows immediately that the truth of (9.3.1) for $\bar{G}$ implies the truth of (9.3.1) for G. We have $\mathscr{Z}_{\bar{L}}^0 \cap \mathscr{Z}_{\bar{L}'}^0 \cap \mathscr{Z}_{\bar{G}} = \{e\}$ and (9.4.3) follows.

We now show:

(9.4.4) *Theorem* 9.2 *holds under the following assumption*: *there exist parabolic subgroups* P, P' *of* G, *defined over* F_q, *having* L, L' *as Levi subgroups*.

We may assume that L, L' are not conjugate in G (see (9.4.1)), so that K_0, K_0' are clean (see (9.1.1)). We shall use the description (8.2.3) of K; note that in our case, the map ψ is defined over F_q. We deduce that

$$\chi_{K,\varphi}(g)=|P^F|^{-1}\sum_{\substack{x\in G^F\\ x^{-1}gx\in\Sigma U_P}}\chi_{\mathscr{E},\varphi_0}(\pi_P(x^{-1}gx))\qquad (g\in G^F).$$

Similarly, we have

$$\chi_{K',\varphi'}(g)=|P'^F|^{-1}\sum_{\substack{x'\in G^F\\ x'^{-1}gx'\in\Sigma' U_{P'}}}\chi_{\mathscr{E}',\varphi_0'}(\pi_{P'}(x'^{-1}gx')).$$

It follows that

$$\begin{aligned}&|G^F|^{-1}\sum_{g\in G^F}\chi_{K,\varphi}(g)\,\chi_{K',\varphi'}(g)\\ &\quad=|G^F|^{-1}\,|P^F|^{-1}|\,P'^F|^{-1}\\ &\quad\times\sum_{\substack{g\in G^F\\ x,x'\in G^F\\ x^{-1}gx\in\Sigma U_P\\ x'^{-1}gx'\in\Sigma' U_{P'}}}\chi_{\mathscr{E},\varphi_0}(\pi_P(x^{-1}gx))\,\chi_{\mathscr{E}',\varphi_0'}(\pi_{P'}(x'^{-1}gx')).\end{aligned}$$

We partition he last sum into partial sums according to the $P-P'$ double coset of $x^{-1}x'$. The partial sums corresponding to double cosets PnP' such that $n^{-1}Pn$, P' do not have a common Levi subgroup are zero. This follows from the identity: $\sum_{u\in U_Q^F}\chi_{\mathscr{E},\varphi}(gu)=0$ valid for any F-stable parabolic $Q\subsetneq L$ and any $g\in Q^F$ (which follows from the fact that K_0 is strongly cuspidal and clean) and from the analogous identity for $\mathscr{E}'$.

Consider the partial sum corresponding to a double coset Pn_0P' such that $n_0^{-1}Pn_0$, P' have a common Levi subgroup. We may assume that $n_0^{-1}Ln_0=L'$; our partial sum can be rewritten as

$$|L^F|^{-1}\,|L'^F|^{-1}\sum_{\substack{n\in\theta^F\\ n\in Pn_0P'}}\sum_{\xi\in\Sigma^F}\chi_{\mathscr{E},\varphi_0}(\xi)\,\chi_{\mathscr{E}',\varphi_0'}(n\xi n^{-1})$$

and (9.4.4) follows.

9.5

We now prove that Theorem 9.2 holds for G under the assumption that Theorem 9.3 holds for G replaced by $Z_G^0(s)$ where s is any semisimple element of G^F.

We shall evaluate the left-hand side of (9.2.1) using Theorem 8.5. We have

$$|G^F|^{-1}\sum_{g\in G^F}\chi_{K,\varphi}(g)\,\chi_{K',\varphi'}(g)$$
$$=|G^F|^{-1}\sum_{\substack{s\in G^F\\ \text{semis.}\\ x,x'\in G^F\\ x^{-1}sx\in\Sigma_1\\ x'^{-1}sx'\in\Sigma_1'}} f(s,x,x')\,|L_x^F|\,|L'^F_{x'}|\,|Z^0(s)^F|^{-2}\,|L^F|^{-1}\,|L'^F|^{-1},$$

where

$$f(s,x,x')=\sum_{\substack{u\in Z^0(s)^F\\ \text{unipotent}}} Q_{L_x,Z^0(s),C_x,\mathscr{F}_x,\varphi_x}(u)\,Q_{L'_{x'},Z^0(s),C'_{x'},\mathscr{F}'_{x'},\varphi'_{x'}}(u).$$

(The notations L_x, C_x,..., are as in 8.4; these depend on s. The notations $L'_{x'}$, $C'_{x'}$,..., are defined similarly in terms of L', Σ',....) By our assumption, we have

$$f(s,x,x')=|Z^0(s)^F|\,|L_x^F|^{-1}\,|L'^F_{x'}|^{-1}$$
$$\times\sum_{\substack{n\in Z^0(s)^F\\ nL_xn^{-1}=L'_{x'}\\ nC_xn^{-1}=C'_{x'}}}\ \sum_{v\in C^F}\chi_{\mathscr{F}_x,\varphi_x}(v)\,\chi_{\mathscr{F}'_{x'},\varphi'_{x'}}(nvn^{-1}).$$

(To be able to apply our assumption, we must first verify that the appropriate complexes on L_x, $L'_{x'}$ are strongly cuspided. This follows from the assumption (9.1.1) together with 7.11(b).) Note that

$$\chi_{\mathscr{F}_x,\varphi_x}(v)=\chi_{\mathscr{E},\varphi_0}(x^{-1}svx),$$
$$\chi_{\mathscr{F}'_{x'},\varphi'_{x'}}(nvn^{-1})=\chi_{\mathscr{E}',\varphi'_0}(x'^{-1}snvn^{-1}x').$$

Note also that for $n\in Z^0(s)$, the condition $nL_xn^{-1}=L'_{x'}$ is equivalent to the condition $nxLx^{-1}n^{-1}=x'L'x'^{-1}$ (since s is isolated in xLx^{-1}) and the condition $nC_xn^{-1}=C'_{x'}$ is equivalent to the condition $nx\Sigma x^{-1}n^{-1}=x'\Sigma' x'^{-1}$. Hence

$$|G^F|^{-1}\sum_{g\in G^F}\chi_{K,\varphi}(g)\,\chi_{K',\varphi'}(g)$$
$$=|G^F|^{-1}\,|L^F|^{-1}\,|L'^F|^{-1}$$
$$\times\sum_{\substack{s\in G^F\\ x,x'\in G^F\\ x^{-1}sx\in\Sigma_1\\ x'^{-1}sx'\in\Sigma_1'}}|Z^0(s)^F|^{-1}\sum_{\substack{n\in Z^0(s)^F\\ nxLx^{-1}n^{-1}=x'L'x'^{-1}\\ nx\Sigma x^{-1}n^{-1}=x'\Sigma'x'^{-1}}}$$
$$\times\sum_{\substack{v\in s^{-1}x\Sigma x^{-1}\cap Z^0(s)^F\\ \text{uni.}}}\chi_{\mathscr{E},\varphi_0}(x^{-1}svx)\,\chi_{\mathscr{E}',\varphi'_0}(x'^{-1}snvn^{-1}x').$$

We now make the change of variable $(x, x', n) \mapsto (x, n, n')$, $n' = x'^{-1}nx$. The condition $nxLx^{-1}n^{-1} = x'L'x'^{-1}$ becomes $n'Ln'^{-1} = L'$; the condition $x'^{-1}sx' \in \Sigma_1$ becomes $n'x^{-1}n^{-1}snxn'^{-1} \in \Sigma'_1$, i.e., $x^{-1}sx \in n'^{-1}\Sigma'_1 n'$. Since $n'^{-1}\Sigma'_1 n' \cap \Sigma_1 = \varnothing$, we must in fact have $n'^{-1}\Sigma'_1 n' = \Sigma_1$, hence $n' \in \theta$. Our sum becomes

$$|G^F|^{-1}\, |L^F|^{-1}\, |L'^F|^{-1} \sum_{\substack{s \in G^F \\ x \in G^F \\ n \in Z^0(s)^F \\ n' \in \theta \\ x^{-1}sx \in \Sigma_1}} |Z^0(s)^F|^{-1} \sum_{\substack{v \in s^{-1}x\Sigma x^{-1} \cap Z^0(s)^F \\ \text{uni}}}$$
$$\times \chi_{\mathscr{E},\varphi_0}(x^{-1}svx)\, \chi_{\mathscr{E}',\varphi'_0}(n'x^{-1}svxn'^{-1}).$$

We now make the change of variable $(s\, x, v) \mapsto (\sigma, x, v')$, $\sigma = x^{-1}sx \in \Sigma_1^F$, $v' = x^{-1}vx \in \sigma\Sigma \cap Z^0(\sigma)^F$. Our sum becomes

$$|L^F|^{-1}\, |L'^F|^{-1} \sum_{\substack{\sigma \in \Sigma_1^F \\ n' \in \theta}} \sum_{\substack{v \in \sigma\Sigma \cap Z^0(\sigma)^F \\ \text{uni}}} \chi_{\mathscr{E},\varphi_0}(\sigma v')\, \chi_{\mathscr{E}',\varphi'_0}(n'\sigma v' n'^{-1})$$
$$= |L^F|^{-1}\, |L'^F|^{-1} \sum_{n' \in \theta} \sum_{\xi \in \Sigma^F} \chi_{\mathscr{E},\varphi_0}(\xi)\, \chi_{\mathscr{E}',\varphi'_0}(n'\xi n'^{-1}),$$

as required.

9.6

We shall now prove that Theorem 9.3 holds for G under the assumption that it holds for groups of dimension strictly smaller than that of G. We can easily reduce the general case to the case where $\mathscr{E}$, $\mathscr{E}'$ are irreducible, which we now assume. By (9.4.3), we may also assume that $\mathscr{Z}^0_L \cap \mathscr{Z}^0_{L'} \cap \mathscr{Z}_G = \{e\}$. The argument in 9.5 can still be partly carried out using 8.5; it gives the following identity:

$$|G^F|^{-1} \sum_{g \in G^F} \chi_{K,\varphi}(g)\, \chi_{K',\varphi'}(g)$$
$$- |L^F|\, |L'^F|^{-1} \sum_{n \in \theta} \sum_{\xi \in \Sigma^F} \chi_{\mathscr{E},\varphi_0}(\xi)\, \chi_{\mathscr{E}',\varphi'_0}(n\xi n^{-1})$$
$$= \sum_{\substack{s \in \mathscr{Z}_G^F \\ s \in \mathscr{E}^0_L \cap \mathscr{Z}^0_{L'}}} \Bigg(|G^F|^{-1} \sum_{\substack{u \in G^F \\ \text{uni}}} \chi_{K,\varphi}(su)\, \chi_{K',\varphi'}(su)$$
$$- |L^F|\, |L'^F|^{-1} \sum_{n \in \theta} \sum_{\xi \in C^F} \chi_{\mathscr{E},\varphi_0}(s\xi)\, \chi_{\mathscr{E}',\varphi'_0}(sn\xi n^{-1}) \Bigg). \qquad (9.6.1)$$

Since $\mathscr{Z}_L^0 \cap \mathscr{Z}_{L'}^0 \cap \mathscr{Z}_G = \{e\}$, by assumption, we must have $s = e$ in the last expression, which is therefore equal to:

$$|G^F|^{-1} \sum_{u \in G^F} Q_{L,G,C,\mathscr{F},\varphi_1}(u)\, Q_{L,G,C',\mathscr{F}',\varphi'_1}(u) \\ - |L^F|\,|L'^F| \sum_{n \in \theta} \sum_{\xi \in C^F} \chi_{\mathscr{F},\varphi_1}(\xi)\, \chi_{\mathscr{F}',\varphi'_1}(n\xi n^{-1}). \tag{9.6.2}$$

We can write $\mathscr{E}$ in the form $\mathscr{G} \boxtimes \mathscr{F}$ (if we identify $\mathscr{Z}_L^0 \cdot C$ with $\mathscr{Z}_L^0 \times C$); here $\mathscr{G}$ is a local system of rank 1 of $\mathscr{Z}_L^0$. Moreover, $\varphi_0: F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ may be identified with $\delta_1 \boxtimes \phi_1$, where $\delta_1: F^*\mathscr{G} \xrightarrow{\sim} \mathscr{G}$. Similarly, we identify $\mathscr{E}'$, ϕ'_0 with $\mathscr{G}' \boxtimes \mathscr{F}'$, $\delta'_1 \boxtimes \phi'_1$ where $\mathscr{G}'$ is a local system of rank 1 on $\mathscr{Z}_{L'}^0$ and $\delta'_1: F^*\mathscr{G}' \xrightarrow{\sim} \mathscr{G}'$.

Note that in (9.3.1), $(\mathscr{E}, \varphi_0)$ and $(\mathscr{E}', \varphi'_0)$ do not enter explicitly; only their restrictions $(\mathscr{F}, \varphi_1)$, $(\mathscr{F}', \varphi'_1)$ to C, C' matter. Hence to prove 9.3 we are free to choose $(\mathscr{G}, \delta_1)$, $(\mathscr{G}', \delta'_1)$ as we please.

Assume first that $(\mathscr{Z}_{L'}^0)^F \neq \{e\}$. We consider a nontrivial character $\theta_1: (\mathscr{Z}_{L'}^0)^F \to \bar{\mathbb{Q}}_l^*$. There is a unique pair $(\mathscr{G}', \delta'_1)$, where $\mathscr{G}'$ is a local system of rank 1 on $\mathscr{Z}_{L'}^0$ and $\delta'_1: F^*\mathscr{G}' \xrightarrow{\sim} \mathscr{G}'$, such that $\chi_{\mathscr{G}',\delta'_1} = \theta_1$. Then $\mathscr{G}'$ is not isomorphic to $\bar{\mathbb{Q}}_l$. We take $\mathscr{G}$ to be the local system $\bar{\mathbb{Q}}_l$ on $\mathscr{Z}_L^0$ and we select any isomorphism $\delta_1: F^*\mathscr{G} \xrightarrow{\sim} \mathscr{G}$. With this choice of $(\mathscr{G}, \delta_1)$, $(\mathscr{G}', \delta'_1)$, the left-hand side of (9.6.1) is zero, by (9.4.2); hence, the expression (9.6.2) is zero. Thus, 9.3 holds for G, $(\mathscr{F}, \phi_1)$ and $(\mathscr{F}', \phi'_1)$. It also holds in the case where $(\mathscr{Z}_{L'}^0)^F = \{e\}$ and $(\mathscr{Z}_L^0)^F \neq \{e\}$, since L, L' play a symmetric role. We are therefore reduced to the case where $(\mathscr{Z}_L^0)^F = (\mathscr{Z}_{L'}^0)^F = \{e\}$.

A torus over F_q which has no rational points over F_q other than e is necessarily an F_q-split torus and we must have $q = 2$. (This fact is also used in [3, Proof of 6.9].) Thus $\mathscr{Z}_L^0$, $\mathscr{Z}_{L'}^0$ are F_q-split tori. It follows that L (resp. L') is a Levi subgroup of a parabolic subgroup P (resp. P') of G, defined over F_q. Therefore, we may use (9.4.4) and we see that the left-hand side of (9.6.1) is zero. Hence, the expression (9.6.2) is also zero, so that 9.3 again holds.

It is clear that the arguments in this section and the previous one provide an inductive proof of both Theorems 9.2 and 9.3.

9.7

We preserve the setup of 9.1. We denote by $\mathscr{E}^\vee$ the local system on Σ dual to $\mathscr{E}$: the stalk $\mathscr{E}_x^\vee$ is equal to $\mathrm{Hom}(\mathscr{E}_x, \bar{\mathbb{Q}}_l)$. We denote by $\varphi_0^\vee: F^*\mathscr{E}^\vee \xrightarrow{\sim} \mathscr{E}^\vee$ the contragredient of φ_0: $F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ (i.e., the isomorphism characterized by the property that for any $\xi \in \Sigma$, $\varphi_0^\vee: \mathscr{E}_{F\xi}^\vee \xrightarrow{\sim} \mathscr{E}_\xi^\vee$ is the isomorphism contragredient to $\varphi_0: \mathscr{E}_{F\xi} \xrightarrow{\sim} \mathscr{E}_\xi$. Assume that $\mathscr{E}$, $\mathscr{E}'$ are irreducible. Let $\theta(\mathscr{E}, \mathscr{E}')$ be the set of all elements $n \in \theta$ such that $\mathrm{ad}(n)^*\mathscr{E}'$ is isomorphic to $\mathscr{E}^\vee$. We associate to $n \in \theta(\mathscr{E}, \mathscr{E}')$ a number $\varepsilon(n) \in \bar{\mathbb{Q}}_l^*$ as follows. Let $\zeta: \mathrm{ad}(n)^*\mathscr{E}' \xrightarrow{\sim} \mathscr{E}^\vee$ be an isomorphism (it is unique up to a nonzero scalar in $\bar{\mathbb{Q}}_l$). Then $\varepsilon(n)$ is characterized by the property that in the diagram

$$\begin{array}{ccc} \mathscr{E}'_{F(n\xi n^{-1})} & \xrightarrow{\zeta} & \mathscr{E}^{\vee}_{F\xi} \\ \downarrow{\scriptstyle \phi'_0} & & \downarrow{\scriptstyle \phi_0^{\vee}} \\ \mathscr{E}'_{n\xi n^{-1}} & \xrightarrow{\zeta} & \mathscr{E}^{\vee}_{\xi} \end{array}$$

we have $\zeta\cdot\phi'_0=\varepsilon(n)\,\phi_0^{\vee}\cdot\zeta$, for all $\xi\in\Sigma$. Clearly, $\varepsilon(n)$ is independent of the choice of ζ. One checks that $\varepsilon(nl)=\varepsilon(n)$ for all $l\in L^F$ so that ε factors through a function on $\theta(\mathscr{E},\mathscr{E}')/L^F$ denoted again ε.

We have the following result:

LEMMA 9.8. *If $n\in\theta$, then*

$$\sum_{\xi\in\Sigma^F}\chi_{\mathscr{E},\varphi_0}(\xi)\,\chi_{\mathscr{E}',\varphi'_0}(n\xi n^{-1})$$

$$=\begin{cases}\varepsilon(n)\,q^{\dim\Sigma-\dim L}\,|L^F| & \text{if } n\in\theta(\mathscr{E}.\mathscr{E}')\\ 0 & \text{otherwise.}\end{cases}$$

Proof. The fact that this sum is zero when $n\notin\theta(\mathscr{E},\mathscr{E}')$ follows from 7.7, exactly as in the proof of (9.4.2). Assume now that $n\in\theta(\mathscr{E},\mathscr{E}')$. Then the local system $\hat{\mathscr{E}}=\mathrm{ad}(n)^*\mathscr{E}'\otimes\mathscr{E}$ is isomorphic to the direct sum $\bar{\mathbb{Q}}_l\oplus\hat{\mathscr{E}}_1$, where $\hat{\mathscr{E}}_1$ is a direct sum of irreducible nonconstant local systems, which are $\mathscr{Z}^0_L\times L$-equivariant. From the proof of 7.7 we see that $H^i_c(\Sigma,\hat{\mathscr{E}}_1)=0$ for all i. It follows that $H^i_c(\Sigma,\hat{\mathscr{E}})\cong H^i_c(\Sigma,\bar{\mathbb{Q}}_l)$.

The isomorphisms φ_0,φ'_0 induce an isomorphism $\hat{\varphi}\colon F^*\hat{\mathscr{E}}\xrightarrow{\sim}\hat{\mathscr{E}}$, which respects the summand $\bar{\mathbb{Q}}_l$ and induces on it $\varepsilon(n)$ times the obvious isomorphism $F^*\bar{\mathbb{Q}}_l\xrightarrow{\sim}\bar{\mathbb{Q}}_l$. By the trace formula for Frobenius maps, our sum is equal to the alternating sum of traces of the Frobenius map on the spaces $H^i_c(\Sigma,\hat{\mathscr{E}})$. Hence it is equal to $\varepsilon(n)\sum_i(-1)^i\,\mathrm{Tr}(F^*,H^i_c(\Sigma,\bar{\mathbb{Q}}_l))$. Consider the map $f\colon\tilde{\Sigma}\to\Sigma$ constructed in 7.7 (for L instead of G) in terms of a base points $y\in\Sigma$. By choosing $y\in\Sigma^F$, we may assume that $\tilde{\Sigma}$ and f are defined over F_q. From the proof of 7.7 we see that $\mathrm{Tr}(F^*,H^i_c(\Sigma,\bar{\mathbb{Q}}_l))=\mathrm{Tr}(F^*,H^i_c(\tilde{\Sigma},\bar{\mathbb{Q}}_l))$. Hence our sum is equal to $\varepsilon(n)\,\Sigma(-1)^i\,\mathrm{Tr}(F^*,H^i_c(\tilde{\Sigma},\bar{\mathbb{Q}}_l))=\varepsilon(n)\,|\tilde{\Sigma}^F|=\varepsilon(n)\,|L^F|\,|\mathscr{Z}^{0F}_L|\cdot|\mathscr{Z}^0_L(y)^F|^{-1}$. By (7.1.2), $\mathscr{Z}^0_L(y)/\mathscr{Z}^0_L$ is a (connected) unipotent group. It follows that $|\mathscr{Z}^0_L(y)^F|\cdot|\mathscr{Z}^{0F}_L|^{-1}=|(\mathscr{Z}^0_L(y)/\mathscr{Z}^0_L)^F|=q^{\dim\mathscr{Z}^0_L(y)-\dim\mathscr{Z}^0_L}=q^{\dim L-\dim\Sigma}$. This completes the proof of the lemma.

Using the lemma, we can now reformulate Theorem 9.2 as follows.

COROLLARY 9.9. *With the assumptions of* 9.2, *we have*

$$|G^F|^{-1} \sum_{g \in G^F} \chi_{K,\varphi}(g)\, \chi_{K',\varphi'}(g)$$
$$= \left(\sum_{n \in \theta(\mathscr{E},\mathscr{E}')/L^F} \varepsilon(n) \right) q^{\dim \Sigma - \dim L}. \tag{9.9.1}$$

In particular, the left-hand side of (9.9.1) *is zero unless there exists* $n \in G^F$ *such that* $nLn^{-1} = L'$, $n\Sigma n^{-1} = \Sigma'$, *and* $\mathrm{ad}(n)^*\mathscr{E}'$ *is isomorphic to* $\mathscr{E}^\vee$.

9.10

Now let $\Sigma = \mathscr{Z}_L^0 C$, $\Sigma' = \mathscr{Z}_{L'}^0 C'$, $\mathscr{F}$, φ_1, $\mathscr{F}'$, φ_1' be as in 9.3. We assume that $\mathscr{F}, \mathscr{F}'$ are irreducible as local systems on C, C'. Let $\mathscr{E}$, φ_0 (resp. $\mathscr{E}'$, φ_0') be the inverse image of $\mathscr{F}$, φ_1 (resp. $\mathscr{F}'$, φ_1') under the canonical map $\Sigma \to C$ resp. $\Sigma' \to C'$. If $n \in \theta$, we have clearly

$$\sum_{\xi \in C^F} \chi_{\mathscr{F},\varphi_1}(\xi)\, \chi_{\mathscr{F}',\varphi_1'}(n\xi n^{-1})$$
$$= |\mathscr{Z}_L^{0F}|^{-1} \sum_{\xi \in \Sigma^F} \chi_{\mathscr{E},\varphi_0}(\xi)\, \chi_{\mathscr{E},\varphi_0'}(n\xi n^{-1}). \tag{9.10.1}$$

With the notations in 9.7, this equals $\varepsilon(n)\, q^{\dim L - \dim \Sigma}\, |L^F|\, |\mathscr{Z}_L^{0F}|^{-1}$ if $\mathrm{ad}(n)^*\mathscr{E}'$ is isomorphic to $\mathscr{E}^\vee$ and is zero otherwise.

We now assume that $L' = L$, $C' = C$, $\mathscr{E}' = \mathscr{E}^\vee$ $\varphi_0' = \varphi_0^\vee$. We shall prove that in this case

$$\varepsilon(n) = 1 \qquad \text{for all} \quad n \in \theta \tag{9.10.2}$$

(notations of 9.7).

According to [4, 9.2], the local system $\pi_*(\tilde{\mathscr{E}}')$ on Y (notation of 8.1) has a canonical direct summand $\mathscr{G}$ which is characterized by the properties

(a) $\mathscr{G}$ is an irreducible local system and it has multiplicity one in $\pi_*(\tilde{\mathscr{E}})$.

(b) $\mathscr{H}_x^0 \mathrm{IC}(\bar{Y}, \mathscr{G}) \neq 0$ for x in a dense subset of the set of unipotent elements in $\bar{Y}$.

Clearly $F^*\mathscr{G}$ also satisfies (a) and (b). Hence the isomorphism $\varphi' \colon F^*\pi_*(\tilde{\mathscr{E}}') \xrightarrow{\sim} \pi_*(\tilde{\mathscr{E}}')$ induced by φ_0' maps $F^*\mathscr{G}$ isomorphically onto $\mathscr{G}$. Now let $\zeta \colon \mathrm{ad}(n)^*\mathscr{E}' \to \mathscr{E}' = \mathscr{E}^\vee$ be an isomorphism. Then ζ induces an isomorphism $\zeta_1 \colon \pi_*(\tilde{\mathscr{E}}') \xrightarrow{\sim} \pi_*(\tilde{\mathscr{E}}')$ (see [4, 3.5] or 10.2), which necessarily preserves the summand $\mathscr{G}$. From the definitions it follows immediately that in the diagram of isomorphism

$$\begin{tikzcd} \pi_*(\tilde{\mathscr{E}}')_{F(y)} \arrow[r, "\zeta_1"] \arrow[d, "\varphi'"'] & \pi_*(\tilde{\mathscr{E}}')_{F(y)} \arrow[d, "\varphi'"] \\ \pi_*(\tilde{\mathscr{E}})_y \arrow[r, "\zeta_1"] & \pi_*(\tilde{\mathscr{E}}')_y \end{tikzcd}$$

we have $\zeta_1 \cdot \phi' = \varepsilon(n)\phi' \cdot \zeta_1$, for any $y \in Y$. The same identity must then hold in the diagram

$$\begin{tikzcd} \mathscr{G}_{F(y)} \arrow[r, "\zeta_1"] \arrow[d, "\varphi'"'] & \mathscr{G}_{F(y)} \arrow[d, "\varphi'"] \\ \mathscr{G}_y \arrow[r, "\zeta_1"] & \mathscr{G}_y \end{tikzcd}$$

Since $\mathscr{G}$ is irreducible, ζ_1 must act on each $\mathscr{G}_y$ as multiplication by a scalar in $\bar{\mathbb{Q}}_l^*$, independent of y. This forces $\varepsilon(n)$ to be equal 1, as stated in (9.10.2).

We note also that, according to [4, 9.2], for any element $g \in N_G(L)$ we have automatically $gCg^{-1} = C'$ and $\mathrm{ad}(g)^*\mathscr{E}' \approx \mathscr{E}'$. Using this and (9.10.1), (9.10.2), we can reformulate Theorem 9.3 as follows:

COROLLARY 9.11. *We make the assumptions of* 9.3; *in addition, we assume that* $\mathscr{F}, \mathscr{F}'$ *are irreducible. Then*

$$|G^F|^{-1} \sum_{\substack{u \in G^F \\ \text{uni}}} Q_{L,G,C,\mathscr{F},\varphi_1}(u)\, Q_{L',G,C',\mathscr{F}',\varphi_1'}(u)$$

$$= \begin{cases} |N_G(L)^F/L^F| \cdot |\mathscr{Z}_L^{0F}|^{-1}\, q^{\dim C - \dim(L/\mathscr{Z}_L^0)} \\ \qquad \text{if} \quad L' = L,\ C' = C,\ \mathscr{F}' = \mathscr{F}^\vee,\ \varphi_1' = \varphi_1^\vee \\ 0 \quad \text{if there is no } g \in G^F \text{ such that} \\ \qquad gLg^{-1} = L',\ gCg^{-1} = C',\ \mathrm{ad}(g)^*\mathscr{F}' \approx \mathscr{F}^\vee. \end{cases}$$

(Here, $\mathscr{F}^\vee$ is the local system on C dual to $\mathscr{F}$ and $\varphi_1^\vee : F^*\mathscr{F}^\vee \xrightarrow{\sim} \mathscr{F}^\vee$ is the contragradient of φ_1, (cf. 9.7).)

10. Orthogonality for Certain Characteristic Functions

10.1

In this chapter, we preserve the assumptions of 8.0. (Note, however, that the definitions in Sections 10.1–10.3 make sense for any algebraically closed ground field k.)

Let L, Σ, $\mathscr{E}$, K be as in 8.1. We assume that $\mathscr{E}$ is irreducible. Let A be an admissible complex on G which is isomorphic to a direct summand of K, (see (8.1.2)). Then $V_A = \mathrm{Hom}(A, K)$ is a finite dimensional $\bar{\mathbb{Q}}_l$-vector space. Let $\mathscr{A} = \mathrm{End}(K)$ be the endomorphism algebra of K in $\mathscr{M}G$: it is a finite dimensional semisimple algebra over $\bar{\mathbb{Q}}_l$. Composition of maps $(\theta, v) \to \theta \cdot v$ $(\theta \in \mathscr{A},\ v \in V_A)$ makes V_A into a left (irreducible) $\mathscr{A}$-moule.

Note that $A \mapsto V_A$ is a 1–1 correspondence between the set of irreducible components of K (up to isomorphism) and the set of irreducible left $\mathscr{A}$-modules (up to isomorphism).

10.2

The algebra $\mathscr{A}$ is at the same time the endomorphism algebra of the local system $\pi_* \tilde{\mathscr{E}}$ on Y (notation of 8.1). We now describe $\mathscr{A}$ following [4, 3.4]. Let $\mathscr{N}$ be the set of all $n \in N_G(L)$ such that $n\Sigma n^{-1} = \Sigma$ and such that $\mathrm{ad}(n)^* \mathscr{E}$ is isomorphic to $\mathscr{E}$, $(\mathrm{ad}(n) g = ngn^{-1})$. Then $\mathscr{N} \supset L$ and we set $\mathscr{W} = \mathscr{N}/L$; it is a finite group.

If $w \in \mathscr{W}$, let $\gamma_w \colon \tilde{Y} \to \tilde{Y}$ be the isomorphism defined by $\gamma_w(g, xL) = (g, xn^{-1}L)$, where n is a representative for w in $\mathscr{N}$.

Let $\mathscr{A}_w$ be the one dimensional $\bar{\mathbb{Q}}_l$-vector space of all homomorphisms of local systems $\tilde{\mathscr{E}} \to \gamma_w^* \tilde{\mathscr{E}}$, over $\tilde{Y}$. Since $\pi_* \gamma_w^* \tilde{\mathscr{E}} = \pi_* \tilde{\mathscr{E}}$, we have a natural imbedding $\mathscr{A}_w \subsetneq \mathrm{End}(\pi_* \tilde{\mathscr{E}}) = \mathscr{A}$; we identify $\mathscr{A}_w$ with its image in $\mathscr{A}$. We then have

$$\mathscr{A} = \bigoplus_{w \in \mathscr{W}} \mathscr{A}_w$$

Under the multiplication in the algebra $\mathscr{A}$, we have $\mathscr{A}_w \cdot \mathscr{A}_{w'} = \mathscr{A}_{ww'}$; moreover the unit element of $\mathscr{A}$ is contained in $\mathscr{A}_e$, where e is the unit element of $\mathscr{W}$. If we choose a basis element θ_w in $\mathscr{A}_w$ for each w, we then have

$$\theta_w \cdot \theta_{w'} = \lambda(w, w') \theta_{ww'}, \qquad \text{where} \quad \lambda(w, w') \in \bar{\mathbb{Q}}_l^*. \tag{10.2.1}$$

In particular, each θ_w is invertible. We also see that $\mathscr{A}_w$ is the group algebra of $\mathscr{W}$, twisted by a 2-cocycle.

10.3

We now state two orthogonality relations for $\mathscr{A}$.

Let $\iota \colon \mathscr{A} \to \mathscr{A}$ be an automorphism of the algebra $\mathscr{A}$. Let $V_1, V_2, \ldots, V_r$ be a set of representatives for the isomorphism classes of irreducible left $\mathscr{A}$-modules V with the following property: there exists an isomorphism $\iota_V \colon V \to V$ of $\bar{\mathbb{Q}}_l$-vector spaces such that $\iota_V(\theta v) = \iota(\theta)\, \iota_V(v)$ for all $\theta \in \mathscr{A}$, $v \in V$. Let us choose such an isomorphism $\iota_{V_i} \colon V_i \to V_i$ for each i $(1 \leqslant i \leqslant r)$; ι_{V_i} are defined uniquely up to a nonzero scalar. We can now state

$$|\mathscr{W}|^{-1} \sum_{w \in \mathscr{W}} \operatorname{Tr}(\theta_w \iota_{V_i}, V_i) \operatorname{Tr}((\theta_w \iota_{V_j})^{-1}, V_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \quad (\text{for any } 1 \leqslant i, j \leqslant r). \end{cases} \tag{10.3.1}$$

$$\sum_{i=1}^{r} \operatorname{Tr}(\theta_w \iota_{V_i}, V_i) \operatorname{Tr}((\theta_{w'} \iota_{V_i})^{-1}, V_i) = \text{trace of the linear map } \theta \mapsto \theta_{w'}^{-1} \iota^{-1}(\theta)\, \theta_w \text{ of } \mathscr{A} \text{ into itself (for any } w, w' \in \mathscr{W}). \tag{10.3.2}$$

Here, all traces are taken over $\bar{\mathbb{Q}}_l$.

The proof of (10.3.1), (10.3.2) is essentially the same as that of the Schur orthogonality relations in the case of ordinary group algebras.

10.4

Assume now that K (see (8.1.2)) is isomorphic to F^*K. Let $\varphi: F^*K \xrightarrow{\sim} K$ be an isomorphism.

If A is an admissible complex of G which is isomorphic to an irreducible component of K, then so is F^*A. Let V_A, V_{F^*A} be the corresponding left $\mathscr{A}$-modules (see 10.3). We define a map $\rho: V_A \to V_{F^*A}$ as follows. Let $v \in V_A = \operatorname{Hom}(A, K)$ and let $F^*(v)$ be the corresponding homomorphism $F^*A \to F^*K$. By definition, $\rho(v) = \varphi \circ F^*(v): F^*A \to K$. Then ρ is an isomorphism of $\bar{\mathbb{Q}}_l$-vector spaces. It is $\mathscr{A}$-semilinear in the following sense: $\rho(\theta v) = \iota(\theta)\, \rho(v)$, where $\iota: \mathscr{A} \to \mathscr{A}$ is the automorphism of the algebra $\mathscr{A}$ defined by $\iota(\theta) = \varphi \circ F^*(\theta) \circ \varphi^{-1}$ $(\theta \in \mathscr{A})$.

If φ_A is an isomorphism $F^*A \xrightarrow{\sim} A$, then the map $V_{F^*A} \to V_A$ defined by $v_1 \mapsto v_1 \circ \varphi_A^{-1}$ is an isomorphism of $\mathscr{A}$-modules and its composition with ρ is an $\mathscr{A}$-semilinear map $\sigma_A: V_A \to V_A$, $\sigma_A(v) = \rho(v) \circ \varphi_A^{-1}$, which is an isomorphism of $\bar{\mathbb{Q}}_l$-vector spaces. (Conversely, if there exists an $\mathscr{A}$-semilinear map $V_A \to V_A$ which is a $\bar{\mathbb{Q}}_l$-isomorphism, then F^*A is isomorphic to A.) We have a natural isomorphism

$$\bigoplus_A (A \otimes V_A) \xrightarrow{\sim} K,$$

where A runs over the set of irreducible components of K (up to isomorphism). It gives rise, for any $g \in G$ and any integer i, to an isomorphism at the level of stalks:

$$\bigoplus_A (\mathscr{H}_g^i(A) \otimes V_A) \xrightarrow{\sim} \mathscr{H}_g^i(K). \tag{10.4.1}$$

This isomorphism can be described as follows: Let $a \in \mathscr{H}_g^i(A)$, $v \in V_A$. Then

v defines a homomorphism $v_g\colon \mathscr{H}^i_g(A) \to \mathscr{H}^i_g(K)$ and $a \otimes v$ corresponds under (10.4.1) to $v_g(a)$.

Now $\varphi\colon F^*K \xrightarrow{\sim} K$ defines an isomorphism $\mathscr{H}^i_{F(g)}(K) \xrightarrow{\sim} \mathscr{H}^i_g(K)$ which will be denoted again φ; similarly, $\varphi_A\colon F^*A \xrightarrow{\sim} A$ defines $\varphi_A\colon \mathscr{H}^i_{F(g)}(A) \xrightarrow{\sim} \mathscr{H}^i_g(A)$.

If $F(g) = g$ and $\varphi_A\colon F^*A \xrightarrow{\sim} A$, it follows from the definitions that the endomorphism $\varphi_A \otimes \sigma_A$ of $\mathscr{H}^i_g(A) \otimes V_A$ is compatible, via (10.4.1), with the endomorphism φ of $\mathscr{H}^i_g(K)$. On the other hand, if $F^*A \not\approx A$, then φ maps the image of $\mathscr{H}^i_g(A) \otimes V_A \subsetneq \mathscr{H}^i_g(K)$ onto a summand corresponding to a different A. It follows that

$$\operatorname{Tr}(\varphi, \mathscr{H}^i_g(K)) = \sum_A \operatorname{Tr}(\varphi_A, \mathscr{H}^i_g(A)) \operatorname{Tr}(\sigma_A, V_A), \tag{10.4.2}$$

sum over a set of representatives A for the isomorphism classes of admissible complexes which are isomorphic to irreducible components of K and which are isomorphic to their inverse image under F; for each such A, we assume chosen an isomorphism $\varphi_A\colon F^*A \xrightarrow{\sim} A$ and we define $\sigma_A(v) = \varphi \circ F^*(v) \circ \varphi_A^{-1}$ as above. (The traces are are taken over $\bar{\mathbb{Q}}_l$.)

If we now replace $\varphi\colon F^*K \to K$ by $\theta_w \circ \varphi$ (see 10.3) for some $w \in W$ and keep φ_A unchanged, then σ_A is changed to $\theta_w \circ \sigma_A$ (θ_w acts on V_A by the $\mathscr{A}$-module structure of V_A). The identity (10.4.2) remains valid and gives:

$$\operatorname{Tr}(\theta_w \circ \varphi, \mathscr{H}^i_g(K)) = \sum_A \operatorname{Tr}(\varphi_A, \mathscr{H}^i_g(A)) \operatorname{Tr}(\theta_w \circ \sigma_A, V_A), \tag{10.4.3}$$

where A, φ_A, σ_A are as in the sum (10.4.2).

We now multiply both sides of (10.4.3) by $\operatorname{Tr}((\theta_w \circ \sigma_{A'})^{-1}, V_{A'})$ (where A' is one of the terms of the summation in (10.4.3)) and we sum over all $w \in \mathscr{W}$. Using (10.3.1), we obtain

$$\begin{aligned}\operatorname{Tr}(\varphi_{A'}, \mathscr{H}^i_g(A'))\\ = |\mathscr{W}|^{-1} \sum_{w \in \mathscr{W}} \operatorname{Tr}(\theta_w \circ \varphi, \mathscr{H}^i_g(K)) \operatorname{Tr}((\theta_w \circ \sigma_{A'})^{-1}, V_{A'})\end{aligned} \tag{10.4.4}$$

for any admissible complex A' which is isomorphic to an irreducible component of K, such that there exists $\varphi_{A'}\colon F^*A' \xrightarrow{\sim} A'$. Taking alternating sum over i in (10.4.4), we obtain the following identity for characteristic functions (see (8.4.1)):

$$\chi_{A,\varphi_A} = |\mathscr{W}|^{-1} \sum_{w \in \mathscr{W}} \operatorname{Tr}((\theta_w \circ \sigma_A)^{-1}, V_A)\, \chi_{K,\theta_w \circ \varphi} \tag{10.4.5}$$

valid for any admissible complex A which is isomorphic to an irreducible component of K and any isomorphism $\varphi_A\colon F^*A \xrightarrow{\sim} A$. (Recall that $\sigma_A(v) = \varphi \circ F^*(v) \circ \varphi_A^{-1}$.)

10.5

Formula (10.4.5) is applicable to any admissible complex A on G such that F^*A is isomorphic to A. Indeed, given such A, we can find $L, \Sigma, \mathscr{E}, K$ as in (8.1.1), (8.1.2) such that A is isomorphic to an irreducible component of K. Then F^*K is obtained from $F^{-1}L$, $L^{-1}\Sigma$, $F^*\mathscr{E}$ in the same way as K is obtained from $L, \Sigma, \mathscr{E}$ (see (8.1.2)) and F^*A is isomorphic to an irreducible component of F^*K. Since $F^*A \approx A$, it follows that A is isomorphic to an irreducible component of F^*K. Using 7.6, we see that there exists $g \in G$ such that $gLg^{-1} = F^{-1}L$, $g\Sigma g^{-1} = F^{-1}\Sigma$, $\mathrm{ad}(g)^*(F^*\mathscr{E}) \approx \mathscr{E}$ ($\mathrm{ad}(g)$: $\Sigma \to F^{-1}\Sigma$, $\mathrm{ad}(g)x = gxg^{-1}$). By Lang's theorem we can write $F(g) = g_1^{-1}F(g_1)$ for some $g_1 \in G$. Let $L_1 = g_1 L g_1^{-1}$, $\Sigma_1 = g_1 \Sigma g_1^{-1}$, $\mathscr{E}_1 = \mathrm{ad}(g_1^1)^*\mathscr{E}$. Then $FL_1 = L_1$, $F\Sigma_1 = \Sigma_1$, $F^*\mathscr{E}_1 \approx \mathscr{E}_1$. Since replacing $(L, \Sigma, \mathscr{E})$ by $(L_1, \Sigma_1, \mathscr{E}_1)$ does not change K, we see that we can assume that $FL = L$, $F\Sigma = \Sigma$, and that there exists an isomorphism φ_0: $F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ of local systems over Σ.

This gives rise to an isomorphism φ: $F^*K \xrightarrow{\sim} K$, as in (8.1.3). The formula (10.4.5) is then applicable to this K and φ.

10.6

Now let w be an element of $\mathscr{W}$; choose a representative n for w in $\mathscr{N}$ and an element $z \in G$ such that $z^{-1}F(z) = n^{-1}$. We set $L^w = zLz^{-1}$, $\Sigma^w = z\Sigma z^{-1}$, $\mathscr{E}^w = \mathrm{ad}(z^{-1})^*\mathscr{E}$ (a local system on Σ^w). Then $FL_w = L^w$ and $F\Sigma^w = \Sigma^w$. We define an isomorphism φ_0^w: $F^*\mathscr{E}^w \xrightarrow{\sim} \mathscr{E}^w$ in terms of φ_0: $F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ and of the fixed basis element θ_w of $\mathscr{A}_w$ (see 10.2), as follows. The basis element θ_w defines for each $\xi \in \Sigma$ and isomorphism of stalks $\mathscr{E}_\xi \xrightarrow{\sim} \mathscr{E}_{n\xi n^{-1}}$. Hence, θ_w defines for each $\xi' \in \Sigma^w$ and isomorphism $\mathscr{E}_{z^{-1}F(\xi')z} \to \mathscr{E}_{nz^{-1}F(\xi')zn^{-1}} = \mathscr{E}_{F(z^{-1}\xi' z)}$. Composing with the isomorphism φ_0: $\mathscr{E}_{F(z^{-1}\xi' z)} \to \mathscr{E}_{z^{-1}\xi' z}$, we get an isomorphism $\mathscr{E}_{z^{-1}F(\xi')z} \to \mathscr{E}_{z^{-1}\xi' z}$, i.e., an isomorphism $\mathscr{E}^w_{F\xi'} \to \mathscr{E}^w_{\xi'}$; this is induced by a well-defined isomorphism φ_0^w: $F^*\mathscr{E}^w \xrightarrow{\sim} \mathscr{E}^w$. We define π^w: $\tilde{Y}^w \to Y^w$, $\tilde{\mathscr{E}}^w$, K^w, φ^w: $F^*K^w \xrightarrow{\sim} K^w$ in terms of $L^w, \Sigma^w, \mathscr{E}^w, \varphi_0^w$ in the same was as π: $\tilde{Y} \to Y$, $\tilde{\mathscr{E}}, K, \varphi$: $F^*K \xrightarrow{\sim} K$ are defined in 8.1 in terms of $L, \Sigma, \mathscr{E}, \varphi_0$.

We have $Y^w = Y$ and the map $(g, xL) \to (g, xz^{-1}L^w)$ is an isomorphism j: $\tilde{Y} \to \tilde{Y}^w$ commuting with the projections π, π^w onto Y. It is clear that $j^*\tilde{\mathscr{E}}^w$ is canonically isomorphic to $\tilde{\mathscr{E}}$. Hence j induces an isomorphism $\pi_*\tilde{\mathscr{E}} \xrightarrow{\sim} \pi^w_*\tilde{\mathscr{E}}^w$ hence an isomorphism j': $K \xrightarrow{\sim} K^w$. One checks from the definitions that the following diagram is commutative

$$\begin{tikzcd}
F^*K \arrow[r, "F^*j'"] \arrow[d, "w \circ \varphi"] & F^*K^w \arrow[d, "\varphi^w"] \\
K \arrow[r, "j'"'] & K^w
\end{tikzcd}$$

It follows that for any $g \in G^F$ and any integer i, we have the equality

$$\operatorname{Tr}(\theta_w \circ \varphi, \mathscr{H}_g^i K) = \operatorname{Tr}(\varphi^w, \mathscr{H}_g^i K^w). \tag{10.6.1}$$

We now replace K by its Verdier dual DK. (Note that DK is obtained from $L, \Sigma, \mathscr{E}^\vee$ in the same way as K is obtained from $L, \Sigma, \mathscr{E}$.) The isomorphism $\varphi_0^\vee : F^*\mathscr{E}^\vee \xrightarrow{\sim} \mathscr{E}^\vee$ (see 9.7) gives rise to an isomorphism $\varphi^\vee : F^*DK \xrightarrow{\sim} DK$. We consider an element $w' \in \mathscr{W}$, and we choose $z' \in G$ such that $z'^{-1}F(z') = n'^{-1}$, where $n' \in \mathscr{N}$ is a representative of w'. We define $L^{w'}, \Sigma^{w'}, (\mathscr{E}^\vee)^{w'}, (DK)^{w'}, (\varphi^v)^{w'}$, as above, in terms of z'. Then we have an identity analogous to (10.5.1):

$$\operatorname{Tr}(\theta_{w'}^\vee \circ \varphi^\vee\ \mathscr{H}_g^i DK) = \operatorname{Tr}((\varphi^\vee)^{w'}, \mathscr{H}_g^i (DK)^{w'}), \tag{10.6.2}$$

where $\theta_{w'}^\vee$ denotes the automorphism of DK contragredient to $\theta_{w'}$. From (10.6.1) and (10.6.2), we deduce that

$$\begin{aligned} &|G^F|^{-1} \sum_{g \in G^F} \chi_{K, \theta_{w'} \circ \varphi}(g)\, \chi_{DK, \theta_{w'}^\vee \circ \varphi^\vee}(g) \\ &\quad = |G^F|^{-1} \sum_{g \in G^F} \chi_{K^w, \varphi^w}(g)\, \chi_{(DK)^{w'}, (\varphi^\vee)^{w'}}(g). \end{aligned} \tag{10.6.3}$$

We now make the assumption that $\operatorname{IC}(\bar{\Sigma}, \mathscr{E})[\dim \Sigma]$ extended to L, by 0 outside $\bar{\Sigma}$, (as well as its Verdier dual) are strongly cuspidal, clean complexes on L. This implies that the analogous statement is true for $\operatorname{IC}(\bar{\Sigma}^w, \mathscr{E}^w)[\dim \Sigma^w]$ extended to L^w, by 0 outside $\bar{\Sigma}^w$, and for its Verdier dual. Hence we may apply 9.9 to evaluate the right-hand side of (10.6.3); we find that it is equal to $(\sum_v \varepsilon(v))\, q^{\dim \Sigma - \dim L}$, where v runs over the set $\theta(\mathscr{E}^w, (\mathscr{E}^\vee)^{w'})/(L^w)^F$ (see (9.7)) and $\varepsilon(v)$ is defined as in 9.7.

The map $v \to \hat{v} = z'^{-1}vz$ is a bijection between $\theta(\mathscr{E}^w, (\mathscr{E}^\vee)^{w'})/(L^w)^F$ and the set of elements $\hat{v} \in \mathscr{W}$ such that $F(\hat{v}) = w'\hat{v}w^{-1}$. Moreover, from the definitions, we see that $\varepsilon(v)$ can be expressed in terms of $\hat{v}$ as follows:

$$\theta_{w'}^{-1} \iota^{-1}(\theta_{\hat{v}})\theta_w = \varepsilon(v)\theta_{\hat{v}} \qquad (w'^{-1}F(\hat{v})w = \hat{v}), \tag{10.6.4}$$

where ι is the automorphism of the algebra $\mathscr{A}$, defined in 10.4.

On the other hand, for arbitrary $\hat{v} \in \mathscr{W}$, we have

$$\theta_{w'}^{-1} \iota^{-1}(\theta_{\hat{v}})\theta_w = \lambda \cdot \theta_{w'^{-1}F(\hat{v})w} \qquad (\lambda \in \bar{\mathbb{Q}}_l^*) \tag{10.6.5}$$

(using (10.2.1) and the identity $\iota(\mathscr{A}_w) = \mathscr{A}_{F^{-1}(w)}$.)

From (10.6.4) and (10.6.5), we see that $\sum_v \varepsilon(v)$ is equal to the trace of the linear map $\theta \to \theta_{w'}^{-1}\iota^{-1}(\theta)\theta_w$ of $\mathscr{A}$ into itself. (The elements $\theta_{\hat{v}}$, $(\hat{v} \in \mathscr{W})$, form a basis of $\mathscr{A}$.) This trace can be expressed as in (10.3.2).

Hence (10.6.3) becomes

$$|G^F|^{-1} \sum_{g \in G^F} \chi_{K,\theta_w \circ \varphi}(g)\, \chi_{DK,\theta_{w'}^{\check{}} \circ \varphi^{\check{}}}(g)$$
$$= q^{\dim \Sigma - \dim L} \sum_A \mathrm{Tr}(\theta_w \sigma_A, V_A)\, \mathrm{Tr}((\theta_{w'} \sigma_A)^{-1}, V_A), \qquad (10.6.6)$$

where A runs over a set of representatives for the isomorphism classes of irreducible perverse sheaves which are components of K and are such that there exists $\varphi_A: F^*A \xrightarrow{\sim} A$. (Then σ_A is defined in terms of φ_A, φ as in 10.4.)

10.7

Let A_1, A_2 be two admissible complexes on G and assume that we are given isomorphisms $\varphi_{A_1}: F^*A_1 \xrightarrow{\sim} A_1$, $\varphi_{A_2}: F^*A_2 \xrightarrow{\sim} A_2$. For $j = 1, 2$, there exist $L_j, \Sigma_j, \mathscr{E}_j$ as in (8.1.1) (with $\mathscr{E}_j$ irreducible) such that A_j is isomorphic to a direct summand of the complex K_j constructed in terms of $L_j, \Sigma_j, \mathscr{E}_j$ in the same way as K is constructed in (8.1.2) in terms of $L, \Sigma, \mathscr{E}$. By 10.5, we may assume that $FL_j = L_j$, $F\Sigma_j = \Sigma_j$ and hat there is an isomorphism $\varphi_{0,j}: F^*\mathscr{E}_j \xrightarrow{\sim} \mathscr{E}_j$ $(j = 1, 2)$. Let $\varphi_j: F^*K_j \xrightarrow{\sim} K_j$ be the isomorphism defined by φ_{0j}, (see (8.1.3)). Let $K_{0j} = \mathrm{IC}(\bar{\Sigma}_j, \mathscr{E}_j)[\dim \Sigma_j]$ extended to L_j by 0 outside $\bar{\Sigma}_j$. We make the following assumption:

(10.7.1) *If L_1 is conjugate in G to L_2, then the complexes K_{0j}, $DK_{0,j}$ are strongly cuspidal and clean on L_j, for $j = 1, 2$.*

We shall denote $(\varphi_{A_1})^{\check{}}: F^*(DA_1) \xrightarrow{\sim} DA_1$ the isomorphism contragredient to $\varphi_{A_1}: F^*A_1 \xrightarrow{\sim} A_1$.

We can now state

THEOREM 10.8. *With the assumptions in* 10.7, *we have*

$$|G^F|^{-1} \sum_{g \in G^F} \chi_{A_1,\varphi_{A_1}}(g)\, \chi_{A_2,\varphi_{A_2}}(g)$$
$$= \begin{cases} 0 & \text{if } A_2 \text{ is not isomorphic t } DA_1 \\ q^{-c} & \text{if } A_1 = DA_2 \text{ and } \varphi_{A_1} = (\varphi_{A_2})^{\check{}}. \end{cases} \qquad (10.8.1)$$

Here, $c = \mathrm{codim}_G \operatorname{supp} A_1$.

Proof. Assume first that $(L_1, \Sigma_1, \mathscr{E}_1^{\check{}})$ is not conjugate in G to $(L_2, \Sigma_2, \mathscr{E}_2)$. Then, by 7.5, A_2 is not isomorphic to DA_1. To show that the left-hand side of (10.8.1) is zero, it is enough, by the trace formula for Frobenius maps to show that $H_c^i(G, A_1 \otimes A_2) = 0$ for all i. Using 7.2 we see that it is enough to check that for any isomorphism $f: L \xrightarrow{\sim} L'$ which can be realized by conjugation by an element of G, we have $H_c^i(L, K_{0,1} \otimes f^*K_{0,2}) = 0$. But if f exists at all then, by our assumption (10.7.1), $K_{0,1}$ and $f^*K_{0,2}$ are clean; since $f^*K_{0,2}$ is not isomorphic to $DK_{0,1}$, the equality $H_c^i(L, K_{0,1} \otimes f^*K_{0,2}) = 0$ follows from 7.8.

We now assume that $(L_1, \Sigma_1, \mathscr{E}_1\check{})$ is conjugate in G to $(L_2, \Sigma_2, \mathscr{E}_2)$. We cab then assume that $L_2 = L_1$, $\Sigma_2 = \Sigma_1$, $\mathscr{E}_2 = \mathscr{E}_1\check{}$; we shall write $L, \Sigma, \mathscr{E}, K, \varphi$ instead of $L_1, \Sigma_1, \mathscr{E}_1, K_1, \varphi_1$.

We shall use the identity (10.4.5) for $\chi_{A_1,\varphi_{A_1}}$. The analogous identity for $\chi_{A_2,\varphi_{A_2}}$ can be written in the following form:

$$\chi_{A_2,\varphi_{A_2}} = |\mathscr{W}|^{-1} \sum_{w' \in \mathscr{W}} \operatorname{Tr}(\theta_{w'} \cdot \sigma_{DA_2}, V_{DA_2})\, \chi_{DK,\theta_{\check{w}'} \circ \varphi\check{}}, \tag{10.8.2}$$

where $\sigma_{DA_2}\colon V_{DA_2} \to V_{DA_2}$ is defined in terms of $\varphi_{A_2}\colon F^*DA_2 \simeq DA_2$ and $\varphi\check{}\colon F^*DK \simeq DK$.

$$\begin{aligned} &|G^F|^{-1} \sum_{g \in G^F} \chi_{A_1,\varphi_{A_1}}(g)\, \chi_{A_2,\varphi_{A_2}}(g) \\ &\quad = |\mathscr{W}|^{-2} \sum_{w, w' \in \mathscr{W}} \operatorname{Tr}((\theta_w \sigma_{A_1})^{-1}, V_{A_1}) \operatorname{Tr}(\theta_{w'} \sigma_{DA_2}, V_{DA_2}) \\ &\qquad \times |G^F|^{-1} \sum_{g \in G^F} \chi_{K,\theta_w \circ \varphi}(g)\, \chi_{DK,\theta_{\check{w}'} \circ \varphi\check{}}(g). \end{aligned}$$

Using (10.6.6), we see that the last expression is equal to

$$\begin{aligned} &|\mathscr{W}|^{-2} \sum_{\substack{w, w' \in \mathscr{W} \\ A}} \operatorname{Tr}((\theta_w \sigma_{A_1})^{-1}, V_{A_1}) \operatorname{Tr}(\theta_{w'} \sigma_{DA_2}, V_{DA_2}) \\ &\qquad \times \operatorname{Tr}(\theta_w \sigma_A, V_A) \operatorname{Tr}(\theta_{w'} \sigma_A)^{-1}, V_A) \cdot q^{-c} \end{aligned}$$

(where A runs over the set described in (10.6.6)). Using now (10.3.1) twice, we see that this equals 0 if $A_1 \not\simeq DA_2$ and it equals q^{-c} if $A_1 = DA_2$ and $\varphi_{A_1} = \varphi\check{}_{A_2}$. This completes the proof of the theorem.

10.9

We shall state a variant of Theorem 10.8. We keep the notations and assumptions of 10.7. In addition, we assume that $\Sigma_j = \mathscr{Z}^0_{L_j} \cdot C_j$, where C_j is a unipotent class in L_j and that $(\mathscr{E}_j, \varphi_{0,j})$ is the inverse image under the projection $\Sigma_j \to C_j$ of $(\mathscr{F}, \varphi_{1,j})$, where $\mathscr{F}$ is a L-equivariant irreducible local system on C_j and $\varphi_{1,j}\colon F^*\mathscr{F} \simeq \mathscr{F}$, $(j = 1, 2)$.

We then have

THEOREM 10.9.

$$|G^F|^{-1} \sum_{u \in G^F_{\mathrm{uni}}} \chi_{A_1,\varphi_{A_1}}(u)\, \chi_{A_2,\varphi_{A_2}}(u)$$

$$= \begin{cases} 0 & \text{if } (L_1, C_1, \mathscr{F}_1) \text{ is not } G\text{-conjugate to } (L_2, C_2, \mathscr{F}_2\check{}) \\ |\mathscr{W}|^{-1} \sum_{w \in \mathscr{W}} \operatorname{Tr}((\theta_w \sigma_{A_1})^{-1}, V_{A_1}) \operatorname{Tr}(\theta_w \sigma_{DA_2}, V_{DA_2})\, |\mathscr{Z}^{0F_w}_{L^w}|^{-1} q^{-c} & \\ \quad \text{if } L_1 = L_2,\ C_1 = C_2,\ \mathscr{F}_1 = \mathscr{F}_2\check{},\ \text{and } \varphi_{1,1} = \varphi\check{}_{1,2}. & \end{cases} \tag{10.9.1}$$

(The notations in the last sum are as follows: we set $L = L_1 = L_2$, $\Sigma = \Sigma_1 = \Sigma_2$, $\mathscr{E} = \mathscr{E}_1 = \mathscr{E}_2\check{}$, $\varphi_0 = \varphi_{0,1}$: $F^*\mathscr{E}_1 \xrightarrow{\sim} \mathscr{E}_1$. Then $\mathscr{W}$, K, φ: $F^*K \xrightarrow{\sim} K$, V_{A_1}, V_{DA_2}, $\sigma_{V_{A_1}}$, $\sigma_{V_{DA_2}}$ are defined in terms of $L, \Sigma, \mathscr{E}, \varphi_0$ as in 8.1, 10.2, 10.4, and we set $L^w = zLz^{-1}$, where $z^{-1}F(z)$ is a representative for w^{-1} in $N_G(L)$. We set $c = \operatorname{codim}_G \operatorname{supp} A_1$.)

Proof. Assume first that $(L_1, C_1, \mathscr{F}_1\check{})$ is not G conjugate to $(L_2, C_2, \mathscr{F}_2)$. By the trace formula for Frobenius maps it is enough to show that $H^i_c(G_{\text{uni}}, A_1 \otimes A_2) = 0$ for all i, where G_{uni} is the variety of unipotent elements in G. Using (7.3.1) we see tha it is enough to check that for any isomorphism f: $L_1 \xrightarrow{\sim} L_2$ which can be realized by an element of G, we have $H^i_c((L_1)_{\text{uni}}, K_{0,1} \otimes f^*K_{0,2}) = 0$ for all i. If such f exists at all then, by the assumption (10.7.1), $K_{0,1}$ and $f^*K_{0,2}$ are clean. We may clearly assume that they have the same support (i.e., $f\Sigma = \Sigma'$). By 7.8, we have $H^i_c(L_1, K_{0,1} \otimes f^*K_{0,2}) = 0$ for all i, hence $H^i_c(\mathscr{Z}^0_{L_1} \cdot C_1, \mathscr{E}_1 \otimes f^*\mathscr{E}_2) = 0$ for all i. This implies that $H^i_c(C_1, \mathscr{F}_1 \otimes f^*\mathscr{F}_2) = 0$ for all i, and hence $H^i_c((L_1)_{\text{uni}}, K_{0,1} \otimes f^*K_{0,2}) = 0$ for all i, as required.

We now assume that $L_1 = L_2 = L$, $\Sigma_1 = \Sigma_2 = \Sigma$, $\mathscr{E}_1 = \mathscr{E}_2\check{} = \mathscr{E}$, $\varphi_{0,1} = \varphi_{0,2}\check{}$. We shall use the following analogue of (10.6.6):

$$|G^F|^{-1} \sum_{\substack{u \in G^F \\ \text{uni}}} \chi_{K,\theta_w \circ \varphi}(u)\, \chi_{DK,\theta_{w'}\check{} \circ \varphi\check{}}(u) = q^{-c} |\mathscr{Z}^{0F}_{L^w}|^{-1} \#\{\hat{v} \in \mathscr{W} \mid w'^{-1} F(\hat{v}) w = \hat{v}\}. \tag{10.9.2}$$

The proof is entirely parallel to that of (10.6.6); it uses 9.11 instead of 9.9. (Note that in the present case there is a canonical choice for the basis θ_w of $\mathscr{A}$, see [4, Sect. 9]; it satisfies $\theta_w \theta_{w'} = \theta_{ww'}$ and $\iota(\theta_w) = \theta_{F^{-1}(w)}$.)

Using (10.9.2), (10.4.5) for χ_{A_1,φ_1} and (10.8.2) we see that

$$\begin{aligned}
&|G^F|^{-1} \sum_{\substack{u \in G^F \\ \text{uni}}} \chi_{A_1,\varphi_{A_1}}(u)\, \chi_{A_2,\varphi_{A_2}}(u) \\
&= |\mathscr{W}|^{-2} \sum_{w,w' \in \mathscr{W}} \operatorname{Tr}((\theta_w \sigma_{A_1})^{-1}, V_{A_1}) \operatorname{Tr}(\theta_{w'} \sigma_{DA_2}, V_{DA_2}) \\
&\quad \times |G^F|^{-1} \sum_{\substack{u \in G^F \\ \text{uni}}} \chi_{K,\theta_w \circ \varphi}(u)\, \chi_{DK,\theta_{w'}\check{} \circ \varphi\check{}}(u) \\
&= |\mathscr{W}|^{-2} \sum_{\substack{w,w' \in \mathscr{W} \\ \hat{v} \in \mathscr{W} \\ w'^{-1}F(\hat{v})w = \hat{v}}} \operatorname{Tr}((\theta_w \sigma_{A_1})^{-1}, V_{A_1}) \operatorname{Tr}(\theta_{w'} \sigma_{DA_2}, V_{DA_2}) |\mathscr{Z}^{0F}_{L^w}|^{-1} q^{-} \\
&= |\mathscr{W}|^{-1} \sum_{w \in \mathscr{W}} \operatorname{Tr}(\theta_w \sigma_{A_1})^{-1}, V_{A_1}) \operatorname{Tr}(\theta_w \sigma_{DA_2}, V_{DA_2}) |\mathscr{Z}^{0F}_{L^w}|^{-1} q^{-c}.
\end{aligned}$$

(We have used the following fact:

$$\begin{aligned}\operatorname{Tr}(\theta_{F(\dot{v})w\dot{v}^{-1}}\sigma_{DA_2}, V_{DA_2}) &= \operatorname{Tr}(\theta_{F(\dot{v})}\theta_w\sigma_{DA_2}\theta_{F(\dot{v})}^{-1}, V_{DA_2})\\ &= \operatorname{Tr}(\theta_w\sigma_{DA_2}, V_{DA_2}).)\end{aligned}$$

This completes the proof of the theorem.

References

5. G. Lusztig, Character sheaves, I, *Adv. in Math.* **56** (1985), 193–237.
6. G. Lusztig, Characters of reductive groups over a finite field, Ann. of Math. Studies, Vol. 107, Princeton Univ. Press, Princeton, N.J., 1984.
7. T. A. Springer, Trigonometric sums, Green functions of finite groups and representations of Weyl groups, *Invent. Math.* **36** (1976), 173–207.

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Character Sheaves III

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Contents. 11. Some invariants of character sheaves. 12. The complexes $\bar{K}_w^{\mathscr{L}}$. 13. Principal series representations. 14. A disjointness theorem for cohomology sheaves and its applications. 15. Induction, restriction, and duality. 16. The two-sided cell attached to a character sheaf.

This paper is part of a series [5, 13] devoted to the study of a class $\hat{G}$ of irreducible perverse sheaves (called character sheaves) on a connected reductive algebraic group G. (The numbering of chapters, sections, and references will continue that of [5, 13].)

This paper is a step towards the classification of character sheaves on G. One of the main results is the following one: under certain assumptions, there is a natural surjective map with finite fibers from $\hat{G}$ to the set of all pairs $(\mathscr{L}, c)$ (up to conjugacy by the Weyl group), where $\mathscr{L}$ is a tame local system on the maximal torus and c is a "two-sided cell' in the stabilizer $W'_{\mathscr{L}}$ of $\mathscr{L}$ in the Weyl group. The assumptions made on G are

(a) G is *clean* (see (13.9.2));

(b) for any $\mathscr{L}$, the pair $(G, \mathscr{L})$ satisfies the *parity condition* (15.13).

These assumptions are actually statements about cuspidal character sheaves and are trivially satisfied when $G = GL_n$. In the general case, the assumptions will be verified (in good characteristic) in another paper in this series.

The main results of this paper are rather similar to results in [6] (especially the disjointness theorem [6, 6.17]). The proofs in the present case must proceed in a quite different way, although towards the end the two proofs become almost identical.

The following convention will be used in this paper: From 12.2 to 14.14 the ground field k will be assumed to be $\bar{F}_q$. Several results in these sections are valid for arbitrary k; they can be reduced, by general principles, to the case $\bar{F}_q$. We shall mark such results by a $(*)$. In the other sections, k is arbitrary (algebraically closed).

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11. Some Invariants of Character Sheaves

11.1. In this chapter, k is any algebraically closed field. Let $\mathscr{L}, \mathscr{L}' \in \mathscr{S}(T)$ (see (2.2). We have the following result.

Proposition 11.2. (a) *If $\mathscr{L}, \mathscr{L}'$ are in the same W-orbit in $\mathscr{S}(T)$ (see 2.2), then the sets $\hat{G}_{\mathscr{L}}, \hat{G}_{\mathscr{L}'}$ (see 2.10) coincide.*

(b) *If $x \in W$ is such that $\mathscr{L}' = (x^{-1})^* \mathscr{L}$ then the map $w \to xwx^{-1}$ is an isomorphism $W'_{\mathscr{L}} \xrightarrow{\sim} W'_{\mathscr{L}'}$ (see 2.2) and for any $w \in W'_{\mathscr{L}}$ we have $\sum_i (-1)^i\, {}^pH^i(K_w^{\mathscr{L}}) = \sum_i (-1)^i\, {}^pH^i(K_{xwx^{-1}}^{\mathscr{L}'})$ (equality in the Grothendieck group $\mathscr{K}G$ of $\mathscr{M}G$; see 6.3).*

(c) *If $\mathscr{L}, \mathscr{L}'$ are not in the same W-orbit in $\mathscr{S}(T)$, then the sets $\hat{G}_{\mathscr{L}}, \hat{G}_{\mathscr{L}'}$ are disjoint.*

(d) *The Verdier duality $D: \mathscr{M}G \to \mathscr{M}G$ defines a bijection $\hat{G}_{\mathscr{L}} \to \hat{G}_{\mathscr{L}^{-1}}$. It takes cuspidal character sheaves to cuspidal character sheaves.*

Proof. To prove (a) and (b) we may assume that $\mathscr{L}' = s_0^* \mathscr{L}$ where s_0 is a simple reflection in W.

If $\mathscr{L}' = \mathscr{L}$, then (a) is obvious and (b) follows from 6.5. Assume now that $\mathscr{L}' \neq \mathscr{L}$. Let $\mathbf{s} = (s_1, s_2, ..., s_r)$ be a sequence in S such that $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$. Let $\mathbf{s}(1)$ be the sequence $(s_0, s_0, s_1, s_2, ..., s_r)$ and let $\mathbf{s}(2)$ be the sequence $(s_0, s_1, s_2, ..., s_r, s_0)$. From the results in 2.15, it follows that ${}^pH^i(K_{\mathbf{s}(1)}^{\mathscr{L}}) = {}^pH^{i-2}(K_{\mathbf{s}}^{\mathscr{L}})(-1)$ (using the fact that $s_0 \notin W'_{\mathscr{L}}$) and from 2.19 it follows that ${}^pH^i(K_{\mathbf{s}(1)}^{\mathscr{L}}) = {}^pH^i(K_{\mathbf{s}(2)}^{\mathscr{L}'})$. It follows that ${}^pH^{i-2}(K_{\mathbf{s}}^{\mathscr{L}})(-1) = {}^pH^i(K_{\mathbf{s}(2)}^{\mathscr{L}'})$, so that $\hat{G}_{\mathscr{L}} \subset \hat{G}_{\mathscr{L}'}$. (The reverse inclusion is proved in a similar way.) This argument implies also that $\sum (-1)^i\, {}^pH^i(K_{\mathbf{s}}^{\mathscr{L}}) = \sum (-1)^i\, {}^pH^i(K_{\mathbf{s}(2)}^{\mathscr{L}'})$ in $\mathscr{K}(G)$. Using 6.5, this equality can be rewritten as $\sum (-)^i\, {}^pH^i(K_w^{\mathscr{L}}) = \sum (-1)^i\, {}^pH^i(K_{s_0 w s_0}^{\mathscr{L}'})$ where $w = s_1 s_2 \cdots s_r$. Thus, (a) and (b) are proved.

We now prove (d). Let $\mathbf{s} = (s_1, s_2, ..., s_r)$ be a sequence in S such that $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$. The Verdier dual of ${}^pH^i(\bar{K}_{\mathbf{s}}^{\mathscr{L}})$ can be determined as follows:

$$\begin{aligned}
D({}^pH^i(\bar{K}_{\mathbf{s}}^{\mathscr{L}})) &= {}^pH^{-i}(D(\bar{K}_{\mathbf{s}}^{\mathscr{L}})) && \text{(see [1, 2.1.16])} \\
&= {}^pH^{-i}(D(\bar{\pi}_{\mathbf{s}})_! \bar{\mathscr{L}}) && \text{(see 2.8)} \\
&= {}^pH^{-i}((\bar{\pi}_{\mathbf{s}})_! D\bar{\mathscr{L}}) && \text{(since } \bar{\pi}_{\mathbf{s}} \text{ is proper)} \\
&= {}^pH^{-i}((\bar{\pi}_{\mathbf{s}})_! (\overline{\mathscr{L}^{-1}}))[2d] && \text{(where } d = \dim \bar{Y}_{\mathbf{s}}) \\
&= {}^pH^{2d-i}((\bar{\pi}_{\mathbf{s}})_! (\overline{\mathscr{L}^{-1}})) \\
&= {}^pH^{2d-i}(\bar{K}_{\mathbf{s}}^{\mathscr{L}^{-1}}).
\end{aligned} \tag{11.2.1}$$

The first statement in (d) follows from (11.2.1) and the definition 2.10. If $A \in \hat{G}_{\mathscr{L}}$ is a cuspidal character sheaf, then DA is cuspidal in the sense of (7.1.1). (See the proof of 7.6.) Since DA is a character sheaf, it must also be cuspidal in the sense of 3.10 (see (7.1.6)), and (d) is proved. We now prove (c).

If $\hat{G}_{\mathscr{L}}$ and $\hat{G}_{\mathscr{L}'}$ are not disjoint then there exist two sequences $\mathbf{s} = (s_1, s_2, \ldots, s_r)$, $\mathbf{s}' = (s'_1, s'_2, \ldots, s'_{r'})$ in S such that $s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$, $s'_1 s'_2 \cdots s'_{r'} \in W'_{\mathscr{L}'}$ and an irreducible perverse sheaf A on G such that A is a direct summand of both ${}^pH^i(\bar{K}_{\mathbf{s}}^{\mathscr{L}})$ and ${}^pH^{i'}(\bar{K}_{\mathbf{s}'}^{\mathscr{L}'})$ for some i, i'. Then DA is a direct summand of ${}^pH^{i''}(\bar{K}_{\mathbf{s}}^{\mathscr{L}^{-1}})$ for some i'' (see (11.2.1)). By (7.4.2), we have $H_c^0(G, DA \otimes A) \neq 0$, hence $H_c^0(G, {}^pH^{i''}(\bar{K}_{\mathbf{s}}^{\mathscr{L}^{-1}}) \otimes {}^pH^{i'}(\bar{K}_{\mathbf{s}'}^{\mathscr{L}'})) \neq 0$. Since $\bar{K}_{\mathbf{s}}^{\mathscr{L}^{-1}}, \bar{K}_{\mathbf{s}'}^{\mathscr{L}'}$ are semisimple (1.12, 2.17(a)) it follows that $H_c^j(G, \bar{K}_{\mathbf{s}}^{\mathscr{L}^{-1}} \otimes \bar{K}_{\mathbf{s}'}^{\mathscr{L}'}) \neq 0$, for some j.

Using the method in 2.13, 2.14, we see that, by replacing, if necessary, s, s′ by subsequences, we have $H_c^{j'}(G, K_{\mathbf{s}}^{\mathscr{L}^{-1}} \otimes K_{\mathbf{s}'}^{\mathscr{L}'}) \neq 0$, for some j'. Using the method in 2.15, 2.16, we deduce that there exist $w \in W'_{\mathscr{L}}$, $w' \in W'_{\mathscr{L}'}$ such that $H_c^{j''}(G, K_w^{\mathscr{L}^{-1}} \otimes K_{w'}^{\mathscr{L}'}) \neq 0$, for some j''. It is therefore enough to prove the following result.

LEMMA 11.3. *If $\mathscr{L}, \mathscr{L}'$ are not in the same W-orbit then $H_c^i(G, K_w^{\mathscr{L}^{-1}} \otimes K_{w'}^{\mathscr{L}'}) = 0$ for all $w \in W'_{\mathscr{L}}$, $w' \in W'_{\mathscr{L}'}$ and all integers i.*

Proof. An equivalent statement is (with the notations of 2.4)

$$H_c^i(Y_w \underset{G}{\times} Y_{w'}, \widetilde{(\mathscr{L}^{-1})} \boxtimes \tilde{\mathscr{L}}') = 0$$

for all $w \in W'_{\mathscr{L}}$, $w' \in W'_{\mathscr{L}'}$ and all i. The variety

$$\begin{aligned} &Y_w \underset{G}{\times} Y_{w'} \\ &\quad = \{(g, B', B'') \in G \times \mathscr{B} \times \mathscr{B} \mid (B', gB'g^{-1}) \in O(w), B'', gB''g^{-1}) \in O(w')\} \end{aligned}$$

can be partitioned into finitely many locally closed pieces Z_y ($y \in W$); the piece Z_y is defined by the condition $(B', B'') \in O(y)$. It is then enough to show that

$$H_c^i(Z_y, \widetilde{(\mathscr{L}^{-1})} \boxtimes \tilde{\mathscr{L}}') = 0 \qquad (\forall i), \tag{11.3.1}$$

for all $y \in W$. (We denote the restriction of $\widetilde{(\mathscr{L}^{-1})} \boxtimes \tilde{\mathscr{L}}'$ to a subvariety of $Y_w \times_G Y_{w'}$ again by $\widetilde{(\mathscr{L}^{-1})} \boxtimes \tilde{\mathscr{L}}'$.)

Let us map Z_y (for fixed $y \in W$) to the space

$$\begin{aligned} R = \{&(B', B'', B''', B^{\mathrm{IV}}) \in \mathscr{B} \times \mathscr{B} \times \mathscr{B} \times \mathscr{B} \mid \\ &(B', B'') \in O(y), (B''', B^{\mathrm{IV}}) \in O(y), \\ &(B, B''') \in O(w), (B'', B^{\mathrm{IV}}) \in O(w')\} \end{aligned}$$

by $(g, B', B'') \mapsto (B', B'', gB'g^{-1}, gB''g^{-1})$. The Leray spectral sequence of the map $Z_y \to R$ shows that (11.3.1) is a consequence of the following statement.

Let ψ_y be any fibre of the map $Z_y \to R$ described above. Then $H_c^i(\psi_y, (\widetilde{\mathscr{L}^{-1}}) \times \tilde{\mathscr{L}}') = 0$ for all i. (11.3.2)

Consider the fibre ψ_y at $(B', B'', B''', B^{\mathrm{IV}}) = (x_1 B x_1^{-1}, x_2 B x_2^{-1}, x_3 B x_3^{-1}, x_4 B x_4^{-1})$. Let $g_0 \in G$ be such that $g_0 x_1 B x_1^{-1} g_0^{-1} = x_3 B x_3^{-1}$, $g_0 x_2 B x_2^{-1} g_0^{-1} = x_4 B x_4^{-1}$. We can assume that $x_3 = g_0 x_1$, $x_4 = g_0 x_2$. A point in ψ_y is completely determined by its g-component. Thus, we may identify

$$\begin{aligned}\psi_y &= \{ g \in G \mid g x_1 B x_1^{-1} g^{-1} = g_0 x_1 B x_1^{-1} g_0^{-1}, g x_2 B x_2^{-1} g^{-1} = g_0 x_2 B x_2^{-1} g_0^{-1} \} \\ &= \{ g \in G \mid g_0^{-1} g \in x_1 B x_1^{-1} \cap x_2 B x_2^{-1} \}.\end{aligned}$$

Here x_1, x_2 are two fixed elements of G such that $x_1^{-1} x_2 = B \dot{y} B$ ($\dot{y} \in N(T)$ represents y). The map $\tau_y : \psi_y \to T$ defined by $x_1^{-1} g_0^{-1} g g_0 x_1 \in \tau_y(g) \cdot U$ makes ψ_y into an affine space bundle over T and one checks that the local system $(\widetilde{\mathscr{L}^{-1}}) \boxtimes \widetilde{\mathscr{L}'}$ on ψ_y is isomorphic to the inverse image under τ_y of the local system $\mathscr{L}^{-1} \otimes (y^{-1})^* \mathscr{L}'$ on T. Hence to prove (11.3.2) it is enough to prove that $H_c^i(T, \mathscr{L}^{-1} \otimes (y^{-1})^* \mathscr{L}') = 0$ for all i. By assumption, $\mathscr{L}$ and $\mathscr{L}'$ are in different W-orbits. It follows that $\mathscr{L}_1 = \mathscr{L}^{-1} \otimes (y^{-1})^* \mathscr{L}'$ is a non-constant local system of rank 1 on T, which belongs to $\mathscr{S}(T)$ (see 2.2). We are reduced to proving the following statement. For any $\mathscr{L}_1 \in \mathscr{S}(T)$, $\mathscr{L}_1$ non-constant, and any i we have $H_c^i(T, \mathscr{L}_1) = 0$. This follows from (1.11.1) and the Künneth formula. This completes the proof of the lemma and hence that of Proposition 11.2.

COROLLARY 11.4. *There is well-defined map $\hat{G} \to \{W\text{-orbits in } \mathscr{S}(T)\}$ given by attaching to $A \in \hat{G}$ the W-orbit of $\mathscr{L}$, where $A \in \hat{G}_{\mathscr{L}}$.*

11.5. Let K be an H-equivariant perverse sheaf on the variety X, where H is a connected algebraic group (see 1.9). Let H_1 be a closed subgroup of H which acts trivially on X. Then we have a natural homomorphism $H_1/H_1^0 \to \mathrm{Aut}(K)$. In the case where K is irreducible, the group $\mathrm{Aut}(K)$ is canonically isomorphic to $\bar{\mathbb{Q}}_l^*$, hence we have a natural homomorphism $\gamma: H_1/H_1^0 \to \bar{\mathbb{Q}}_l^*$; note that each of the sheaves $\mathscr{H}^i K$ is H-equivariant and that the induced action of H_1/H_1^0 on any stalk of $\mathscr{H}^i K$ is a multiple of the character γ.

11.6. We shall apply this in the case where $H = G \times T$ and $H_1 = \mathscr{Z}_G$

($=$ centre of G), imbedded diagonally into $G \times T$. We take X to be one of the varieties in the diagram (see 2.5, 2.6)

$$T \leftarrow \dot{Y}_{\mathbf{s}} \rightarrow Y_{\mathbf{s}} \hookrightarrow \bar{Y}_{\mathbf{s}} \xrightarrow{\bar{\pi}_{\mathbf{s}}} G \tag{11.6.1}$$

where $\mathbf{s}$ is a sequence $(s_1, s_2, \ldots, s_r)$ in S such that $w = s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$. The action of H is defined as follows:

—on T by (g_0, t_0): $t \rightarrow w^{-1}(t_0)\, t t_0^{-1}$;

—on $\dot{Y}_{\mathbf{s}}$ by (g_0, t_0): $(g, h_0 U, h_1 B, \ldots, h_r B) \rightarrow (g_0 g g_0^{-1}, g_0 h_0 t_0^{-1} U, g_0 h_1 B, \ldots, g_0 h_r B)$;

—on $Y_{\mathbf{s}}$ and $\bar{Y}_{\mathbf{s}}$ by (g_0, t_0): $(g, B_0, B_1, \ldots, B_r) \rightarrow (g_0 g g_0^{-1}, g_0 B_0 g_0^{-1}, g_0 B_1 g_0^{-1}, \ldots, g_0 B_r g_0^{-1})$;

—on G by (g_0, t_0): $g \rightarrow g_0 g g_0^{-1}$.

Each of the maps in (11.6.1) is H-equivariant and $H_1 = \mathscr{Z}_G$ acts trivially on each of the varieties in (11.6.1). By (2.2.2), the local system $\mathscr{L}$ on T is H-equivariant. Let γ_0: $H_1/H_1^0 \rightarrow \bar{\mathbb{Q}}_l^*$ be the character by which H_1/H_1^0 acts on each stalk of $\mathscr{L}$. The local system $\dot{\mathscr{L}}$ on $\dot{Y}_{\mathbf{s}}$ (see (2.5) is H-equivariant and the induced action of H_1/H_1^0 on $\dot{\mathscr{L}}$ is via the character γ_0 on each stalk of $\dot{\mathscr{L}}$. The local system $\tilde{\mathscr{L}}$ on $Y_{\mathbf{s}}$ (see 2.5) is H-equivariant and the induced action of H_1/H_1^0 on $\tilde{\mathscr{L}}$ is via a character γ_1 on each stalk of $\tilde{\mathscr{L}}$. Since $\dot{\mathscr{L}}$ is the inverse image of $\tilde{\mathscr{L}}$ under $\dot{Y}_{\mathbf{s}} \rightarrow Y_{\mathbf{s}}$, it follows that $\gamma_1 = \gamma_0$.

The constructible sheaf $\bar{\mathscr{L}}$ on $\bar{Y}_{\mathbf{s}}$ (see 2.8) is H-equivariant, and is irreducible as a perverse sheaf (after a shift). Hence the induced action of H_1/H_1^0 on $\bar{\mathscr{L}}$ is via a character γ_2 on each stalk of $\bar{\mathscr{L}}$. Since $\bar{\mathscr{L}} \mid Y_{\mathbf{s}} = \tilde{\mathscr{L}}$, it follows that $\gamma_2 = \gamma_0$.

It follows that for each of the H-equivariant constructible sheaves $\mathscr{H}^i((\bar{\pi}_{\mathbf{s}})_! \bar{\mathscr{L}})$ the induced action of H_1/H_1^0 is via the character γ_0 on each stalk. Since $(\bar{\pi}_{\mathbf{s}})_! \bar{\mathscr{L}} = \bar{K}_{\mathbf{s}}$ is semisimple (1.12, 2.17(a)) it follows that for each of the H-equivariant constructible sheaves $\mathscr{H}^i({}^pH^j(\bar{K}_{\mathbf{s}}))$ the induced action of H_1/H_1^0 is via the character γ_0 on each stalk. The same is then true for $\mathscr{H}^i(A)$ where A is any irreducible direct summand of ${}^pH^j(\bar{K}_{\mathbf{s}})$ (in $\mathscr{M}(G)$). It follows that for such A (which is necessarily H-equivariant as a perverse sheaf) the corresponding homomorphism $H_1/H_1^0 \rightarrow \bar{\mathbb{Q}}_l^* = \mathrm{Aut}(A)$ is given by γ_0.

Let us write $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\psi})$, as in 2.2. (We recall that λ: $T \rightarrow k^*$, $n \geqslant 1$ is invertible in k and ψ: $\mu_n \rightarrow \bar{\mathbb{Q}}_l^*$.) As $w \in W'_{\mathscr{L}}$, we have $w(\lambda) = \lambda \cdot \lambda_1^n$ for some character λ_1: $T \rightarrow k^*$. We show that

$$\gamma_0(z) = \psi(\lambda_1(z)) \qquad \text{for all} \quad z \in \mathscr{Z}_G. \tag{11.6.2}$$

(Note that $\lambda_1(t) \in \mu_n$ whenever $t \in T$ is fixed by w, and in particular $\lambda_1(z) \in \mu_n$ if $z \in \mathscr{Z}_G$. We also have $\lambda_1(z) = 1$ if $z \in \mathscr{Z}_G^0$.)

Consider the n-fold covering $T' \to^{\pi} T$ where $T' = \{(t, z) \in T \times k^* \mid \lambda(t) = z^n\}$. The action $t_0: t \to w^{-1}(t_0)\, t t_0^{-1}$ of T on T lifts to an action $t_0: (t, z) \to (w^{-1}(t_0)\, t t_0^{-1}, z\lambda_1(t_0))$ of T on T'. Since $\mathscr{L}$ is the local system associated to the principal μ_n-covering $T' \to^{\pi} T$ and the character $\psi: \mu_n \to \bar{\mathbb{Q}}_l^*$, it follows that $\gamma(z)$ is given by (11.6.2) for any z.

We therefore have the following result:

PROPOSITION 11.7. *Let A be a character sheaf of G. Assume that A is a component of ${}^pH^i(\bar{K}_{\mathbf{s}})$ where $\mathbf{s} = (s_1, s_2, ..., s_r)$ is a sequence in S such that $w = s_1 s_2 \cdots s_r \in W'_{\mathscr{L}}$, $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\psi}) \in \mathscr{S}(T)$. Let $\gamma: \mathscr{Z}_G/\mathscr{Z}_G^0 \to \bar{\mathbb{Q}}_l^*$ be the character associated to the G-equivariant perverse sheaf A, as in* 11.5 *with $(K, H, H_1) = (A, G, \mathscr{Z}_G^0)$. Let $\lambda_1: T \to k^*$ be the character defined by $w(\lambda) = \lambda \cdot \lambda_1^n$. Then*

$$\gamma(z) = \psi(\lambda_1(z))$$

for all $z \in \mathscr{Z}_G$.

11.8. We define for any $\mathscr{L} \in \mathscr{S}(T)$ a map

$$\alpha: W'_{\mathscr{L}}/W_{\mathscr{L}} \to \mathrm{Hom}(\mathscr{Z}_G/\mathscr{Z}_G^0, \bar{\mathbb{Q}}_l^*) \tag{11.8.1}$$

as follows. Write $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\psi})$ as in 2.2 and let $\lambda_w: T \to k^*$ be the character defined by $w(\lambda) = \lambda\lambda_w^n$. We define $\alpha(w): \mathscr{Z}_G/\mathscr{Z}_G^0 \to \bar{\mathbb{Q}}_l^*$ by $\alpha(w)(z) = \psi(\lambda_w(z))$, $z \in \mathscr{Z}_G$. (Note that $\lambda_w(z) \in \mu_n$ for $z \in \mathscr{Z}_G$ and $\lambda_w(z) = 1$ for $z \in \mathscr{Z}_G^0$.) When $w \in W_{\mathscr{L}}$, then λ_w is in the root lattice, hence $\lambda_w(z) = 1$ for $z \in \mathscr{Z}_G$ and $\alpha(w) = 1$. It is easy to check that α is a homomorphism and it is independent of the choice of λ, n, ψ.

We now prove that

$$\text{the homomorphism } \alpha \text{ is injective.} \tag{11.8.2}$$

Assume that $w \in W'_{\mathscr{L}}$ is such that $\psi(\lambda_w(z)) = 1$ for all $z \in \mathscr{Z}_G$, i.e., such that $\lambda_w(z) = 1$ for all $z \in \mathscr{Z}_G$. We must prove that $w \in W_{\mathscr{L}}$. This is clear if $\mathscr{Z}_G$ is connected since then, as it is well known, we have $W_{\mathscr{L}} = W'_{\mathscr{L}}$. In the general case, we imbed G into $\tilde{G} = (G \times T)/\mathscr{Z}_G$ (where $\mathscr{Z}_G$ is imbedded diagonally into $G \times T$) by $g \to (g, 1)\, \mathscr{Z}_G$. Then $\tilde{G}$ has connected centre ($\approx T$) and has maximal torus $\tilde{T} = (T \times T)/\mathscr{Z}_G$. The Weyl group of $\tilde{G}$ with respect to $\tilde{T}$ may be naturally identified with that of G with respect to T; the action of w on $\tilde{T}$ is $(t, t')\, \mathscr{Z}_G \to (w(t), t')\, \mathscr{Z}_G$.

We extend $\lambda: T \to k^*$ to a character $\tilde{\lambda}: \tilde{T} \to k^*$ by $\tilde{\lambda}((t, t')\, \mathscr{Z}_G) = \lambda(t)\, \lambda(w(t'))^{-1}$. We have $w(\tilde{\lambda}) = \tilde{\lambda} \cdot \tilde{\lambda}_w^n$ where $\tilde{\lambda}_w: T \to k^*$ is defined by $\tilde{\lambda}_w((t, t')\, \mathscr{Z}_G) = \lambda_w(t)$. (Note that $\tilde{\lambda}_w$ is well defined, since $\lambda_w(z) = 1$ for all $z \in \mathscr{Z}_G$, by our assumption.) Since $\tilde{G}$ has connected centre, the equality

$w(\tilde{\lambda}) = \tilde{\lambda} \cdot \tilde{\lambda}_w^n$ implies that w is a product of reflections $s_i \in W$ each of which satisfies $s_i(\tilde{\lambda}) = \tilde{\lambda}\tilde{\lambda}_i^n$ ($\tilde{\lambda}_i$: $T \to k^*$). Restricting to T, we find $s_i(\lambda) = \lambda \cdot \lambda_i^n$ where $\lambda_i = \tilde{\lambda}_i \mid T$. It follows that $w \in W_{\mathscr{L}}$ (see 2.3), as required.

We can now state

PROPOSITION 11.9. *Let us fix $\mathscr{L} \in \mathscr{S}(T)$. There is a well-defined map $\hat{G}_{\mathscr{L}} \to W'_{\mathscr{L}}/W_{\mathscr{L}}$ given by $A \to wW_{\mathscr{L}}$ where $w \in W'_{\mathscr{L}}$ is any element such that $w = s_1 s_2 \cdots s_r$ ($s_i \in S$), and A is an irreducible component of ${}^pH^i(\bar{K}_{\mathbf{s}}^{\mathscr{L}})$, $\mathbf{s} = (s_1, s_2,..., s_r)$, for some i.*

Proof. Assume that A is also a component of ${}^pH^j(\bar{K}_{\mathbf{s}'}^{\mathscr{L}})$ where $\mathbf{s}' = (s'_1,..., s'_{r'})$ is a sequence in S such that $w' = s'_1 s'_2 \cdots s'_{r'} \in W'_{\mathscr{L}}$. We must prove that $wW_{\mathscr{L}} = w'W_{\mathscr{L}}$. Let γ: $\mathscr{Z}_G/\mathscr{Z}_G^0 \to \bar{\mathbb{Q}}_l^*$ be as in 11.7. It is an invariant of A. From 11.7, it follows that $\alpha(wW_{\mathscr{L}}) = \gamma$, $\alpha(w'W_{\mathscr{L}}) = \gamma$. Since α is injective (11.8.2), we deduce that $wW_{\mathscr{L}} = w'W_{\mathscr{L}}$, as required.

COROLLARY 11.10. *Let us fix $\mathscr{L} \in \mathscr{S}(T)$. If $A \in \hat{G}_{\mathscr{L}}$ is a component of both ${}^pH^i(K_w^{\mathscr{L}})$ and ${}^pH^j(K_{w'}^{\mathscr{L}})$ $(w, w' \in W'_{\mathscr{L}})$ then $wW_{\mathscr{L}} = w'W_{\mathscr{L}}$. Let γ: $\mathscr{Z}_G/\mathscr{Z}_G^0 \to \bar{\mathbb{Q}}_l^*$, λ, n, ψ be as in 11.7. Then $\gamma(z) = \psi(\lambda_1(z))$ $(z \in \mathscr{Z}_G)$, where $w(\lambda) = \lambda \cdot \lambda_1^n$.*

Proof. Let $\mathbf{s} = (s_1, s_2,..., s_r)$, $\mathbf{s}' = (s'_1, s'_2,..., s'_{r'})$ be sequences in S such that $s_1 s_2 \cdots s_r = w$, $r = l(w)$, $s'_1 s'_2 \cdots s'_{r'} = w'$, $r' = l(w')$.

As in 2.11, we see that $K_w^{\mathscr{L}} = K_{\mathbf{s}}^{\mathscr{L}}$, hence A is a component of ${}^pH^i(K_{\mathbf{s}}^{\mathscr{L}})$. We define $I_{\mathbf{s}}$ and $\mathbf{s}_J$, $(J \subset I_{\mathbf{s}})$, as in 2.6, 2.12, and let J be a maximal subset of $I_{\mathbf{s}}$ with the property that A is a component of ${}^pH^h(K_{\mathbf{s}_J}^{\mathscr{L}})$ for some h. From the proof of (2.14.1) we see that A must also be a component of ${}^pH^{h'}(\bar{K}_{\mathbf{s}_J}^{\mathscr{L}})$. We define $I_{\mathbf{s}'}$, $\mathbf{s}'_{J'}$ $(J' \subset I_{\mathbf{s}'})$ in terms of $\mathscr{L}'$, $\mathbf{s}'$ in the same way as $I_{\mathbf{s}}$, $\mathbf{s}_J$ $(J \subset I_{\mathbf{s}})$ were defined in terms of $\mathscr{L}$, $\mathbf{s}$; we then see that A must be a component of ${}^pH^{h'}(\bar{K}^{\mathscr{L}}{}_{\mathbf{s}'_{J'}})$ for some h' and some $J' \subset I_{\mathbf{s}'}$. From 11.9 it follows that the product of the terms in $\mathbf{s}_J$ and the analogous product for $\mathbf{s}'_{J'}$ are in the same $W_{\mathscr{L}}$-coset. On the other hand, from the definition of $I_{\mathbf{s}}$ (2.12), it is clear that the product of terms in $\mathbf{s}_J$ is in the same $W_{\mathscr{L}}$-coset as the product of terms in $\mathbf{s}$, which is w. Similarly, the product of terms in $\mathbf{s}'_{J'}$ is in the same $W_{\mathscr{L}}$-coset as w'. This shows that w, w' are in the same $W_{\mathscr{L}}$-coset. The last statement in the corollary follows from 11.7.

12. THE COMPLEXES $\bar{K}_w^{\mathscr{L}}$

12.1. Let $\mathscr{L} \in \mathscr{S}(T)$ and let $w \in W'_{\mathscr{L}}$. We shall complete the diagram

$$Y_w \xrightarrow{\pi_w} G \tag{12.1.1}$$

of 2.4 into a diagram

$$\begin{tikzcd} \hat{Y}_w \arrow[r] \arrow[d, hook] & Y_w \arrow[r, "\pi_w"] \arrow[d, hook] & G \\ \hat{\bar{Y}}_w \arrow[r] & \bar{Y}_w \arrow[ur, "\bar{\pi}_w"'] & \end{tikzcd} \tag{12.1.2}$$

where

$$\bar{Y}_w = \{(g, B') \in G \times \mathscr{B} \mid (B', gB'g^{-1}) \in \overline{O(w)})\},$$

$$\hat{Y}_w = \{(g, h) \in G \times G \mid h^{-1}gh \in BwB\},$$

$$\hat{\bar{Y}}_w = \{(g, h) \in G \times G \mid h^{-1}gh \in \overline{BwB}\},$$

$$\bar{\pi}_w(g, B') = g,$$

$\hat{Y}_w \to Y_w$ and $\hat{\bar{Y}}_w \to \bar{Y}_w$ is $(g, h) \to (g, hBh^{-1})$ (a principal fibration with group B), and $\hat{Y}_w \hookrightarrow \hat{\bar{Y}}_w$, $Y_w \hookrightarrow \bar{Y}_w$ are the obvious imbeddings. (Here, $\overline{O(w)}$ is the closure of $O(w)$ in $\mathscr{B} \times \mathscr{B}$ and $\overline{BwB}$ is the closure of BwB in G.) Then $\bar{\pi}_w$ is a proper map, $\hat{Y}_w$ is open dense in $\hat{\bar{Y}}_w$, hence Y_w is open dense in $\bar{Y}_w$.

Let $\dot{w}$ be a representative for w in $N(T)$, let $\hat{\mathscr{L}}_{\dot{w}}$ be the inverse image of $\mathscr{L}$ under the map $\hat{Y}_w \to T$ given by $(g, h) \to \mathrm{pr}_{\dot{w}}(h^{-1}gh)$ (see 2.4); then $\hat{\mathscr{L}}_{\dot{w}}$ is B-equivariant for the free B-action on $\hat{Y}_w$ given by right translation on the h-factor. Hence there is a canonical local system $\tilde{\mathscr{L}}_{\dot{w}}$ on Y_w whose inverse image under $\hat{Y}_w \to Y_w$ is $\hat{\mathscr{L}}_{\dot{w}}$. This is the same as the local system $\tilde{\mathscr{L}}$ on Y_w defined in 2.4. Its isomorphism class is independent of the choice of representative $\dot{w}$. (However, we shall want to consider rational structures for this local system and for that the choice of $\dot{w}$ does play a role.)

Let $J_{\dot{w}}^{\mathscr{L}} = \mathrm{IC}(\bar{Y}_w, \tilde{\mathscr{L}}_{\dot{w}}) \in \mathscr{D}\bar{Y}_w$, $\hat{J}_{\dot{w}}^{\mathscr{L}} = \mathrm{IC}(\hat{\bar{Y}}_w, \hat{\mathscr{L}}_{\dot{w}}) \in \mathscr{D}(\hat{\bar{Y}}_w)$; then $\hat{J}_{\dot{w}}^{\mathscr{L}}$ is canonically the inverse image of $J_{\dot{w}}^{\mathscr{L}}$ under $\hat{\bar{Y}}_w \to \bar{Y}_w$. Define

$$\bar{K}_{\dot{w}}^{\mathscr{L}} = (\bar{\pi}_w)_! J_{\dot{w}}^{\mathscr{L}} \in \mathscr{D}G, \qquad K_{\dot{w}}^{\mathscr{L}} = (\pi_w)_! \tilde{\mathscr{L}}_{\dot{w}} \in \mathscr{D}G. \tag{12.1.3}$$

The isomorphism class of $\bar{K}_{\dot{w}}^{\mathscr{L}}$, $K_{\dot{w}}^{\mathscr{L}}$ depends only on w, not on $\dot{w}$; when we are interested only in its isomorphism class, we shall write $\bar{K}_w^{\mathscr{L}}$, $K_w^{\mathscr{L}}$ instead of $\bar{K}_{\dot{w}}^{\mathscr{L}}$, $K_{\dot{w}}^{\mathscr{L}}$. (The notation $K_w^{\mathscr{L}}$ is compatible with that in 2.4.)

12.2. Assume now that $k = \bar{F}_q$ and that we are given an F_q-rational structure on G such that B, T are defined over F_q and T is split over F_q. We assume also that $\mathscr{L}^{\otimes (q-1)} \cong \bar{\mathbb{Q}}_l$.

The varieties and maps in (12.1.2) are naturally defined over F_q; we shall denote by a subscript zero the corresponding schemes over F_q. The local system $\mathscr{L}$ is also defined over F_q. More precisely, let us write $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\psi})$ as in 2.2. (Here $\lambda \colon T \to k^*$ is a character, $n \geqslant 1$ is an integer dividing $q - 1$, and $\psi: \mu_n \to \bar{\mathbb{Q}}_l^*$.) Then there is a unique F_q-rational structure on $\mathscr{L}$ such that the trace of the Frobenius map at the stalk of $\mathscr{L}$ at $t \in T(F_q)$ is $\psi(\lambda(t)^{(q-1)/n})$.

We assume that $\dot{w} \in N(T)(F_q)$. Then $\hat{\mathscr{L}}_{\dot{w}}$, $\tilde{\mathscr{L}}_{\dot{w}}$ inherit natural F_q-structures from $\mathscr{L}$. (These F_q-structures depend on $\dot{w}$, since the map $\hat{Y}_w \to T$ used to define $\hat{\mathscr{L}}_{\dot{w}}$ depends on $\dot{w}$.) It follows that $J_{\dot{w}}^{\mathscr{L}}$, $\hat{J}_{\dot{w}}^{\mathscr{L}}$, $\bar{K}_{\dot{w}}^{\mathscr{L}}$, $K_{\dot{w}}^{\mathscr{L}}$ can naturally be regarded as objects in the derived category of mixed complexes over the F_q-scheme $(\bar{Y}_w^{\mathscr{L}})_0$, $(\hat{\bar{Y}}_w^{\mathscr{L}})_0$, G_0, G_0, respectively. It follows also that ${}^pH^i(K_{\dot{w}}^{\mathscr{L}})$, ${}^pH^i(\bar{K}_{\dot{w}}^{\mathscr{L}})$ can naturally be regarded as mixed perverse sheaves on G_0.

$J_{\dot{w}}$ *is a pure complex of weight* 0 (by Gabber's theorem [1, 5.3.4]). (12.2.1)

$\bar{K}_{\dot{w}}^{\mathscr{L}}$ *is a pure complex of weight* 0 (by Deligne's theorem [2, 6.2.6], by (12.2.1), and by the properness of $\bar{\pi}_w$). (12.2.2)

12.3. Write $W'_{\mathscr{L}} = \Omega_{\mathscr{L}} \cdot W_{\mathscr{L}}$ as in 5.1 and let $\bar{l}: W'_{\mathscr{L}} \to \mathbb{N}$ be the length function defined in 5.1. Let $Z = w \cdot W_{\mathscr{L}}$ be the $W_{\mathscr{L}}$-coset containing w and let w_1 be the unique element in $Z \cap \Omega_{\mathscr{L}}$. Under the assumptions in 12.2, we say that a subset $\dot{Z}$ of $N(T)(F_q)$ is a *coherent lifting* of Z if it has the following properties:

(a) The natural map $N(T) \to W$ defines a bijection $\dot{Z} \xrightarrow{\sim} Z$ (we denote by $\dot{y}$ the element in $\dot{Z}$ corresponding to $y \in Z$).

(b) For any $y \in Z$, the element $(\dot{w}_1)^{-1}\dot{y}$ can be written as a product $n_1 n_2 \cdots n_r$, where $r = \bar{l}(w_1^{-1}y)$; each n_i is a representative of $N(T)(F_q)$ for a simple reflection of $W_{\mathscr{L}}$ of the form $u \cdot u' \cdot u''$, where u, u', u'' belong to the union of the corresponding two root subgroups (over F_q).

(The notion of coherent lifting appeared in the work of Kilmoyer on principal series representations of Chevalley groups over F_q; see also [6, 1.23].)

We can now state the following result which is analogous to [6, 2.4].

THEOREM 12.4. *Assume that we are in the setup* 12.2. *Assume that* $w \in W'_{\mathscr{L}}$ *and that* $\dot{Z}$ *is a coherent lifting* (12.3) *of* $Z = wW_{\mathscr{L}}$. *Let* $y \in W$ *be an element such that* $y \leqslant w$, *for the standard partial order of* W, *so that* $\hat{Y}_y \subset \hat{\bar{Y}}_w$ *and* $Y_y \subset \bar{Y}_w$. *The restriction* $\mathscr{H}^i(J_{\dot{w}}) \mid Y_y$ *is a local system with finite monodromy. It is zero unless* $y \in Z$ *and* i *is even. If these conditions are satisfied and if* $\dot{y}$, $\dot{w}$ *are chosen in* $\dot{Z}$, *then it admits a filtration* (*defined over* F_q) *by local systems, with all subquotients isomorphic* (*over* F_q) *to* $\mathscr{L}_{\dot{y}} \otimes \bar{\mathbb{Q}}_l(-i/2)$ *and with a number of steps equal to* $n_{y,w,i}$, *where* $n_{y,w,i}$ *is the coefficient of* $X^{i/2}$ *in* $X^{(1/2)(l(w) - l(y) - \bar{l}(w) + \bar{l}(y))} P_{w_1^{-1}y, w_1^{-1}w}(X)$, *where* $P_{\alpha,\beta}$ *are the polynomials of* [12] *for the Coxeter group* $W_{\mathscr{L}}$.

Proof. We shall deduce the theorem from Theorem 1.24 in [6]. Recall that $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\psi})$. Let $N_w = BwB \times k^*$, $P_w = N_w/B$ (where B acts on N_w by b: $(g, z) \to (gb^{-1}, \lambda(b)z)$), and let $N_w \to P_w$ be the canonical map. Consider the principal μ_n-covering of P_w (defined in terms of $\dot{w} \in \dot{Z}$):

$$(U/U \cap \dot{w}U\dot{w}^{-1}) \times k^* \to P_w, \qquad (u, z) \to B\text{-orbit of } (u\dot{w}, z^n). \tag{12.4.1}$$

Let $\mathscr{E}_{\dot{w}}$ be the local system on P_w attached to the μ_n-covering (12.4.1) and to $\psi: \mu_n \to \bar{\mathbb{Q}}_l^*$. It is in a natural way a local system defined over F_q. Let $\bar{N}_w = \overline{BwB} \times k^*$, $\bar{P}_w = \bar{N}_w/B$, where B acts on $\bar{N}_w$ by the same formula as on N_w. Then N_w is open dense in $\bar{N}_w$, hence P_w is open dense in $\bar{P}_w$. We have $N_y \subset \bar{N}_w$, $P_y \subset \bar{P}_w$ in a natural way. The following result is proved in [6, 1.24]:

> The restriction of $\mathscr{H}^i(\mathrm{IC}(\bar{P}_w, \mathscr{E}_{\dot{w}})) \mid P_y$ is a local system (defined over F_q) with finite monodromy. It is zero unless $y \in Z$ and i is even. If these conditions are satisfied then it admits a filtration by local systems (defined over F_q) with all subquotients isomorphic (over F_q) to $\mathscr{E}_{\dot{y}} \otimes \bar{\mathbb{Q}}_l(-i/2)$ (where $\dot{y} \in Z$) and with a number of steps equal to $n_{y,w,i}$. (12.4.2)

We now prove

> In the diagram $Y_w \leftarrow \hat{Y}_w \xleftarrow{\mathrm{pr}_1} \hat{Y}_w \times k^* \xrightarrow{\alpha} N_w \to P_w$ where $\alpha(g, x, z) = (x^{-1}gx, z^n)$, the inverse image of $\tilde{\mathscr{L}}_{\dot{w}}$ (to $\hat{Y}_w \times k^*$) and the inverse image of $\mathscr{E}_{\dot{w}}$ (to $\hat{Y}_w \times k^*$) are isomorphic (as local systems defined over F_q). (12.4.3)

First we note that the inverse image of $\tilde{\mathscr{L}}_{\dot{w}}$ under $\hat{Y}_w \to Y_w$ is $\hat{\mathscr{L}}_{\dot{w}}$ and this is the local system associated to the following principal μ_n-covering of $\hat{Y}_w$:

$$\{(g, x, \xi) \in G \times G \times k^* \mid x^{-1}gx \in BwB, \lambda(\mathrm{pr}_{\dot{w}}(x^{-1}gx)) = \xi^n\} \to \hat{Y}_w,$$
$$(g, x, \xi) \mapsto (g, x). \tag{12.4.4}$$

(From this we see that $\hat{\mathscr{L}}_{\dot{w}}$ is B-equivariant for the free B-action on $\hat{Y}_w$ given by b: $(g, h) \to (g, hb^{-1})$; indeed this B-action on $\hat{Y}_w$ lifts to a B-action on the space (12.4.4), b: $(g, x, \xi) \to (g, xb^{-1}, \xi\lambda_1(t_b))$ where $\lambda_1: T \to k^*$ is the character defined by $w(\lambda) = \lambda \cdot \lambda_1^n$ and t_b is the T-component of $b \in B$.)

Next we note that the inverse image of $\mathscr{E}_{\dot{w}}$ under $N_w \to P_w$ is the local system associated to the following principal μ_n-covering of N_w:

$$N_w \to N_w, \qquad (g, z) \mapsto (g, \lambda(\mathrm{pr}_{\dot{w}}(g))^{-1} z^n). \tag{12.4.5}$$

(Indeed, we have a cartesian diagram

$$\begin{tikzcd} N_w \arrow[r, "(12.4.5)"] \arrow[d, "\beta"'] & N_w \arrow[d] \\ (U/U\cap \dot{w}U\dot{w}^{-1})\times k^* \arrow[r, "(12.4.1)"] & P_w \end{tikzcd}$$

where $\beta(u\dot{w}tu', z)=(u, z)$, $u, u'\in U$, $t\in T$.) Hence to prove (12.4.3), it is enough to show that the μ_n-coverings (12.4.4) and (12.4.5) have the same inverse image under $\hat{Y}_w\times k^*\to\hat{Y}_w$, $\hat{Y}_w\times k^*\to N_w$. This follows from the cartesian diagram

$$\begin{tikzcd} \{(g,x,z,\xi)\in G\times G\times k^*\times k^* \mid x^{-1}gx\in BwB, \lambda(\mathrm{pr}_{\dot{w}}(x^{-1}gx))=\xi^n\} \arrow[r, "\gamma"] \arrow[d, "\delta"'] & N_w \arrow[d, "(12.4.5)"] \\ \{(g,x,z)\in G\times G\times k^*\mid x^{-1}gx\in BwB\} \arrow[r, "\varepsilon"] & N_w \end{tikzcd}$$

where $\gamma(g, x, z, \xi) = (x^{-1}gx, z\xi)$, $\delta(g, x, z, \xi) = (g, x, z)$, $\varepsilon(g, x, z) = (x^{-1}gx, z^n)$. Thus, (12.4.3) is proved.

Consider the diagram

$$\begin{tikzcd} Y_y \arrow[d, hook] & \hat{Y}_y\times k^* \arrow[l] \arrow[r] \arrow[d, hook] & P_y \arrow[d, hook] \\ \bar{Y}_w & \hat{\bar{Y}}_w\times k^* \arrow[l] \arrow[r] & \bar{P}_w \end{tikzcd}$$

Consider the following three statements:

(a) the statement (12.4.2);

(b) the statement obtained from (12.4.2) by replacing $P_y, \bar{P}_w$ by $\hat{Y}_y\times k^*$, $\hat{\bar{Y}}_w\times k^*$ and $\mathscr{E}_{\dot{y}}, \mathscr{E}_{\dot{w}}$ by their inverse images to $\hat{Y}_y\times k^*$, $\hat{Y}_w\times k^*$;

(c) the statement obtained from (12.4.2) by replacing $P_y, \bar{P}_w$ by $\hat{Y}_y\times k^*$, $\hat{\bar{Y}}_w\times k^*$ and $\mathscr{E}_{\dot{y}}, \mathscr{E}_{\dot{w}}$ by the inverse images of $\tilde{\mathscr{L}}_{\dot{y}}, \tilde{\mathscr{L}}_{\dot{w}}$ under $Y_y\leftarrow\hat{Y}_y\times k^*$, $Y_w\leftarrow\hat{Y}_w\times k^*$;

(d) the statement obtained from (12.4.2) by replacing $P_y, \bar{P}_w$ by Y_y, $\bar{Y}_w$ and $\mathscr{E}_{\dot{y}}, \mathscr{E}_{\dot{w}}$ by $\tilde{\mathscr{L}}_{\dot{y}}, \tilde{\mathscr{L}}_{\dot{w}}$.

Then (a)$\Rightarrow$(b) since $\bar{Y}_w\times k^*\to\bar{P}_w$ is a locally trivial fibration with smooth fibres. Moreover, (c)$\Leftrightarrow$(d), since $\hat{\bar{Y}}_w\times k^*\to\bar{Y}_w$ is a locally trivial fibration with smooth and connected fibres. From (12.4.3), it follows that (b)$\Leftrightarrow$(c). Thus, (a)$\Rightarrow$(d). Since the statement (a) is true, it follows that the statement (d) is true and the theorem is proved.

COROLLARY 12.5. *The following identity holds in the Grothendieck group of mixed perverse sheaves over G_0 (for $y \leqslant w$):*

$$
\begin{aligned}
&{}^pH^j((\pi_y)_!\,(\mathscr{H}^i(J_{\dot w}) \mid Y_y)) \\
&\quad = \begin{cases} n_{y,w,i}\,{}^pH^j((\pi_y)_!\,(\tilde{\mathscr{L}}_{\dot y}))(-i/2), & \text{if } y \in Z,\ \dot y \in \dot Z, \text{ and } i = \text{even} \\ 0, & \text{if } y \notin Z \text{ or } i = \text{odd}. \end{cases}
\end{aligned}
$$

We now prove:

PROPOSITION 12.6. *With the assumptions of* 12.4, *the following identity holds in the Grothendieck group of mixed perverse sheaves over G_0:*

$$
\begin{aligned}
\sum_j (-1)^j\,{}^pH^j(\bar K^{\mathscr{L}}_{\dot w}) &= \sum_j (-1)^j\,{}^pH^j(K^{\mathscr{L}}_{\dot w}) \\
&\quad + \sum_{\substack{y<w \\ y\in Z}} \sum_j \sum_{\substack{i \\ \text{even}}} (-1)^j\, n_{y,w,i}\,{}^pH^j(K^{\mathscr{L}}_{\dot y})(-i/2)
\end{aligned}
$$

where $\dot y$ are chosen in $\dot Z$ for all $y \in Z$.

Proof. Consider the distinguished triangle

$$(\tau_{\leqslant i-1} J_{\dot w},\ \tau_{\leqslant i} J_{\dot w},\ \mathscr{H}^i J_{\dot w}[-i])$$

in the derived category of mixed complexes on $(\bar Y_w)_0$ ($\tau_{\leqslant i}$ denotes ordinary truncation). Apply to it $(\bar\pi_w)_!$; we get a distinguished triangle

$$((\bar\pi_w)_!\,\tau_{\leqslant i-1} J_{\dot w},\ (\bar\pi_w)_!\,\tau_{\leqslant i} J_{\dot w},\ (\bar\pi_w)_!\,\mathscr{H}^i J_{\dot w}[-i]). \tag{12.6.1}$$

This implies the following identity in the Grothendieck group of mixed perverse sheaves on G:

$$
\begin{aligned}
&\sum_j (-1)^{i+j}\,{}^pH^j((\bar\pi_w)_!\,(\mathscr{H}^i J_{\dot w})) \\
&\quad = \sum_j (-1)^j\,{}^pH^j((\bar\pi_w)_!\,\tau_{\leqslant i} J_{\dot w}) - \sum_j (-1)^j\,{}^pH^j((\bar\pi_w)_!\,\tau_{\leqslant i-1} J_{\dot w}).
\end{aligned}
$$

Taking now the sum over all integers i in a large interval, we get

$$\sum_{i,j} (-1)^{i+j}\,{}^pH^j((\bar\pi_w)_!\,(\mathscr{H}^i J_{\dot w})) = \sum_j (-1)^j\,{}^pH^j(\bar K_{\dot w}). \tag{12.6.2}$$

Consider now the partition $\bar Y_w = \bigcup_{y \leqslant w} Y_y$. Let

$$Z_a = \bigcup_{\substack{y \leqslant w \\ l(y) \leqslant a}} Y_y.$$

Then $Z_0 \subset Z_1 \subset Z_2 \subset \cdots$ are closed subsets of $\bar{Y}_w$. Let $\phi_a\colon Z_a \hookrightarrow \bar{Y}_w$, $\psi_a\colon Z_a - Z_{a-1} \hookrightarrow \bar{Y}_w$ be the inclusions. We have a distinguished triangle

$$((\psi_a)_! \psi_a^* \mathscr{H}^i J_{\dot{w}}, (\phi_a)_! \phi_a^* \mathscr{H}^i J_{\dot{w}}, (\phi_{a-1})_! \phi_{a-1}^* \mathscr{H}^i J_{\dot{w}})$$

in the derived category of mixed perverse sheaves over $(\bar{Y}_w)_0$.

Applying to it $(\bar{\pi}_w)_!$ we get a distinguished triangle

$$((\bar{\pi}_w)_! (\psi_a)_! \psi_a^* \mathscr{H}^i J_{\dot{w}}, (\bar{\pi}_w)_! (\phi_a)_! \phi_a^* \mathscr{H}^i J_{\dot{w}}, (\bar{\pi}_w)_! (\phi_{a-1})_! \phi_{a-1}^* \mathscr{H}^i J_{\dot{w}}). \tag{12.6.3}$$

This implies the identity

$$\begin{aligned}\sum_j (-1)^j\, {}^pH^j((\bar{\pi}_w)_! (\phi_a)_! \phi_a^* \mathscr{H}^i J_{\dot{w}}) \\ -\sum_j (-1)^j\, {}^pH^j((\bar{\pi}_w)_! (\phi_{a-1})_! (\phi_{a-1})^* \mathscr{H}^i J_{\dot{w}}) \\ =\sum_j (-1)^j\, {}^pH^j((\bar{\pi}_w)_! (\psi_a)_! \psi_a^* \mathscr{H}^i J_{\dot{w}}).\end{aligned}$$

We have

$$\begin{aligned}(\phi_a)_! \phi_a^* \mathscr{H}^i J_{\dot{w}} = \mathscr{H}^i J_{\dot{w}}, \qquad & \text{for} \quad a >> 0 \\ = 0, \qquad & \text{for} \quad a < 0;\end{aligned}$$

hence by taking the sum over all integers a in some large interval we get

$$\begin{aligned}&\sum (-1)^j\, {}^pH^j((\bar{\pi}_w)_! \mathscr{H}^i J_{\dot{w}}) \\ &= \sum_a \sum_j (-1)^j\, {}^pH^j((\bar{\pi}_w)_! (\psi_a)_! \psi_a^* \mathscr{H}^i J_{\dot{w}}) \\ &= \sum_{y \leqslant w} \sum_j (-1)^j\, {}^pH^j((\pi_y)_! (\mathscr{H}^i J_{\dot{w}} \mid Y_y)) \\ &= \begin{cases} \displaystyle\sum_{\substack{y \leqslant w \\ y \in Z}} \sum_j (-)^j n_{y,w,i}\, {}^pH^j(K_{\dot{y}})(-i/2), & \text{if} \quad i = \text{even} \\ 0, & \text{if} \quad i = \text{odd}, \end{cases}\end{aligned}$$

the last step being given by 12.5. This, together with (12.6.2), gives the proposition.

PROPOSITION 12.7*. *Let $\mathscr{L} \in \mathscr{S}(T)$ and let Z be a $W_{\mathscr{L}}$-coset in $W'_{\mathscr{L}}$. Let A be an irreducible perverse sheaf on G. The following two conditions are equivalent*:

(a) *A is a constituent of ${}^pH^i(K_w^{\mathscr{L}})$ for some $w \in Z$ and some i.*

(b) *A is a constituent of ${}^pH^i(\bar{K}_w^{\mathscr{L}})$ for some $w \in Z$ and some i.*

Proof. We use notations and results in the proof of 12.6. It is enough to verify the following statement.

$$\text{Let } w\in Z \text{ be such that } (A\colon {}^pH^i(K_y^{\mathscr{L}}))=0 \text{ for any } i \text{ and for any } y\in Z,\ y<w. \text{ Then } (A\colon {}^pH^j(K_w^{\mathscr{L}}))=(A\colon {}^pH^j(\bar{K}_w^{\mathscr{L}})), \text{ for any } j. \tag{12.7.1}$$

(Here, for any perverse sheaf K on G, $(A\colon K)$ denotes the multiplicity of A in a Jordan–Holder series of K.)

From the long perverse cohomology exact sequence associated to (12.6.3) and the hypothesis of (12.7.1) we see that

$$(A\colon {}^pH^j((\bar{\pi}_w)_!\,(\phi_a)_!\,\phi_a^*\mathscr{H}^iJ_{\dot{w}}))=(A\colon {}^pH^j((\bar{\pi}_w)_!\,(\phi_{a-1})_!\,\phi_{a-1}^*\mathscr{H}^iJ_{\dot{w}})),$$

for any i, j and any $a<l(w)$; in particular,

$$(A\colon {}^pH^j(\bar{\pi}_w)_!\,(\phi_a)_!\,\phi_a^*\mathscr{H}^iJ_{\dot{w}})=0 \qquad \text{for} \quad a=l(w)-1 \tag{12.7.2}$$

(since it is zero for $a<0$).

The same long exact sequence and (12.7.2) show that

$$(A\colon {}^pH^j((\bar{\pi}_w)_!\,(\phi_a)_!\,\phi_a^*\mathscr{H}^iJ_{\dot{w}}))=(A\colon {}^pH^j((\bar{\pi}_w)_!\,(\psi_a)_!\,\psi_a^*\mathscr{H}^iJ_{\dot{w}})),$$

if $a=l(w)$, hence

$$(A\colon {}^pH^j((\bar{\pi}_w)_!\,\mathscr{H}^iJ_{\dot{w}}))=\begin{cases}(A\colon {}^pH^j(K_{\dot{w}})) & \text{if } i=0\\ 0, & \text{if } i\neq 0.\end{cases} \tag{12.7.3}$$

Next, we consider the long perverse cohomology exact sequence associated to (12.6.1). For $i>0$, we see that

$$(A\colon {}^pH^j((\bar{\pi}_w)_!\,\tau_{\leqslant i-1}J_{\dot{w}}))=(A\colon {}^pH^j((\bar{\pi}_w)_!\,\tau_{\leqslant i}J_{\dot{w}})).$$

It follows that

$$(A\colon {}^pH^j(\bar{\pi}_w)_!\,\tau_{\leqslant 0}J_{\dot{w}})=(A\colon {}^pH^j(\bar{K}_{\dot{w}})).$$

The same long exact sequence for $i=0$ gives

$${}^pH^j((\bar{\pi}_w)_!\,\tau_{\leqslant 0}J_{\dot{w}})={}^pH^j((\bar{\pi}_w)_!\,\mathscr{H}^0J_{\dot{w}}).$$

It follows that

$$(A\colon {}^pH^j((\bar{\pi}_w)_!\,\mathscr{H}^0J_{\dot{w}}))=(A\colon {}^pH^j\bar{K}_{\dot{w}}).$$

Combining with (12.7.3) we see that $(A\colon {}^pH^j(K_{\dot{w}}))=(A\colon {}^pH^j(\bar{K}_{\dot{w}}))$. This completes the proof of the proposition.

PROPOSITION 12.8*. *Let* $w \in W'_{\mathscr{L}}$. *Then* $\bar{K}^{\mathscr{L}}_w \in \mathscr{D}G$ *is semisimple* (see 1.12).

Proof. This follows from (12.2.2) and the decomposition theorem [1, 5.4.5].

12.9. Let $\hat{W}'_{\mathscr{L}}$ denote the set of isomorphism classes of irreducible $\bar{\mathbb{Q}}_l[W'_{\mathscr{L}}]$-modules. With each $E \in \hat{W}'_{\mathscr{L}}$ one can associate canonically an $H'_{\mathscr{L}} \otimes_{\mathscr{A}} \bar{\mathbb{Q}}_l[u^{1/2}, u^{-1/2}]$-module $E(u)$ as in [14, 1.1, 1.2; 6, 3.3]; the corresponding modules $E(u) \otimes \bar{\mathbb{Q}}_l(u^{1/2})$ form a complete set of irreducible representations of $H'_{\mathscr{L}} \otimes_{\mathscr{A}} \bar{\mathbb{Q}}_l(u^{1/2})$. Under the specialization $\mathscr{A} \to \bar{\mathbb{Q}}_l$, $u^{1/2} \to q^{1/2}$, where $q^{1/2}$ is a fixed square root of q in $\bar{\mathbb{Q}}_l$, $E(u)$ becomes an $H'_{\mathscr{L}}(q) = H'_{\mathscr{L}} \otimes_{\mathscr{A}} \bar{\mathbb{Q}}_l$-module $E(q)$ and the $E(q)$ form again a complete set of irreducible representations of $H'_{\mathscr{L}}(q)$. It is clear that

> Any $\mathscr{A}$-linear function $f: H'_{\mathscr{L}} \to \bar{\mathbb{Q}}_l(u^{1/2})$ such that $f(h_1 h_2) = f(h_2 h_1)$ for all $h_1, h_2 \in H'_{\mathscr{L}}$ is a $\bar{\mathbb{Q}}_l(u^{1/2})$-linear combination of $\mathscr{A}$-linear functions f_E of the form $f_E(T_w) = \mathrm{Tr}(T_w, E(u))$ $(E \in \hat{W}'_{\mathscr{L}})$. (The trace is taken over $\bar{\mathbb{Q}}_l[u^{1/2}, u^{-1/2}]$.) (12.9.1)

For each $w \in W'_{\mathscr{L}}$ we set

$$\begin{aligned} C_w &= \sum (-1)^{\bar{l}(w) - \bar{l}(y)} P_{w_1^{-1}y, w_1^{-1}w}(u^{-1})\, u^{\bar{l}(w)/2 - \bar{l}(y)} T_y \in H'_{\mathscr{L}} \\ C'_w &= \sum P_{w_1^{-1}y, w_1^{-1}w}(u)\, u^{-\bar{l}(w)/2} T_y \in H'_{\mathscr{L}} \end{aligned} \tag{12.9.2}$$

(the sums are taken over all $y \in wW_{\mathscr{L}}$; w_1 is the unique element in $\Omega_{\mathscr{L}} \cap wW_{\mathscr{L}}$ and $P_{\alpha,\beta}$ are the polynomials defined in [12] for the Coxeter group $W_{\mathscr{L}}$).

We shall need some properties of $\mathrm{Tr}(T_w, E(u))$. (Compare [6, 3.3, (6.9.5)].)

$$\mathrm{Tr}(T_w, E(u)) \in \zeta \cdot \mathbb{Z}[u^{1/2}], \tag{12.9.3}$$

where ζ is a root of 1 of order dividing $|\Omega_{\mathscr{L}}|$, which depends only on E and on the $W_{\mathscr{L}}$-coset of $w \in W'_{\mathscr{L}}$.

$$\mathrm{Tr}(T^{-1}_{w^{-1}}, E(u)) = \overline{\mathrm{Tr}(T_w, E(u))} \tag{12.9.4}$$

where the bar denotes the involution of the ring $\bar{\mathbb{Q}}_l[u^{1/2}, u^{-1/2}]$ which is identity on $\bar{\mathbb{Q}}_l$ and takes $u^{i/2}$ to $u^{-i/2}$.

$$\mathrm{Tr}(T_{w^{-1}}, E(u)) = \mathrm{Tr}(T_w, E^*(u)) \tag{12.9.5}$$

where E^* is the representation of $W'_{\mathscr{L}}$ dual to E.

$$\sum_{w\in W'_{\mathscr{L}}} u^{-\bar{l}(w)}\operatorname{Tr}(T_w, E(u))\operatorname{Tr}(T_{w^{-1}}, E'(u))$$
$$=\begin{cases}\left(\sum_{w\in W'_{\mathscr{L}}} u^{\bar{l}(w)}\right) D_{E_1}(u)^{-1}\cdot \dim E_1, & \text{if } E\approx E'\\ 0, & \text{if } E\not\approx E'.\end{cases} \tag{12.9.6}$$

Here $D_{E_1}(u)\in Q[u]$ denotes the "formal dimension" (or generic degree) of any irreducible $W_{\mathscr{L}}$-module E_1 appearing in the restriction of E to $W_{\mathscr{L}}$.

$$\operatorname{Tr}(T_w, (E\otimes\varepsilon)(u)) = (-1)^{l(w)} u^{\bar{l}(w)}\operatorname{Tr}(T^{-1}_{w^{-1}}, E(u)), \tag{12.9.7}$$

where $\varepsilon\colon W'_{\mathscr{L}}\to \pm 1$ is the restriction to $W'_{\mathscr{L}}$ of the sign representation $w\to(-1)^{l(w)}$ of W.

PROPOSITION 12.10*. *There is a unique function*

$$\hat{G}_{\mathscr{L}}\times \hat{W}'_{\mathscr{L}}\to \bar{\mathbb{Q}}_l(u^{1/2}),\qquad (A, E)\mapsto c_{A,E}(u)$$

such that

$$\sum_i (-1)^i (A: {}^pH^i(\bar{K}^{\mathscr{L}}_{\dot{w}}))\, u^{i/2} = \sum_{E\in \hat{W}'_{\mathscr{L}}} u^{(1/2)(\dim G + l(w))} c_{A,E}(u)\operatorname{Tr}(C'_w, E(u)) \tag{12.10.1}$$

for all $w\in W'_{\mathscr{L}}$, $A\in \hat{G}_{\mathscr{L}}$. (Identity in $\bar{\mathbb{Q}}_l[u^{1/2}, u^{-1/2}]$.) For any A and E there exists a root ζ of 1 of order dividing $|\Omega_{\mathscr{L}}|$ and an integer $f\geqslant 1$ such that $\zeta^{-1}u^f(\sum_{w\in W'_{\mathscr{L}}} u^{\bar{l}(w)})\, c_{A,E}(u)\in \mathbb{Q}[u^{1/2}]$. In particular, the identity (12.10.1) can be specialized for $u^{1/2}=q^{s/2}$ $(s=1, 2, 3,...)$.

Proof. The uniqueness of the $c_{A,E}(u)$ satisfying (12.10.1) is clear. We now prove the existence. We shall consider the mixed complex $\bar{K}^{\mathscr{L}}_{\dot{w}}$ as in 12.2. From (12.2.2) and [1, 5.4.4] it follows that ${}^pH^i(\bar{K}^{\mathscr{L}}_{\dot{w}})$ is pure of weight i. By the definition of χ_u (see 6.3) it then follows that the left-hand side of (12.10.1) is equal to $\chi_u(\bar{K}^{\mathscr{L}}_{\dot{w}})$. Using now 12.6 and the additivity of χ_u (which follows from [1, 5.3.5]) we get the following identity in $\mathscr{K}(G)\otimes_{\mathbb{Z}}\mathscr{A}$ (see 6.3):

$$\chi_u(\bar{K}^{\mathscr{L}}_{\dot{w}}) = \sum_{\substack{y\in wW_{\mathscr{L}}\\ y\leqslant w}}\left(\sum_{\substack{i\\ \text{even}}} n_{y,w,i}u^{i/2}\right)\chi_u(K^{\mathscr{L}}_{\dot{y}})$$
$$= \sum_{\substack{y\in wW_{\mathscr{L}}\\ y\leqslant w}} u^{(1/2)(l(w)-l(y)-\bar{l}(w)+\bar{l}(y))} P_{w_1^{-1}y, w_1^{-1}w}(u)\,\chi_u(K^{\mathscr{L}}_{\dot{y}})$$

(notations of (12.9.2)).

By (6.3.2) this is equal to

$$\sum_{y \leqslant w} u^{(1/2)(\dim G + l(w) - \bar{l}(w))} P_{w_1^{-1}y, w_1^{-1}w}(u)\, \varepsilon'(T_y), \tag{12.10.2}$$

where $\varepsilon'\colon H'_{\mathscr{L}} \to \mathscr{K}(G) \otimes \mathscr{A}$ satisfies $\varepsilon'(h_1 h_2) = \varepsilon'(h_2 h_1)$ for all $h_1, h_2 \in H'_{\mathscr{L}}$. By (12.9.1) we can write

$$\varepsilon'(T_y) = \sum_{A \in \hat{G}_{\mathscr{L}}} \sum_{E \in \hat{W}'_{\mathscr{L}}} c_{A,E}(u) \operatorname{Tr}(T_y, E(u))\, A \qquad (\forall y \in W'_{\mathscr{L}}), \tag{12.10.3}$$

where $c_{A,E}(u) \in \bar{\mathbb{Q}}_l(u^{1/2})$.

Now (12.10.2) becomes

$$\sum_{A \in \hat{G}_{\mathscr{L}}} \sum_{E \in \hat{W}'_{\mathscr{L}}} u^{(1/2)(\dim G + l(w))} c_{A,E}(u) \operatorname{Tr}(C'_w, E(u))\, A,$$

as desired.

By the definition of $c_{A,E}(u)$, we have:

$$\sum_{E \in \hat{W}'_{\mathscr{L}}} c_{A,E}(u) \operatorname{Tr}(T_y, E(u)) = \pi_{y,A}(u) \qquad (\forall y \in W'_{\mathscr{L}})$$

where $\pi_{y,A}(u) \in \mathbb{Z}[u^{1/2}, u^{-1/2}]$ is zero unless y is in a fixed $W_{\mathscr{L}}$-coset of $W'_{\mathscr{L}}$ (depending on A). (Here we use 11.10.) Applying now (12.9.6), we get

$$\begin{aligned} c_{A,E}(u) \cdot \sum_{y \in W'_{\mathscr{L}}} u^{-\bar{l}(y)} \operatorname{Tr}(T_y, E(u)) \operatorname{Tr}(T_{y^{-1}}, E(u)) \\ = \sum_{y \in W'_{\mathscr{L}}} \pi_{y,A}(u) \operatorname{Tr}(T_{y^{-1}}, E(u)). \end{aligned} \tag{12.10.4}$$

In the right-hand side of this equality we may assume (by the previous remark) that y runs through only one $W_{\mathscr{L}}$-coset of $W'_{\mathscr{L}}$; using (12.9.3), it follows that the right-hand side of (12.10.4) is in $\zeta \cdot \mathbb{Z}[u^{1/2}, u^{-1/2}]$ where ζ is a root of 1 of order dividing $|\Omega_{\mathscr{L}}|$. By (12.9.6), the factor multiplying $c_{A,E}(u)$ in the left-hand side of (12.10.4) is a divisor in $Q[u, u^{-1}]$ of $\sum_{w \in W'_{\mathscr{L}}} u^{\bar{l}(w)}$. The proposition follows.

12.11. The function $c_{A,E}(u)$ is defined with respect to $\mathscr{L}$. If we replace $\mathscr{L}$ by $\mathscr{L}^{-1}$, then $c_{DA,E}(u)$ is defined in the same way with respect to $\mathscr{L}^{-1}$. (See 11.2(d) and note that $W'_{\mathscr{L}} = W'_{\mathscr{L}^{-1}}$.) We have

$$c_{DA,E}(u) = c_{A,E}(u). \tag{12.11.1}$$

Indeed, let $d = \dim Y_w$. We have $D({}^pH^i\bar{K}_w^{\mathscr{L}}) = {}^pH^{2d-i}\bar{K}_w^{\mathscr{L}^{-1}} = {}^pH^i\bar{K}_w^{\mathscr{L}^{-1}}$. (The first equality is proved as in (11.2.1); the second equality is the hard Lefschetz theorem of [1, 5.4.10].) Now (12.11.1) follows from (12.10.1).

12.12. Let us write $c_{A,E}(u)=\zeta P$ where ζ is a root of 1 and $P\in\mathbb{Q}(u^{1/2})$ (see 12.10). We show that

$$c_{A,E^*}(u)=\zeta^{-1}P. \tag{12.12.1}$$

To prove this, we use the following observation. The representation E^* can be obtained from E by applying an element γ in the Galois group of $\bar{\mathbb{Q}}_l$ over $\mathbb{Q}$ which takes each root of 1 to its inverse. Since the construction $E\to E(u)$ is compatible with the action of this Galois group, it follows that $\mathrm{Tr}(T_y, E^*(u))$ is obtained from $\mathrm{Tr}(T_y, E(u))$ by applying γ to each coefficient. The effect of γ on $c_{A,E}(u)$ can be determined from (12.10.4). Since $\pi_{y,A}(u)$ in (12.10.4) has integral coefficients, it is invariant under γ. From (12.10.4) it then follows that γ carries $c_{A,E}(u)$ to $c_{A,E^*}(u)$, and (12.12.1) follows.

13. Principal Series Representations

13.1. In this chapter we assume that we are given an F_q-rational structure on G such that B, T are defined over F_q and T is split over F_q. We also assume given $\mathscr{L}=\lambda^*(\mathscr{E}_{n,\psi})\in\mathscr{S}(T)$ as in 2.2 and that n divides $q-1$. We regard $\mathscr{L}$ as a local system defined over F_q, as in 12.2. Let $F\colon G\to G$ denote the Frobenius map.

13.2. Let $\mathscr{P}$ be the vector space of all functions $f\colon G^F/U^F\to\bar{\mathbb{Q}}_l$. It is a G^F-module:

$$(g_1 f)(gU^F)=f(g_1^{-1}gU^F),\qquad f\in\mathscr{P},\ g,\ g_1\in G^F.$$

If $\bar{n}\in N(T)^F$, we define $\tau_{\bar{n}}\colon\mathscr{P}\to\mathscr{P}$ by

$$(\tau_{\bar{n}}f)(gU^F)=\sum_{\substack{g'U^F\in G^F/U^F\\ g^{-1}g'\in U\bar{n}U}} f(g'U^F).$$

Then $(\tau_{\bar{n}})_{\bar{n}\in N(T)^F}$ is a basis for $\mathrm{End}_{G^F}(\mathscr{P})$, and each $\tau_{\bar{n}}$ is invertible. Let $\theta\colon T^F\to\bar{\mathbb{Q}}_l^*$ be the character defined by

$$\theta(t)=\psi(\lambda(t)^{(q-1)/n}),\qquad t\in T^F. \tag{13.2.1}$$

We define

$$\mathscr{P}^\theta=\{f\in\mathscr{P}\mid \tau_t f=\theta(t)^{-1}f,\ \forall t\in T^F\}.$$

Then $\mathscr{P}^\theta$ is a G^F-submodule of $\mathscr{P}$ and the maps $\tau_{\bar{n}}\colon\mathscr{P}^\theta\to\mathscr{P}^\theta$ (where $\bar{n}$ runs through a set of representatives in $N(T)^F$ for the elements in $W'_{\mathscr{L}}$) form a basis for $\mathrm{End}_{G^F}(\mathscr{P}^\theta)$.

The structure of the algebra $\mathrm{End}_{G^F}(\mathscr{P}^\theta)$ has been described by Kilmoyer and Howlett [11]. We shall now describe their result.

Let $(W'_{\mathscr{L}})^{\cdot}=\{\dot{w}\mid w\in W'_{\mathscr{L}}\}$ be a set of representatives in $N(T)^F$ for the elements $w\in W'_{\mathscr{L}}$ with the following properties:

(a) $\{\dot{y}\mid y\in W_{\mathscr{L}}\}$ is a coherent lifting (see 12.3) of $W_{\mathscr{L}}$ such that $\dot{e}=e$.

(b) $\{\dot{x}\mid x\in\Omega_{\mathscr{L}}\}$ is any lfting of $\Omega_{\mathscr{L}}$ (see 5.1) such that $\dot{e}=e$.

(c) If $x\in\Omega_{\mathscr{L}}$ and $y\in W_{\mathscr{L}}$, then $(xy)^{\cdot}=\dot{x}\dot{y}$. (13.2.2)

Choose an algebra homomorphism (preserving unit) $h\colon \mathrm{End}_{G^F}(\mathscr{P}^\theta)\to\bar{\mathbb{Q}}_l$. (The existence of h is equivalent to the existence of an irreducible G^F-module appearing in $\mathscr{P}^\theta$ with multiplicity 1; this follows from the fact that the restriction of $\mathscr{P}^\theta$ to U^F contains any generic 1-dimensional representation of U^F exactly once.)

We define a $\bar{\mathbb{Q}}_l$-linear map $\zeta\colon H'_{\mathscr{L}}(q)\to\mathrm{End}_{G^F}(\mathscr{P}^\theta)$ by

$$\zeta(T_{xy})=q^{-(1/2)(l(xy)-l(x)-\bar{l}(y))}h(\tau_{\dot{x}})^{-1}\,\tau_{\dot{x}\dot{y}}\colon\mathscr{P}^\theta\to\mathscr{P}^\theta \qquad (13.2.3)$$

($x\in\Omega_{\mathscr{L}}$, $y\in W_{\mathscr{L}}$), where $\dot{x}$, $\dot{y}$, h are as above, $H'_{\mathscr{L}}$ is the Hecke algebra over $\mathscr{A}$ defined in 6.1, and $H'_{\mathscr{L}}(q)$ is its specialization (12.9). Then

ζ is an algebra isomorphism (*see* [11]). (13.2.4)

For future reference, we note also the formula

$$(q^{-l(x)/2}\tau_{\dot{x}})(q^{-l(x')/2}\tau_{\dot{x}'})=q^{-l(xx')/2}\tau_{\dot{x}\dot{x}'}\colon\mathscr{P}^\theta\to\mathscr{P}^\theta, \qquad (13.2.5)$$

valid for any representatives $\dot{x}$, $\dot{x}'$ in $N(T)^F$ for $x, x'\in\Omega_{\mathscr{L}}$.

13.3. For each $w\in W'_{\mathscr{L}}$, the complex $K^{\mathscr{L}}_{\dot{w}}\in\mathscr{D}G$ ($\dot{w}\in(W'_{\mathscr{L}})^{\cdot}$, see (13.2.2)) comes naturally from an object in the derived category of mixed perverse sheaves on G_0 (over F_q), cf. 12.2. Hence for each $g\in G^F$ we may consider the alternating sum (see (8.4.1)):

$$\chi_{K^{\mathscr{L}}_{\dot{w}},F}(g)=\sum_i(-1)^i\,\mathrm{Tr}(F,\mathscr{H}^i_g(K^{\mathscr{L}}_{\dot{w}})), \qquad (13.3.1)$$

where F denotes the map induced by the Frobenius map. Similarly, we may consider

$$\chi_{\bar{K}^{\mathscr{L}}_{\dot{w}},F}(g)=\sum_i(-1)^i\,\mathrm{Tr}(F,\mathscr{H}^i_g(\bar{K}^{\mathscr{L}}_{\dot{w}})). \qquad (13.3.2)$$

We now prove

PROPOSITION 13.4. *With the notations in* 13.2, 13.3, *we have for any* $g \in G^F$:

$$\chi_{K_{\dot{w}}^{\mathscr{L}},F}(g) = \mathrm{Tr}(\tau_{\dot{w}}\, g, \mathscr{P}^{\theta}). \tag{13.4.1}$$

Proof. From the definition of $K_{\dot{w}}^{\mathscr{L}}$ (see (12.1.3)) we see that

$$\begin{aligned}
\chi_{K_{\dot{w}}^{\mathscr{L}},F}(g) &= \sum_i (-1)^i \,\mathrm{Tr}(F, H_c^i(\pi_w^{-1}(g), \tilde{\mathscr{L}}_w)) \\
&= \sum_{\substack{B' \in \mathscr{B}^F \\ (g,B') \in Y_w}} \mathrm{Tr}(F, \text{stalk of } \tilde{\mathscr{L}}_{\dot{w}} \text{ at } (g, B') \in Y_w) \\
&\qquad \text{(by the Lefschetz fixed point formula for the variety } \pi_w^{-1}(g)) \\
&= |B^F|^{-1} \sum_{\substack{h \in G^F \\ h^{-1}gh \in BwB}} \mathrm{Tr}(F, \text{stalk of } \hat{\mathscr{L}}_{\dot{w}} \text{ at } (g, h) \in \hat{Y}_w) \\
&= |B^F|^{-1} \sum_{\substack{h \in G^F \\ h^{-1}gh \in BwB}} \mathrm{Tr}(F, \text{stalk of } \mathscr{L} \text{ at } \mathrm{pr}_{\dot{w}}(h^{-1}gh) \in T) \\
&= |B^F|^{-1} \sum_{\substack{h \in G^F \\ h^{-1}gh \in BwB}} \psi(\lambda(\mathrm{pr}_{\dot{w}}(h^{-1}gh))^{(q-1)/n}) \qquad \text{(see 12.2)} \\
&= |B^F|^{-1} \sum_{\substack{h \in G^F \\ h^{-1}gh \in BwB}} \theta(\mathrm{pr}_{\dot{w}}(h^{-1}gh)) \qquad \text{(see 13.2.1)} \\
&= \mathrm{Tr}(\tau_{\dot{w}}\, g, \mathscr{P}^{\theta}).
\end{aligned}$$

COROLLARY 13.5. *The sum*

$$|G^F|^{-1} \sum_{g \in G^F} \chi_{K_{\dot{w}}^{\mathscr{L}}}(g)\, \chi_{K_{\dot{w}'}^{\mathscr{L}^{-1}}}(g)$$

is equal to the trace of the linear transformation of the Hecke algebra $H'_{\mathscr{L}}(q)$ (*see* 12.9) *given on the basis elements* T_z ($z \in W'_{\mathscr{L}}$) *by*

$$T_z \to T_{w'^{-1}} \cdot T_z \cdot T_w \cdot q^{(1/2)(l(w) + l(w') - \bar{l}(w) - \bar{l}(w'))}.$$

(*Here* w' *is another element in* $W'_{\mathscr{L}} = W'_{\mathscr{L}^{-1}}$, $\dot{w}'$ *is its representative in* $(W'_{\mathscr{L}})^{\cdot}$, *and* $\mathscr{L}^{-1} = (\lambda^{-1})^* (\mathscr{E}_{n,\psi})$ *is regarded as a local system* (*over* F_q) *on* T *such that the trace of Frobenius on the stalk at* $t \in T^F$ *is* $\psi(\lambda^{-1}(t)^{(q-1)/n}) = \theta^{-1}(t)$.) *In particular, our sum is zero unless* $wW_{\mathscr{L}} = w'W_{\mathscr{L}}$.

Proof. The sum in the corollary is equal (by (13.4.1) to

$$\begin{aligned}
&|G^F|^{-1} \sum_{g \in G^F} \mathrm{Tr}(\tau_{\dot{w}}\, g, \mathscr{P}^{\theta})\, \mathrm{Tr}(\tau_{\dot{w}'}\, g, \mathscr{P}^{\theta^{-1}}) \\
&\quad = \mathrm{Tr}(\tau_{\dot{w}} \otimes \tau_{\dot{w}'}, (\mathscr{P}^{\theta} \otimes \mathscr{P}^{\theta^{-1}})^{G^F}),
\end{aligned} \tag{13.5.1}$$

where $\mathscr{P}^{\theta^{-1}}$ is defined just as $\mathscr{P}^{\theta}$, replacing θ by θ^{-1}, and G^F acts diagonally on $\mathscr{P}^{\theta} \otimes \mathscr{P}^{\theta^{-1}}$. Under the natural duality $\mathscr{P}^{\theta} \otimes \mathscr{P}^{\theta^{-1}} \to \bar{\mathbb{Q}}_l$ given by $f, f' \to \sum_{gU^F \in G^F/U^F} (ff')(gU^F)$, the automorphism $\tau_{\dot{w}'}$ of $\mathscr{P}^{\theta^{-1}}$ corresponds to the contragredient of $\tau_{(\dot{w}')^{-1}}: \mathscr{P}^{\theta} \to \mathscr{P}^{\theta}$. Hence (13.5.1) is equal to the trace of the linear transformation of $\mathrm{End}_{G^F}(\mathscr{P}^{\theta})$ into itself given by $\xi \to \tau_{(\dot{w}')^{-1}} \xi \tau_{\dot{w}}$. Using the isomorphism (13.2.4), we see that it is enough to prove the identity

$$\tau_{(\dot{w}')^{-1}} \zeta(T_z)\, \tau_{\dot{w}} = \zeta(T_{w'^{-1}} T_z T_w) \cdot q^{(1/2)(l(w) + l(w') - \bar{l}(w) - \bar{l}(w'))} \cdot \alpha \qquad (z \in W'_{\mathscr{L}}), \tag{13.5.2}$$

where $\alpha \in \bar{\mathbb{Q}}_l^*$ satisfies $\alpha = 1$ whenever $wW_{\mathscr{L}} = w'W_{\mathscr{L}}$.

Using the definition (13.2.3) and the fact that ζ is an algebra homomorphism, we see that (13.5.2) is equivalent to the formula

$$h(\tau_{(\dot{x})^{-1}}, \tau_{\dot{x}}) = q^{l(x)} \qquad (x \in \Omega_{\mathscr{L}})$$

and this follows by applying h to (13.2.5) with $\dot{x}' = (\dot{x})^{-1}$.

PROPOSITION 13.6. *In the setup of* 13.4, *we have for any* $g \in G^F$:

$$\chi_{\bar{K}^{\mathscr{L}}_{\dot{w}}}(g) = \sum_{u \in wW_{\mathscr{L}}} P_{x^{-1}u, x^{-1}w}(q) \cdot q^{(1/2)(l(w) - l(u) - \bar{l}(w) + \bar{l}(u))} \chi_{K^{\mathscr{L}}_{\dot{u}}}(g),$$

where $P_{\alpha,\beta}$ denote the polynomials [12] *for the Coxeter group $W_{\mathscr{L}}$, x is the unique element in $\Omega_{\mathscr{L}} \cap wW_{\mathscr{L}}$, and the representatives $\dot{u}$ are taken in $(W'_{\mathscr{L}})^{\boldsymbol{\cdot}}$ (see* (13.2.2)).

Proof. There is a natural spectral sequence

$$E_2^{a,b} = \mathscr{H}^a({}^pH^b \bar{K}^{\mathscr{L}}_{\dot{w}}) \Rightarrow \mathscr{H}^{a+b}(\bar{K}^{\mathscr{L}}_{\dot{w}})$$

in the category of mixed constructible sheaves on G_0. Taking stalks at g, we get a spectral sequence $E_2^{a,b} = \mathscr{H}^a_g({}^pH^b \bar{K}^{\mathscr{L}}_{\dot{w}}) \Rightarrow \mathscr{H}^{a+b}_g(\bar{K}^{\mathscr{L}}_{\dot{w}})$. Taking alternating sums of traces of the Frobenius map, we get

$$\chi_{\bar{K}^{\mathscr{L}}_{\dot{w}}, F}(g) = \sum_{a,b} (-1)^{a+b} \operatorname{Tr}(F, \mathscr{H}^a_g({}^pH^b \bar{K}^{\mathscr{L}}_{\dot{w}})). \tag{13.6.1}$$

Similarly, we have

$$\chi_{K^{\mathscr{L}}_{\dot{w}}, F}(g) = \sum_{a,b} (-1)^{a+b} \operatorname{Tr}(F, \mathscr{H}^a_g({}^pH^b K^{\mathscr{L}}_{\dot{w}})). \tag{13.6.2}$$

The desired formula follows now from (13.6.1), (13.6.2), and 12.6.

COROLLARY 13.7. *In the setup of* 13.5, *the sum*

$$|G^F|^{-1} \sum_{g \in G^F} \chi_{K^{\mathscr{L}}_{\dot{w},F}}(g)\, \chi_{K^{\mathscr{L}^{-1}}_{\dot{w}',F}}(g)$$

is equal to the trace of the linear transformation of the Hecke algebra $H'_{\mathscr{L}}(q)$ (*see* 12.9) *into itself given on the basis elements* T_z $(z \in W'_{\mathscr{L}})$ *by*

$$T_z \to q^{(1/2)(l(w)+l(w'))} C'_{w'^{-1}} T_z C'_{w}.$$

(*Here* C'_{w}, $C'_{w'^{-1}}$ *are as in* (12.9.2).)

Proof. This follows immediately from 13.5 and 13.6.

13.8. We recall that $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\psi})$ is fixed.

In the rest of this chapter, we shall assume not only that n divides $q-1$, but also that every complex $A \in \hat{G}_{\mathscr{L}}$ is isomorphic to F^*A, where F is the Frobenius map corresponding to the F_q-structure on G. If n only divides $q-1$, then each ${}^pH^i(\bar{K}^{\mathscr{L}}_{\dot{w}})$ $(i \in \mathbb{Z},\ \dot{w} \in (W_{\mathscr{L}})^{\cdot})$ is defined over F_q (see 12.2), hence F^* defines a permutation of the set $\hat{G}_{\mathscr{L}}$. Replacing q by a power, if necessary, we may therefore assume that F^* acts trivially on the set $\hat{G}_{\mathscr{L}}$.) (13.8.0)

In the rest of this chapter we shall assume chosen a specific isomorphism $\phi_A: F^*A \xrightarrow{\sim} A$, for any $A \in \hat{G}_{\mathscr{L}}$, with the following property: for any $r \geqslant 1$ and any $g \in Y^{F^r}_{L,\Sigma}$ (supp $A = \bar{Y}_{L,\Sigma}$ of dimension d), $\phi^r_A: \mathscr{H}^{-d}_g A \to \mathscr{H}^{-d}_g A$ has all eigenvalues of the form root of 1 times $q^{(\dim G - d)r/2}$. (13.8.1)

(such ϕ_A exists, since $A \mid Y_{L,\Sigma}$ is a local system with finite monodromy.) Since ${}^pH^i(\bar{K}^{\mathscr{L}}_{\dot{w}})$ is semisimple, we may write canonically

$$ {}^pH^i(\bar{K}^{\mathscr{L}}_{\dot{w}}) = \bigoplus_A (A \otimes V_{A,i,\dot{w}}) \tag{13.8.2}$$

(A runs through $\hat{G}_{\mathscr{L}}$), where $V_{A,i,\dot{w}}$ are finite-dimensional vector spaces over $\bar{\mathbb{Q}}_l$ endowed with endomorphisms $\psi_A: V_{A,i,\dot{w}} \to V_{A,i,\dot{w}}$ such that under (13.8.2) the map $\phi_A \otimes \psi_A$ corresponds to the isomorphism $F^*({}^pH^i(\bar{K}^{\mathscr{L}}_{\dot{w}})) \xrightarrow{\sim} {}^pH^i(\bar{K}^{\mathscr{L}}_{\dot{w}})$ arising from the fact that ${}^pH^i(\bar{K}^{\mathscr{L}}_{\dot{w}})$ comes from a mixed perverse sheaf on G_0. Passing to stalks, it follows that

$$\mathrm{Tr}(F, \mathscr{H}^j_g\, {}^pH^i(\bar{K}^{\mathscr{L}}_{\dot{w}})) = \sum_A \mathrm{Tr}(\phi_A, \mathscr{H}^j_g A)\, \mathrm{Tr}(\psi_A, V_{A,i,\dot{w}})$$

for all $g \in G^F$, all i, j, and all $w \in W'_{\mathscr{L}}$. Taking alternating sums over i and j, and using (13.6.1), we find

$$\chi_{\bar{K}^{\mathscr{L}}_{\dot{w}},F}(g) = \sum_A (\chi_{A,\phi_A}(g) \sum_i (-1)^i \operatorname{Tr}(\psi_A, V_{A,i,\dot{w}})) \tag{13.8.3}$$

(A runs through $\hat{G}_{\mathscr{L}}$); see (8.4.1).

From (12.2.2) and [1, 5.4.4], it follows that ${}^pH^i\bar{K}^{\mathscr{L}}_{\dot{w}}$ is pure of weight i. From [1, 5.3.4] and by the choice of ϕ_A, we see that (A, ϕ_A) is pure of weight $\dim G$. From (13.8.2) we can now deduce that the endomorphism ψ_A of $V_{A,i,\dot{w}}$ is pure of weight $i - \dim G$. In other words:

The eigenvalues of $\psi_A \colon V_{A,i,\dot{w}} \to V_{A,i,\dot{w}}$ are algebraic numbers all of whose complex conjugates have absolute value $q^{(i-\dim G)/2}$. (13.8.4)

13.9. We now replace $\mathscr{L}$ by $\mathscr{L}^{-1}$ in 13.8. We have $W'_{\mathscr{L}} = W'_{\mathscr{L}^{-1}}$, and for each $w' \in W'_{\mathscr{L}}$, ${}^pH^i(K^{\mathscr{L}^{-1}}_{\dot{w}'})$ is defined over F_q, where $\dot{w}' \in (W'_{\mathscr{L}})^{\cdot}$. (The F_q-rational structure on $\mathscr{L}^{-1}$ is taken as in 13.5.) The set $\hat{G}_{\mathscr{L}^{-1}}$ consists of $\{DA \mid A \in \hat{G}_{\mathscr{L}}\}$ (see (11.2.2)) and for $A \in \hat{G}_{\mathscr{L}}$, we define $\phi'_{DA} \colon F^*DA \simeq DA$ to be the contragredient of $\phi_A \colon F^*A \simeq A$, times $q^{\dim G - d}$ (where d is as in (13.8.1)). We define $V'_{DA,i,\dot{w}'}$ and its endomorphism ψ'_{DA} as in (13.8.2):

$${}^pH^i(\bar{K}^{\mathscr{L}^{-1}}_{\dot{w}'}) = \bigoplus_A (DA \otimes V'_{DA,i,\dot{w}'});$$

ψ'_{DA} is defined in terms of ϕ'_{DA}, just as ψ_A is defined in terms of ϕ_A. We then have for all $g \in G^F$

$$\chi_{\bar{K}^{\mathscr{L}^{-1}}_{\dot{w}'},F}(g) = \sum_A (\chi_{DA,\phi'_{DA}}(g) \sum_i (-1)^i \operatorname{Tr}(\psi'_{DA}, V'_{DA,i,\dot{w}'})).$$

Multiplying this with (13.8.3) and summing over all $g \in G^F$, we find

$$\begin{aligned} &|G^F|^{-1} \sum_{g \in G^F} \chi_{\bar{K}^{\mathscr{L}}_{\dot{w}},F}(g)\, \chi_{\bar{K}^{\mathscr{L}^{-1}}_{\dot{w}'},F}(g) \\ &= \sum_{A,A'} \Bigg(|G^F|^{-1} \sum_{g \in G^F} (\chi_{A,\phi_A}(g)\, \chi_{DA',\phi'_{DA'}}(g)) \\ &\quad \times \sum_{i,j} (-1)^{i+j} \operatorname{Tr}(\psi_A, V_{A,i,\dot{w}}) \operatorname{Tr}(\psi'_{DA'}, V'_{DA',j,\dot{w}'}) \end{aligned} \tag{13.9.1}$$

(here A, A' run over $\hat{G}_{\mathscr{L}}$).

We shall say that G is *clean* if for any Levi subgroup L of a parabolic subgroup in G, any cuspidal character sheaf of L is clean (see (7.7). (13.9.2)

If we assume that G is clean, then the sum $|G^F|^{-1}$ $\sum_{g\in G^F}\chi_{A,\phi_A}(g)\,\chi_{DA',\phi_{DA'}}(g)$ in (13.9.1) is equal, by 10.8 and the choice of ϕ_A, $\phi'_{DA'}$, to 1 if $A=A'$ and to 0 if $A\neq A'$. Hence, for G clean, the right-hand side of (13.9.1) is equal to

$$\sum_A \sum_{i,j} (-1)^{i+j}\operatorname{Tr}(\psi_A, V_{A,i,\dot w})\operatorname{Tr}(\psi'_{DA}, V'_{DA,j,\dot w'})$$

(A runs over $\hat G_{\mathscr L}$). The resulting identity clearly remains true if F_q is replaced by an extension F_{q^s}. Thus, we have the identity

$$|G^{F^s}|^{-1}\sum_{g\in G^{F^s}}\chi_{\bar K_{\dot w}^{\mathscr L},F^s}(g)\,\chi_{\bar K_{\dot w'}^{\mathscr L^{-1}},F^s}(g)=\sum_h (-1)^h\rho_{h,s},\tag{13.9.3}$$

where

$$\rho_{h,s}=\sum_A \sum_{\substack{i,j\\ i+j=h}}\operatorname{Tr}(\psi_A^s, V_{A,i,\dot w})\operatorname{Tr}((\psi'_{DA})^s, V'_{DA,j,\dot w'})\tag{13.9.4}$$

(A runs over $\hat G_{\mathscr L}$). According to 13.7, the left-hand side of (13.9.3) is equal to $\Pi(q^s)$, where Π is a polynomial with integral coefficients depending on $\mathscr L$, w, w', but not on q or s. On the other hand, from (13.8.4) and the analogous property $\psi'_{DA}\colon V'_{DA,j,\dot w'}\to V'_{DA,j,\dot w'}$, it follows that $\rho_{h,s}$ is of the form $\sum_r(\alpha_{h,r})^s$ where $\alpha_{h,r}$ are algebraic numbers all of whose complex conjugates have absolute value $q^{(h/2)-\dim G}$. From the identity

$$\sum_h\Big((-1)^h\sum_r(\alpha_{h,r})^s\Big)=\Pi(q^s)$$

(valid for all $s\geqslant 1$) it follows then that, for fixed h, the set $\{\alpha_{h,r}\}$ is empty if h is odd and that $\alpha_{h,r}=q^{(h/2)-\dim G}$ if h is even. This implies that $V_{A,i,\dot w}\otimes V_{DA,j,\dot w'}$ is zero for $i+j$ odd ($A\in\hat G_{\mathscr L}$). It also implies that, for $i+j$ even, any eigenvalue of ψ_A on $V_{A,i,\dot w}$ multiplied by any eigenvalue of ψ'_{DA} on $V'_{DA,j,\dot w'}$ gives $q^{(i+j)/2-\dim G}$.

Since, for $A\in\hat G_{\mathscr L}$, we have $V'_{DA,j,\dot w'}\neq 0$ for some j and $w'\in W'_{\mathscr L}$, it follows that the parity of i such that $V_{A,i,\dot w}\neq 0$ for some $w\in W'$ is an invariant of A and that, for an eigenvalue ξ of ψ_A on $V_{A,i,\dot w}$, the product $\xi q^{-i/2+(\dim G)/2}$ is also an invariant of A. Thus, we have the following result, which is analogous to [6, 2.18].

THEOREM 13.10. *In the setup of* 13.8, *we assume that* G *is clean* (*see* (13.9.2)).

(a) *With each* $A\in\hat G_{\mathscr L}$, *one can associate a sign* $\varepsilon_A\in\{\pm1\}$ *with the following property: if* A *is a component of* ${}^pH^i(\bar K_{\dot w}^{\mathscr L})$ ($w\in W'_{\mathscr L}$), *then* $(-1)^{i+\dim G}=\varepsilon_A$.

(b) *With each* $A \in \hat{G}_{\mathscr{L}}$, *one can associate an element* $\xi_A \in \bar{\mathbb{Q}}_l^*$ (*depending on the choice of lifting* $(W'_{\mathscr{L}})^{\cdot}$ *in* (13.2.2), *the choice of* ϕ_A *in* (13.8.1), *and the choice of a square root of* q *in* $\bar{\mathbb{Q}}_l$) *such that for any* i *and any* $w \in W'_{\mathscr{L}}$, *all eigenvalues of* $\phi_A\colon V_{A,i,\dot{w}} \to V_{A,i,\dot{w}}$ (*see* (13.8.2)) *are equal to* $\xi_A q^{(i-\dim G)/2}$; *moreover,* ξ_A *is an algebraic number all of whose complex conjugates have absolute value* 1. *The eigenvalues of* $\psi'_{DA}\colon V'_{DA,i,\dot{w}'} \to V'_{DA,i,\dot{w}'}$ (*see* 3.9) *are equal to* $\xi_A^{-1} q^{(i-\dim G)/2}$.

13.11. Consider the identity (13.9.3). Its left-hand side is equal to the trace of the linear map $H'_{\mathscr{L}}(q^s) \to H'_{\mathscr{L}}(q^s)$ given by $T_z \to q^{(s/2)(l(w)+l(w'))} C'_{w'^{-1}} T_z C'_w$. Hence it is equal to $q^{(s/2)(l(w)+l(w'))} \sum_{E \in \hat{W}'_{\mathscr{L}}} \operatorname{Tr}(C'_{w'^{-1}}, E(q^s)) \operatorname{Tr}(C'_w, E(q^s))$ and hence to

$$q^{(s/2)(l(w)+l(w'))} \sum_{E \in \hat{W}'_{\mathscr{L}}} \operatorname{Tr}(C'_{w'}, E^*(q^s)) \operatorname{Tr}(C'_w, E(q^s)), \qquad (13.11.1)$$

where E^* is the representation of $W'_{\mathscr{L}}$ dual to E.

The right-hand side of (13.9.3) can be expressed as in (13.9.4), and under the assumptions of 13.10 can be written as

$$\begin{aligned}
&\sum_{\substack{A\\ i,j}} (-1)^{i+j} \xi_A^s q^{s(i-\dim G)/2} \xi_A^{-s} q^{s(j-\dim G)/2} \\
&\qquad \times \dim V_{A,i,\dot{w}} \dim V'_{DA,j,\dot{w}'} \\
&= \sum_A \left(\sum_i (-1)^i q^{s(i-\dim G)/2} (A\colon {}^pH^i \bar{K}_w^{\mathscr{L}}) \right) \\
&\qquad \times \left(\sum_j (-1)^j q^{s(j-\dim G)/2} (DA\colon {}^pH^i K_{w'}^{\mathscr{L}^{-1}}) \right) \\
&= \sum_A \left(q^{sl(w)/2} \sum_{E \in \hat{W}'_{\mathscr{L}}} c_{A,E}(q^s) \operatorname{Tr}(C'_w, E(q^s)) \right) \\
&\qquad \times \left(q^{sl(w')/2} \sum_{E' \in \hat{W}'_{\mathscr{L}}} c_{DA,E'}(q^s) \operatorname{Tr}(C'_{w'}, E'(q^s)) \right)
\end{aligned}$$

(by (12.10.1), with $u^{1/2} = q^{s/2}$). This equals

$$\begin{aligned}
&q^{(s/2)(l(w)+l(w'))} \sum_{E,E' \in \hat{W}'_{\mathscr{L}}} \left(\sum_{A \in \hat{G}_{\mathscr{L}}} c_{A,E}(q^s)\, c_{DA,E'}(q^s) \right) \\
&\qquad \times \operatorname{Tr}(C'_w, E(q^s)) \operatorname{Tr}(C'_{w'}, E'(q^s)). \qquad (13.11.2)
\end{aligned}$$

Thus (13.9.3) can be expressed as the equality of (13.11.1) and (13.11.2), for all $w, w' \in W'_{\mathscr{L}}$. Since the functions $f_{E,E'}\colon W'_{\mathscr{L}} \times W'_{\mathscr{L}} \to \bar{\mathbb{Q}}_l$,

$$(w, w') \mapsto \operatorname{Tr}(C'_w, E(q^s)) \operatorname{Tr}(C'_{w'}, E'(q^s))$$

$(E, E' \in \hat{W}_{\mathscr{L}})$ are linearly independent, it follows that

$$\sum_{A \in \hat{G}_{\mathscr{L}}} c_{A,E}(q^s)\, c_{DA,E'}(q^s) = \begin{cases} 1, & \text{if } E' = E^* \\ 0, & \text{otherwise.} \end{cases}$$

Since here s is any integer $\geqslant 1$, this identity remains true when q^s is replaced by the indeterminate u.

Using now (12.11.1) we get the following result:

Proposition 13.12. *Under the assumptions of* 13.10, *we have, for any* $E, E' \in \hat{W}'_{\mathscr{L}}$,

$$\sum_{A \in \hat{G}_{\mathscr{L}}} c_{A,E}(u)\, c_{A,E'}(u) = \begin{cases} 1, & \text{if } E' = E^* \\ 0, & \text{otherwise.} \end{cases}$$

14. A Disjointness Theorem for Cohomology Sheaves and Its Applications

14.1. In 3.11, we have defined a canonical partition of G into finitely many locally closed smooth irreducible subvarieties $Y_{(L,\Sigma)}$ stable by conjugation. We shall prove the following result.

Theorem 14.2*. *Let* $Y = Y_{(L,\Sigma)}$ *be a piece in the partition* 3.11 *of* G *and let* A' *be any admissible complex on* G *(see* (7.1.10)*).*

(a) *The cohomology sheaves* $\mathscr{H}^i A'$ *restricted to* Y *are local systems with finite monodromy.*

(b) *Assume that* G *is clean (see* (13.9.2)*), that* A *is a character sheaf of* G *whose support is equal to* $\bar{Y}$, *and that* A' *is a character sheaf of* G *which is not isomorphic to* A. *Let* $d = \dim Y$, *so that* $\mathscr{H}^{-d}(A)$ *is the only cohomology sheaf of* A *which is non-zero on* Y. *(It is an irreducible local system on* Y.*) Then for any* i, *the local system* $\mathscr{H}^i(A') \mid Y$ *has no irreducible direct summands isomorphic to* $\mathscr{H}^{-d}(A) \mid Y$.

Proof. We refer to 8.1 for the notations L, Σ, Σ_{reg}, $Y = Y_{(L,\Sigma)}$. Let Z be the variety of all pairs $(M, \bar{s})$ where M is a closed subgroup of G and $\bar{s}$ is a semisimple element of $M/\mathscr{Z}^0_M$ such that there is an element of G conjugating M to L and $\bar{s}$ to the semisimple part of some element in the image of Σ in $L/\mathscr{Z}_L^0$.

Let ρ: $Y \to Z$ be defined by $\rho(g) = (M, \bar{s})$, where M is the unique conjugate of L containing $Z^0(g_s)$ and $\bar{s}$ is the image of g_s in $M/\mathscr{Z}_M^0$. Then ρ is a locally trivial fibration. Moreover, G acts naturally on Y and Z by conjugation (compatibly with ρ) and the action on Z is transitive. Since $\mathscr{H}^i(A') \mid Y$ is G-equivariant, we see that in order to prove (a) it is enough to show that the restriction of $\mathscr{H}^i(A')$ to some fibre of ρ is a local system with finite monodromy. We shall consider the fibre Φ of ρ at $(L, \bar{s})$, where $su = us$ is an element of Σ_{reg} (s semisimple, u unipotent) and $\bar{s}$ is the image of s in $L/\mathscr{Z}_L^0$.

Note that, if $Z_G^0(zs) \subset L$ ($z \in \mathscr{Z}_L^0$), then $Z_G^0(zs) = Z_G^0(s)$. (Both are equal to $Z_L^0(s)$.) It follows that Φ is the set of all $g \in L$ which are of the form $g = zsv$, where $z \in \mathscr{Z}_L^0$ is such that $Z_G^0(zs) \subset L$, v is unipotent in $Z_G^0(s)$, and zsv is conjugate to an element in Σ_{reg} (s is fixed).

We now define $\hat{\Phi}$ to be the set of all $g \in L$ which are of the form $g = zsv$, where $z \in \mathscr{Z}_L^0$, v is unipotent in $Z_G^0(s)$, and zsv is conjugate to an element in Σ (s is fixed).

It is clear that Φ is an open dense subset of $\hat{\Phi}$ and that $s^{-1}\hat{\Phi} \subset Z_G^0(s)$.

Let Γ be the isotropy group of $(L, \bar{s})$ for the transitive action of G on Z by conjugation. Note that $Z_G^0(s)$ is contained in Γ as a subgroup of finite index. The action of $\mathscr{Z}_L^0 \times \Gamma$ on $\hat{\Phi}$ given by (z, γ): $g \to z\gamma g\gamma^{-1}$ is clearly transitive. Hence $\hat{\Phi}$ (or $s^{-1}\hat{\Phi}$) is a union of finitely many $\mathscr{Z}_L^0 \times Z_G^0(s)$-orbits.

Now let K' be a perverse sheaf on G obtained by inducing a cuspidal admissible complex from a Levi subgroup of a parabolic subgroup, and such that A' is a direct summand of K' (see (7.1.10)). Let ε: $Z_G^0(s) \to sZ_G^0(s)$ be multiplication by s. In (8.8.4) we have constructed an isomorphism

$$(\varepsilon^* K' \mid \mathscr{U})[-\delta] \approx \left(\bigoplus_{\alpha} K'_\alpha \right) \Big| \mathscr{U} \qquad (\text{in } \mathscr{D}\mathscr{U}), \tag{14.2.1}$$

where $\mathscr{U}$ is an open subset of $Z_G^0(s)$ as in 8.6 and K'_α are finitely many perverse sheaves on $Z_G^0(s)$ of the same type as K'. More precisely:

> K'_α is obtained by inducing an irreducible cuspidal perverse sheaf $K'_{\alpha,0}$ from a Levi subgroup $(= L'_\alpha \cap Z_G^0(s))$ of a parabolic subgroup of $Z_G^0(s)$, where L'_α is a Levi subgroup (containing s) of some parabolic subgroup of G such that $\operatorname{supp} K' = \bar{Y}_{(L'_\alpha, \Sigma'_\alpha)}$. (14.2.2)
>
> The support of K'_α is the closure of the piece in the partition 3.11 of $Z_G^0(s)$ corresponding to the pair $(L'_\alpha \cap Z_G^0(s), \{v \in Z_G^0(s) \text{ unipotent} \mid sv \in \Sigma'_\alpha\} \cdot \mathscr{Z}^0_{L'_\alpha \cap Z^0(s)}$. (14.2.3)

Now A' is a direct summand of K', hence $(\varepsilon^*A' \mid \mathscr{U})[-\delta]$ is a direct summand of $(\bigoplus_\alpha K'_\alpha) \mid \mathscr{U}$ (in $\mathscr{D}\mathscr{U}$, hence in $\mathscr{M}\mathscr{U}$). Since $\mathscr{U}$ is open in $Z^0(s)$, and $\bigoplus_\alpha K'_\alpha$ is a semisimple perverse sheaf, there exists a direct summand of $\bigoplus_\alpha K'_\alpha$ whose restriction to $\mathscr{U}$ is isomorphic to $(\varepsilon^*A' \mid \mathscr{U})[-\delta]$. (We use the fact that any irreducible perverse sheaf on $Z^0(s)$ gives upon restriction to $\mathscr{U}$ either 0 or an irreducible perverse sheaf on $\mathscr{U}$.) Now, any direct summand of $\bigoplus_\alpha K'_\alpha$ is a direct sum of irreducible admissible complexes of $Z^0(s)$. Thus, $(\varepsilon^*A' \mid \mathscr{U})[-\delta]$ is isomorphic to a direct sum of admissible complexes on $Z^0(s)$, restricted to $\mathscr{U}$. Since any admissible complex A_1 on $Z^0(s)$ is equivariant for the action (z, g_0): $g \to z^n g_0 g g_0^{-1}$ of $\mathscr{Z}_L^0 \times Z_G^0(s)$ on $Z_G^0(s)$ (for some $n \geqslant 1$) and since $s^{-1}\hat{\Phi}$ is a union of finitely many orbits for this action, it follows that $\mathscr{H}^i(A_1) \mid s^{-1}\hat{\Phi}$ must be a local system with finite monodromy, hence $\mathscr{H}^i(A_1) \mid s^{-1}\hat{\Phi} \cap \mathscr{U}$ is a local system with finite monodromy (for all i). Since $(\varepsilon^*A' \mid \mathscr{U})[-\delta]$ is isomorphic to a direct sum of such $A_1 \mid \mathscr{U}$, it follows that $\mathscr{H}^i(\varepsilon^*A') \mid s^{-1}\hat{\Phi} \cap \mathscr{U}$ and $\mathscr{H}^i(A') \mid \hat{\Phi} \cap s\mathscr{U}$ are local systems with finite monodromy. In particular $\mathscr{H}^i(A') \mid \Phi \cap s\mathscr{U}$ is a local system with finite monodromy.

The set $\mathscr{U}$ considered above depends on s; we now denote it $\mathscr{U}(s)$. When s runs over the set of all $\sigma \in L$ such that σ is mapped to $\bar{s}$ under $L \to L/\mathscr{Z}_L^0$, and such that $Z_G^0(\sigma) \subset L$, the sets $\Phi \cap \mathscr{U}(s)$ form an open covering of Φ. Since $\mathscr{H}^i(A')$ is a local system with finite monodromy when restricted to any of the open sets of this covering, it follows that $\mathscr{H}^i(A') \mid \Phi$ is a local system with finite monodromy. This completes the proof of (a).

We now begin the proof of (b). We may assume that supp A' contains $\bar{Y}$ as a proper subset; otherwise, there is nothing to prove. Let K be a perverse sheaf on G obtained by inducing a cuspidal admissible complex of L to G, such that A is a direct summand of K.

Let $su \in \Sigma_{\text{reg}}$be as in the proof of (a) and let Φ_1 be the set of all $g \in L$ which are of the form $g = zsv$ where $z \in \mathscr{Z}_L^0$ is such that $Z_G^0(zs) \subset L$ and $v \in C$ ($C = Z^0(s)$-cnjugacy class of u). Then Φ_1 is a connected component of Φ above. If ε and $\mathscr{U} \subset Z^0(s)$ are as above, we may assume that, besides (14.2.1), there is an isomorphism

$$(\varepsilon^*K \mid \mathscr{U})[-\delta] \approx \left(\bigoplus_\beta K_\beta \right) \Big| \mathscr{U} \qquad (\text{in } \mathscr{D}\mathscr{U}), \tag{14.2.4}$$

where K_β are finitely many perverse sheaves on $Z^0(s)$ satisfying properties similar to those satisfied by K'_α in (14.2.2), (14.2.3). In our case, however, each K_β must necessarily be a cuspidal perverse sheaf, with support equal to $\bar{V}$, where $V = \mathscr{Z}^0_{Z^0(s)} \cdot C$.

Using our assumption that G is clean, together with 7.11 and 11.2(d), we see that each $K'_{\alpha,0}$ (as well as $DK'_{\alpha,0}$) is a strongly cuspidal, clean, irreducible perverse sheaf on $Z_G^0(s)$. Similarly, each K_β (as well as DK_β) is a strongly cuspidal, clean, irreducible perverse sheaf on $Z_G^0(s)$.

It is enough to show that the local systems $\mathscr{H}^i(K') \mid \Phi_1 \cap s\mathscr{U}$, $\mathscr{H}^j(K) \mid \Phi_1 \cap s\mathscr{U}$ have no common irreducible direct summand, for any i, j. Using (14.2.1) and (14.2.3), we see that it is enough to show that the local systems $\mathscr{H}^i K'_\alpha \mid s^{-1}\Phi_1 \cap \mathscr{U}$, $\mathscr{H}^j K_\beta \mid s^{-1}\Phi_1 \cap \mathscr{U}$ have no common irreducible direct summand, for all i, j, α, β. Since $\mathscr{H}^i(K'_\alpha)$ is $\mathscr{Z}^0_{Z^0(s)} \times Z^0(s)$-equivariant, it must be a local system on V (which is a single orbit). Similarly, $\mathscr{H}^j(K_\beta)$ is a local system on V. Since $s^{-1}\Phi_1 \cap \mathscr{U}$ is open dense in V and V is irreducible, it is enough to show that the local systems $\mathscr{H}^i K'_\alpha \mid V$, $\mathscr{H}^j K_\beta \mid V$ have no common irreducible direct summand. Since $\operatorname{supp} K_\beta = \bar{V}$, we see that it is enough to check the statement (14.2.5) and Lemma 14.3 below:

$$\text{For any } \alpha, \beta, \text{ we have } \operatorname{supp} K'_\alpha \neq \operatorname{supp} K_\beta. \tag{14.2.5}$$

LEMMA 14.3*. *Let K be an irreducible cuspidal perverse sheaf* (7.1.10) *on G such that DK is strongly cuspidal and clean. Let K' be a perverse sheaf on G obtained by inducing to G a strongly cuspidal, clean, irreducible perverse sheaf of a Levi subgroup of a parabolic subgroup. Let $Y_{(G,\Sigma)} = \Sigma$ be the piece in the partition* 3.11 *of G such that* $\operatorname{supp} K = \bar{\Sigma}$, *and let $\mathscr{E}$ be the irreducible local system $\mathscr{H}^{-d}(K) \mid \Sigma$, $d = \dim \Sigma$. Assume that* $\operatorname{supp} K' \neq \bar{\Sigma}$. *Then the local system $(\mathscr{H}^i K') \mid \Sigma$ (see part* (a) *of Theorem* 14.2) *does not contain $\mathscr{E}$ as a direct summand.*

14.4. *Proof of* (14.2.5). Assume that $\operatorname{supp} K'_\alpha = \operatorname{supp} K_\beta = \bar{V}$. From (14.2.3) it follows that $L'_\alpha \cap Z_G^0(s) = Z_G^0(s)$, and that $su \in \Sigma'_\alpha$. Thus, $Z_G^0(s) \subset L'_\alpha$; since $L = H_G(s)$ (see 3.11), we must have $L \subset L'_\alpha$. From our assumption $\bar{Y} \subset \operatorname{supp} A'$, i.e., $\bar{Y}_{(L,\Sigma)} \subset \bar{Y}_{(L'_\alpha,\Sigma'_\alpha)}$ (see (14.2.2)), it follows by applying the Steinberg map σ (see 7.3) that $\dim \sigma(\bar{Y}_{(L,\Sigma)}) \leqslant \dim \sigma(\bar{Y}_{(L'_\alpha,\Sigma'_\alpha)})$. But it is clear that $\dim \sigma(\bar{Y}_{(L,\Sigma)}) = \dim \mathscr{Z}_L^0$, $\dim \sigma(\bar{Y}_{(L'_\alpha,\Sigma'_\alpha)}) = \dim \mathscr{Z}^0_{L'_\alpha}$, hence $\dim \mathscr{Z}_L^0 \leqslant \dim \mathscr{Z}^0_{L'_\alpha}$. This, together with $L \subset L'_\alpha$, implies $L = L'_\alpha$. From $su \in \Sigma'_\alpha$, it follows that $\bar{\Sigma} \cap \Sigma'_\alpha \neq \varnothing$, hence $\Sigma = \Sigma'_\alpha$, since $L = L'_\alpha$. Thus, we have $\bar{Y}_{(L,\Sigma)} = \bar{Y}_{(L'_\alpha,\Sigma'_\alpha)}$, contradicting our assumption $\bar{Y} \neq \operatorname{supp} A'$, and (14.2.5) is proved.

14.5. *Proof of Lemma* 14.3. If K' is itself cuspidal, then it is clean by assumption, so that $(\mathscr{H}^i K') \mid \Sigma = 0$. Assume now that K' is not cuspidal. Then by 7.2, we have $H_c^j(G, DK \otimes K') = 0$ for all j. Since DK is clean, we must have $H_c^j(\Sigma, DK \otimes K') = 0$ for all j, hence $H_c^j(\Sigma, \mathscr{E}^* \otimes K') = 0$ for all j, where $\mathscr{E}^*$ is the local system on Σ, dual to $\mathscr{E}$. We must prove that the local system $\mathscr{H}^i(\mathscr{E}^* \otimes (K' \mid \Sigma))$ on Σ contains no direct summand isomorphic to $\bar{\mathbb{Q}}_l$.

Assume that $\mathscr{H}^{i_0}(\mathscr{E}^* \otimes (K' \mid \Sigma))$ contains a direct summand isomorphic to $\bar{\mathbb{Q}}_l$ and that i_0 is maximum possible with this property. We shall reach a contradiction as follows. Our assumption implies $H_c^{2d}(\Sigma, \mathscr{H}^{i_0}(\mathscr{E}^* \otimes (K' \mid \Sigma))) \neq 0$. Hence $E_2^{2d, i_0} \neq 0$ in the usual spectral sequence

$$E_2^{p,q} = H_c^p(\Sigma, \mathscr{H}^q(\mathscr{E}^* \otimes (K' \mid \Sigma))) \Rightarrow H_c^{p+q}(\Sigma, \mathscr{E}^* \otimes (K' \mid \Sigma)).$$

In the proof of 7.8, we have seen that $H_c^i(\Sigma, \mathscr{F}) = 0$ for any irreducible $\mathscr{Z}_G^0 \times G$-equivariant (see 2.18(b)) local system $\mathscr{F}$ on Σ, which has no direct summand isomorphic to $\bar{\mathbb{Q}}_l$. In particular, we may take $\mathscr{F} = \mathscr{H}^i(\mathscr{E}^* \otimes (K' \mid \Sigma))$ for $i > i_0$. It follows that $E_2^{p,q} = 0$ if $q > i_0$. It is clear that $E_2^{p,q} = 0$ if $p > 2d$, since $d = \dim \Sigma$. This implies that $E_2^{2d, i_0} = E_3^{2d, i_0} = \cdots = E_\infty^{2d, i_0}$. Since $E_2^{2d, i_0} \neq 0$, it follows that $H_c^{2d + i_0}(\Sigma, \mathscr{E}^* \otimes (K' \mid \Sigma)) \neq 0$, a contradiction.

This completes the proof of Lemma 14.3, and hence that of Theorem 14.2.

14.6. In the rest of this chapter, we assume that we are in the setup of 13.1 and, moreover, that q is large enough so that (13.8.0) is satisfied. We fix $Y = Y_{(L, \Sigma)}$ and $A \in \hat{G}_{\mathscr{L}}$ with support $\bar{Y}$, as in 14.2. From 14.2(a) it follows that there exists a principal covering $\pi: \tilde{Y} \to Y$ with finite group Γ (acting on $\tilde{Y}$ on the left), with $\tilde{Y}$ irreducible such that each of the local systems $\mathscr{H}^i A' | Y$ (for various i and various $A' \in \hat{G}_{\mathscr{L}}$) is associated to π and to a representation of Γ (denoted $[\mathscr{H}^i A']$). For a large enough integer $r \geqslant 1$, both $\tilde{Y}$ and π are defined over F_{q^r} (Y is defined over F_q, as a consequence of (13.8.0) and, in particular, is defined over F_{q^r}). Moreover we can assume that the Frobenius map $\tilde{F}: \tilde{Y} \to \tilde{Y}$ with respect to the F_{q^r}-structure is such that $\tilde{F}(\gamma y) = \gamma \tilde{F}(y)$ for all $\gamma \in \Gamma$ and all $\tilde{y} \in \tilde{Y}$.

There exists an integer $c_0 \geqslant 1$ such that for any integer $c \geqslant c_0$ and any $\gamma \in \Gamma$, we have $\tilde{F}^c \tilde{y} = \gamma \tilde{y}$ for some $\tilde{y} \in \tilde{Y}$, which depends on γ on c. (Indeed, $\gamma^{-1} \tilde{F}^c: \tilde{Y} \to \tilde{Y}$ is the Frobenius map for an $F_{q^{rc}}$-rational structure on $\tilde{Y}$, hence for large enough c it must have some fixed point, since $\tilde{Y}$ is irreducible: the number of its fixed points tends to ∞ as c tends to ∞.) We set (for $c \geqslant c_0$): $y_{c,\gamma} = \pi(\tilde{y})$, where $\tilde{y} \in \tilde{Y}$ is some point such that $\tilde{F}^c \tilde{y} = \gamma \tilde{y}$. Then for any i and any $A' \in \hat{G}_{\mathscr{L}}$, we have

$$\operatorname{Tr}(\phi_{A'}^{rc}, \mathscr{H}_{y_{c,\gamma}}^i(A')) = b_{A', rc, i} \operatorname{Tr}(\gamma, [\mathscr{H}^i(A')]) \qquad (\forall \gamma \in \Gamma, c \geqslant c_0). \tag{14.6.1}$$

Here $\phi_{A'}$ is as in (13.8.1) and $b_{A', rc, i}$ is independent of γ. If $A' = A$, then $b_{A, rc, -d}$ is a root of 1 times $q^{(\dim G - d)(rc/2)}$ ($d = \dim Y$), and we have

$$\begin{aligned} &\operatorname{Tr}((\phi'_{DA})^{rc}, \mathscr{H}_{y_{c,\gamma}}^{-d}(DA)) && (14.6.2) \\ &\quad = b_{A, rc, -d}^{-1} \cdot q^{(\dim G - d) rc} \cdot \operatorname{Tr}(\gamma^{-1}, [\mathscr{H}^{-d}(A)]) \qquad (\forall \gamma \in \Gamma, c \geqslant c_0), \end{aligned}$$

where ϕ'_{DA} is as in 13.9.

By 14.2(b), the Γ-module $[\mathscr{H}^i(A')]$ contains no irreducible components isomorphic to $[\mathscr{H}^{-d}(A)]$, if $A' \neq A$. Hence, from (14.6.1), (14.6.2), and the orthogonality relations for the characters of Γ, we deduce

$$
\begin{aligned}
|\Gamma|^{-1} & \sum_{\gamma\in\Gamma} \chi_{A',\phi_{A'}^{rc}}(y_{c,\gamma})\,\chi_{DA,(\phi'_{DA})^{rc}}(y_{c,\gamma}) \\
&= \sum_i \Big((-1)^i b_{A',rc,i} b^{-1}_{DA,rc,-d} q^{(\dim G - d)rc} \\
&\quad \times |\Gamma|^{-1} \sum_{\gamma\in\Gamma} \mathrm{Tr}(\gamma, [\mathscr{H}^i(A')])\,\mathrm{Tr}(\gamma^{-1}, [\mathscr{H}^{-d}A])\Big) \\
&= \begin{cases} 0, & \text{if } A' \neq A \\ q^{(\dim G-d)rc}, & \text{if } A' = A \end{cases} \quad (c \geqslant c_0). \qquad (14.6.3)
\end{aligned}
$$

14.7. Under the assumptions of 13.10, the identity (13.8.3) can be written as follows:

$$
\begin{aligned}
\chi_{K^{\mathscr{L}}_{\dot w},F}(g) &= \sum_A \chi_{A,\phi_A}(g)\,\xi_A \sum_i (-1)^i q^{(i-\dim G)/2}(A : {}^pH^i\bar{K}^{\mathscr{L}}_{\dot w}) \\
&= \sum_A \chi_{A,\phi_A}(g)\,\xi_A \cdot q^{l(w)/2} \sum_E c_{A,E}(q)\,\mathrm{Tr}(C'_w, E(q) \qquad (g\in G^F).
\end{aligned}
\qquad (14.7.1)
$$

Using 13.6, 13.4, and (13.2.3) we can write

$$
\chi_{K^{\mathscr{L}}_{\dot w},F}(g) = \mathrm{Tr}(q^{(1/2)(l(w)-l(w_1))}h(\tau_{\dot w_1})\,C'_w\,g, \mathscr{P}^\theta) \qquad (g\in G^F), \quad (14.7.2)
$$

where w_1 is the unique element in $\Omega_{\mathscr{L}} \cap wW_{\mathscr{L}}$ and h is as in 13.2. Since $H'_{\mathscr{L}}(q) \cong \mathrm{End}_{G^F}(\mathscr{P}^\theta)$ (see (13.2.4)), we can decompose $\mathscr{P}^\theta$ as a $H'_{\mathscr{L}}(q)\times G^F$-module as $\mathscr{P}^\theta = \bigoplus_{E\in \hat{W}'_{\mathscr{L}}}(E(q)\otimes \mathscr{P}^\theta_E)$, where $\mathscr{P}^\theta_E = \mathrm{Hom}_{H'_{\mathscr{L}}(q)}(E(q), \mathscr{P}^\theta)$ is an irreducible G^F-module. Then (14.7.2) becomes

$$
\chi_{K^{\mathscr{L}}_{\dot w},F}(g) = \sum_{E\in \hat{W}'_{\mathscr{L}}} q^{(1/2)(l(w)-l(w_1))}h(\tau_{\dot w_1})\,\mathrm{Tr}(g, \mathscr{P}^\theta_E)\,\mathrm{Tr}(C'_w, E(q)). \qquad (14.7.3)
$$

From (13.2.5), it follows that $q^{-l(w_1)}h(\tau_{\dot w_1})$ is a root of 1 of order dividing $n|\Omega_{\mathscr{L}}|$. From 11.10 and 12.7 it follows that for $A\in \hat{G}_{\mathscr{L}}$, the $W_{\mathscr{L}}$-coset of an element $w\in W'_{\mathscr{L}}$ such that $(A : {}^pH^i\bar{K}^{\mathscr{L}}_w) \neq 0$ is an invariant of w. It follows that there exists a function $v : \hat{G}_{\mathscr{L}} \to \{\text{roots of 1 of order dividing } n|\Omega_{\mathscr{L}}|\}$ with the following property: if $(A : {}^pH^i\bar{K}^{\mathscr{L}}_w) \neq 0$ $(w\in W'_{\mathscr{L}})$, then $v(A) = q^{-l(w_1)}h(\tau_{\dot w_1})$, where w_1 is the unique element of minimal length in $wW_{\mathscr{L}}$.

Comparing (14.7.1) and (14.7.3) we then have

$$\sum_{A \in \hat{G}_{\mathscr{L}}} \chi_{A,\phi_A}(g)\, \xi_A v(A) \sum_E c_{A,E}(q)\, \mathrm{Tr}(C'_w, E(q))$$
$$= \sum_E \mathrm{Tr}(g, \mathscr{P}_E^\theta)\, \mathrm{Tr}(C'_w, E(q)) \qquad (g \in G^F).$$

Since the functions $w \to \mathrm{Tr}(C'_w, E(q))$ on W ($E \in \hat{W}'_{\mathscr{L}}$) are linearly independent, the previous equality implies

$$\mathrm{Tr}(g, \mathscr{P}_E^\theta) = \sum_{A \in \hat{G}_{\mathscr{L}}} \xi_A v(A)\, c_{A,E}(q)\, \chi_{A,\phi_A}(g) \tag{14.7.4}$$

for all $g \in G^F$ and all $E \in \hat{W}'_{\mathscr{L}}$. This identity holds also if we replace q by a power q^s ($s \geqslant 1$). It implies

$$\sum_{A \in \hat{G}_{\mathscr{L}}} \xi_A^s v_s(A)\, c_{A,E}(q^s)\, \chi_{A,\phi_A^s}(g) \in \mathcal{O} \qquad (\forall g \in G^{F^s}) \tag{14.7.5}$$

where $\mathcal{O}$ is the ring of all cyclotomic integers and $v_s(A)$ is a root of 1 of order dividing $n|\Omega_{\mathscr{L}}|$ and depending possibly on s. We fix $A \in \hat{G}_{\mathscr{L}}$ with support $\bar{Y}$ as in 14.6. We take $s = rc$, $c \geqslant c_0$, and we select $y_{\gamma,c} \in Y$, for $\gamma \in \Gamma$, as in 14.6. Then $\chi_{DA,\phi'^s_{DA}}(y_{\gamma,c})$ is a root of 1 times $q^{s\delta/2}$, where $\delta = \operatorname{codim} \bar{Y}$. Multiply (14.7.5), for $g = y_{\gamma,c}$, by $\chi_{DA,\phi'^s_{DA}}(y_{\gamma,c})$ and sum over all $\gamma \in \Gamma$. We get

$$\sum_{\gamma \in \Gamma} \sum_{A' \in \hat{G}_{\mathscr{L}}} \xi_{A'}^s v_s(A')\, c_{A',E}(q^s)\, \chi_{A',\phi_{A'}^s}(y_{\gamma,c})\, \chi_{DA,\phi'^s_{DA}}(y_{\gamma,c}) \in q^{s\delta/2} \cdot \mathcal{O}.$$

Using now the identity (14.6.3), we deduce

$$|\Gamma|\, \xi_A^s v_s(A)\, c_{A,E}(q^s)\, q^{s\delta} \in q^{s\delta/2} \cdot \mathcal{O},$$

hence

$$|\Gamma|\, \xi_A^s c_{A,E}(q^s)\, q^{s\delta/2} \in \mathcal{O} \qquad (s = rc,\ c \geqslant c_0) \tag{14.7.6}$$

for all $A \in \hat{G}_{\mathscr{L}}$ and all $E \in \hat{W}'_{\mathscr{L}}$. We now prove

LEMMA 14.8. *Under the assumptions of* 13.10, *we have* $c_{A,E}(u) \in \zeta \cdot \mathbb{Q}[u^{1/2}, u^{-1/2}]$, *where* ζ *is a root of* 1 *of order dividing* $|\Omega_{\mathscr{L}}|$ ($A \in \hat{G}_{\mathscr{L}}$, $E \in \hat{W}'_{\mathscr{L}}$).

Proof. Write $c_{A,E}(u) \in \zeta \cdot \mathbb{Q}(u^{1/2})$ with ζ a root of 1, as in 12.10. Let K be a finite Galois extension of $\mathbb{Q}$ of degree a, containing ξ_A and ζ, and let $N\colon K \to \mathbb{Q}$ be the norm map. Then $N(\xi_A)$ has complex absolute value 1 since all complex conjugates of ξ_A have absolute value 1 (see 13.10). It follows that $N(\xi_A) = \pm 1$. Hence by applying N to (14.7.6) we get $|\Gamma|^a (\zeta^{-1} c_{A,E}(q^s))^a q^{as\delta/2} \in \mathcal{O}$.

Let $H(X) \in \mathbb{Q}(X)$ be defined by $H(X) = |\Gamma|^a (\zeta^{-1} c_{A,E}(X^2))^a X^{a\delta}$. Then $H(q^s) \in \mathcal{O}$ for $s = rc$, $c \geqslant c_0/2$. Since $H(q^s)$ is a rational number and an algebraic integer, it is an ordinary integer. Thus we have $H(q^s) \in \mathbb{Z}$ for infinitely many integers $s \geqslant 1$. This implies, as it is well known, that $H(X) \in \mathbb{Q}[X]$. The lemma follows.

We can now prove

THEOREM 14.9*. *Under the assumptions of* 13.10, *we have* $c_{A,E}(u) \in \zeta \cdot \mathbb{Q}$, *where* ζ *is a root of* 1 *of order dividing* $|\Omega_{\mathscr{L}}|$ $(A \in \hat{G}_{\mathscr{L}}, E \in \hat{W}'_{\mathscr{L}})$.

Proof. We fix $E \in \hat{W}'_{\mathscr{L}}$. By 14.8 we can write $c_{A,E}(u) = \zeta_A P_A$, $P_A \in \mathbb{Q}[u^{1/2}, u^{-1/2}]$, $\zeta_A^{|\Omega_{\mathscr{L}}|} = 1$ and by (12.12.1), we have $c_{A,E^*} = \zeta_A^{-1} P_A$.

We now consider the identity 13.12 for $E' = E^*$. It gives

$$\sum_{A \in \hat{G}_{\mathscr{L}}} P_A^2 = 1.$$

This forces each P_A to be a constant (i.e., independent of $u^{1/2}$). The theorem follows.

We remark that the proofs of 14.8 and 14.9 bear some similarity to proofs in the paper [10] of Digne and Michel.

14.10. From now on we shall write $c_{A,E}$ instead of $c_{A,E}(u)$. Let us now specialize the identity (12.10.3) for $u^{1/2} \to 1$. Then $\varepsilon'(T_y) = \chi_u(K_y^{\mathscr{L}})$ becomes

$$\sum_i (-1)^i \, {}^pH^i(K_y^{\mathscr{L}}) = \sum_{A \in \hat{G}_{\mathscr{L}}} \sum_{E \in \hat{W}'_{\mathscr{L}}} c_{A,E} \operatorname{Tr}(y, E) \, A.$$

Using the orthogonality relations for $W'_{\mathscr{L}}$, this can be also written as

$$c_{A,E} = |W'_{\mathscr{L}}|^{-1} \sum_{y \in W'_{\mathscr{L}}} \operatorname{Tr}(y^{-1}, E) \sum_i (-1)^i (A : {}^pH^i(K_y^{\mathscr{L}})),$$

hence

$$c_{A,E} = (-1)^{\dim G} (A : R_E^{\mathscr{L}}), \tag{14.10.1}$$

where we use the following notations:

$\mathscr{K}_0(G) =$ subgroup of the Grothendieck group $\mathscr{K}(G)$ of $\mathscr{M}(G)$ spanned by the character sheaves of G (14.10.2)

$$R_E^{\mathscr{L}} = |W'_{\mathscr{L}}|^{-1} \sum_{y \in W'_{\mathscr{L}}} \operatorname{Tr}(y^{-1}, E) \sum_i (-1)^{i + \dim G} \, {}^pH^i(K_y^{\mathscr{L}}) \in \mathscr{K}_0(G) \otimes \bar{\mathbb{Q}}_l \tag{14.10.3}$$

(for $E \in \hat{W}'_{\mathscr{L}}$ or, more generally, for E an element of $\mathscr{R}(W'_{\mathscr{L}}) \otimes \bar{\mathbb{Q}}_l$, where $\mathscr{R}(W'_{\mathscr{L}})$ is the Grothendieck group of virtual representations of $W'_{\mathscr{L}}$).

> $(:)$ is the symmetric $\bar{\mathbb{Q}}_l$-bilinear form on $\mathscr{K}_0(G) \otimes \bar{\mathbb{Q}}_l$ with values in $\bar{\mathbb{Q}}_l$ such that $(A_1 : A_2) = \delta_{A_1, A_2}$ for any two character sheaves A_1, A_2 on G. (14.10.4)

Substituting (14.10.1) into (12.10.1) we get the following

COROLLARY 14.11*. *Under the assumptions of* 13.10, *we have*

$$\sum_i (-1)^i (A : {}^pH^i\bar{K}_w^{\mathscr{L}}) u^{i/2} = \sum_{E \in \hat{W}'_{\mathscr{L}}} (-1)^{\dim G} u^{(1/2)(\dim G + l(w))} (A : R_E^{\mathscr{L}}) \operatorname{Tr}(C'_w, E(u)) \tag{14.11.1}$$

for all $w \in W'_{\mathscr{L}}$, $A \in \hat{G}_{\mathscr{L}}$. (*Identity in* $\bar{\mathbb{Q}}_l[u^{1/2}, u^{-1/2}]$.)

(Compare with [6, 3.8].)

COROLLARY 14.12*. *Under the assumptions of* 13.10, *for any* $A \in \hat{G}_{\mathscr{L}}$, *there exists* $E \in \hat{W}'_{\mathscr{L}}$ *such that* $(A : R_E^{\mathscr{L}}) \neq 0$.

Proof. There exist $w \in W'_{\mathscr{L}}$ and i such that $(A : {}^pH^i(\bar{K}_w^{\mathscr{L}})) \neq 0$. From (14.11.1) it follows that $(A : R_E^{\mathscr{L}}) \neq 0$ for some $E \in \hat{W}'_{\mathscr{L}}$, as desired.

COROLLARY 14.13*. *Under the assumptions of* 13.10, *we have for any* E, $E' \in \hat{W}'_{\mathscr{L}}$:

$$(R_E^{\mathscr{L}} : R_{E'}^{\mathscr{L}}) = \begin{cases} 1, & \text{if } E' = E^* \\ 0, & \text{otherwise.} \end{cases}$$

Proof. In view of (14.10.1) and (14.10.4), this is just a reformulation of 13.12. (Compare with [6, 3.9].)

COROLLARY 14.14. *Under the assumptions of* 13.10, *the identity* (14.7.4) *can be rewritten as*

$$\operatorname{Tr}(g, \mathscr{P}_E^{\theta}) = (-1)^{\dim G} \sum_{A \in \hat{G}_{\mathscr{L}}} \xi_A v(A)(A : R_E^{\mathscr{L}}) \chi_{A, \phi_A}(g) \qquad (g \in G^F).$$

COROLLARY 14.15. *With the assumptions of* 13.10, *let* s *be a simple reflection in* W *such that* $s \notin W_{\mathscr{L}}$. *Let* $\mathscr{L}' = s^*\mathscr{L}$. *Then for any* $w \in W'_{\mathscr{L}}$ *we have* $sws \in W'_{\mathscr{L}'}$ *and*

$${}^pH^i\bar{K}_w^{\mathscr{L}} = pH^{i + l(sws) - l(w)}\bar{K}_{sws}^{\mathscr{L}'}.$$

Proof. We write the identity (14.11.1) for $(\mathscr{L}, w)$ and for $(\mathscr{L}', sws)$. The right-hand sides of these two identities are related to each other using 11.2(b). Hence we get a relation between the left-hand sides, which is just the desired equality. (Compare with [6, 6.5(i)].)

15. Induction, Restriction, and Duality

15.1. In this chapter G is assumed to be clean (13.9.2). Consider the functors ind, res defined in 3.8, 4.1, respectively, with respect to G, P, where P is a parabolic subgroup of G. We shall denote them ind_P^G, res_P^G. Let Q be another parabolic subgroup of G. Let L, M be Levi subgroups for P, Q, respectively. Let Γ be a set of representatives x for the double cosets $Q\backslash G/P$ such that Q, xPx^{-1} contain a common maximal torus. For $x \in \Gamma$, $M \cap xPx^{-1}$ is a parabolic subgroup of M with Levi subgroup $M \cap xLx^{-1}$; similarly, $x^{-1}Qx \cap L$ is a parabolic subgroup of L with Levi subgroup $x^{-1}Mx \cap L$.

Proposition 15.2. *Let A be a character sheaf of G. Then*

$$\mathrm{res}_P^G \, \mathrm{ind}_Q^G A = \bigoplus_{x \in \Gamma} \mathrm{ind}_{x^{-1}Qx \cap L}^L \, \mathrm{res}_{M \cap xPx^{-1}}^M A \tag{15.2.1}$$

(equality in $\mathscr{M}L$).

(The formula has the following meaning. By 4.8(b), $\mathrm{ind}_Q^G A$ is a direct sum of character sheaves of G, hence, by 6.9(a), $\mathrm{resp}_P^G \, \mathrm{ind}_Q^G A$ is a direct sum of character sheaves of L. By 6.9(a), $\mathrm{res}_{M \cap xPx^{-1}}^M A$ is a direct sum of character sheaves on $M \cap xLx^{-1}$; we transfer it to $x^{-1}Mx \cap L$ using conjugation by x^{-1}; applying to it $\mathrm{ind}_{x^{-1}Qx \cap L}^L$ we get, by 4.8(b), a direct sum of character sheaves of L.).

Proof. The operations res_P^G, ind_P^G are also defined at the level of class functions on groups over a finite field. Thus if G, P, L are defined over F_q, we may define $\underline{\mathrm{ind}}_P^G$: {class functions on $L(F_q)$} $\to$ {class functions on $G(F_q)$} as lifting to $P(F_q)$ via the natural projection $P(F_q) \to L(F_q)$, followed by usual induction from $P(F_q)$ to $G(F_q)$; we may also define $\underline{\mathrm{res}}_P^G$: {class functions on $G(F_q)$} $\to$ {class functions on $L(F_q)$} as restriction to $P(F_q)$ followed by averaging over the fibres of $P(F_q) \to L(F_q)$. These operations are related to the corresponding operations on complexes as follows. If $(A, \phi\colon F^*A \to A)$ is a character sheaf of G defined over F_q, then $\mathrm{res}_P^G A$ has a natural $\phi_1\colon F^*(\mathrm{res}_P^G A) \xrightarrow{\sim} \mathrm{res}_P^G A$ and

$$\chi_{\mathrm{res}_P^G A, \phi_1} = \underline{\mathrm{res}}_P^G(\chi_{A,\phi}).$$

If (A', ϕ') is a character sheaf of L defined over F_q, then $\mathrm{ind}_P^G A'$ has a natural ϕ_1': $F^*(\mathrm{ind}_P^G A) \xrightarrow{\sim} \mathrm{ind}_P^G A$ and $\chi_{\mathrm{ind}_P^G A', \phi_1'} = \underline{\mathrm{ind}}_P^G(\chi_{A',\phi'})$. (These formulas follow immediately from definitions and from the trace formula for Frobenius maps.)

We shall now choose an F_q-rational structure on G such that all groups appearing in 15.1 are defined over F_q, and such that (a) A is defined over F_q and (b) all character sheaves of L which are components of the left- or right-hand side of (15.2.1) are defined over F_q.

Let K_1, K_2 be the two sides of (15.2.1); we have natural isomorphisms ϕ_1: $F^*K_1 \xrightarrow{\sim} K_1$, ϕ_2: $F^*K_2 \xrightarrow{\sim} K_2$.

The analogue of (15.2.1) for $\underline{\mathrm{res}}$, $\underline{\mathrm{ind}}$ is well known. It implies that

$$\chi_{K_1,\phi_1^t}(g) = \chi_{K_2,\phi_2^t}(g) \qquad \text{for all } g \in L(F_{q^t}) \text{ and all } t \geqslant 1. \tag{15.2.2}$$

Now let A' be any character sheaf of L which is a component of K_1 or K_2 and let $\phi_{A'}$ be an isomorphism $F^*A \xrightarrow{\sim} A$. We can write

$$K_1 = \bigoplus_{A'} (A' \otimes V_{A',1}), \qquad K_2 = \bigoplus_{A'} (A' \otimes V_{A',2}),$$

where $V_{A',1}$, $V_{A',2}$ are $\bar{Q}_l$-vector space with natural endomorphisms $\psi_{A',1}$, $\psi_{A',2}$, respectively, such that

$$\chi_{K_i,\phi_i^t}(g) = \sum_{A'} \chi_{A',\phi_{A'}^t}(g)\,\mathrm{Tr}(\psi_{A',i}^t, V_{A',i}) \tag{15.2.3}$$

($i = 1, 2$, $g \in L(F_{q^t})$). Using now the orthogonality formula 10.8 for L (which is applicable since L is clean) we see from (15.2.2) and (15.2.3) that

$$|L^{F^t}|^{-1} \sum_{g \in L^{F^t}} \chi_{K_i,\phi_i^t}(g)\,\chi_{DA',(\phi_{A'}^t)^\vee}(g) = q^{-\mathrm{codim\,supp}\,A'}\,\mathrm{Tr}(\psi_{A',i}^t, V_{A',i}) \tag{15.2.4}$$

for all A', all $t \geqslant 1$, and for $i = 1$ or 2. From (15.2.2) it follows that the left-hand side of (15.2.4) is independent of i. Hence the same is true for the right-hand side

$$\mathrm{Tr}(\psi_{A',1}^t, V_{A',1}) = \mathrm{Tr}(\psi_{A',2}^t, V_{A',2})$$

for all A' and all $t \geqslant 1$.

This remains automatically true for $t = 0$ so that $\dim V_{A',1} = \dim V_{A',2}$ for all A'. It follows that $K_1 \approx K_2$ and the proposition is proved.

15.3. For each subset I of the set S of simple reflections of W, we denote by P_I the parabolic subgroup of G generatted by B and by representatives in $N(T)$ of the simple reflections $s_i \in I$. Let L_I be the unique Levi subgroup of P_I containing T and let W_I be the subgroup of W generated by I.

If $I \subset J$ and A_1 is a character sheaf of L_I, the complex $\mathrm{ind}_{P_I \cap L_J}^{L_J}(A_1)$ will be denoted $i_I^J(A_1)$; it is a direct sum of character sheaves of L_J; see 4.8(b)). By linearity, this extends to a homomorphism $i_I^J\colon \mathscr{K}_0(L_I) \to \mathscr{K}_0(L_J)$ (see (14.10.2)) and to a linear map $i_I^J\colon \mathscr{K}_0(L_I) \otimes \bar{\mathbb{Q}}_l \to \mathscr{K}_0(L_J) \otimes \bar{\mathbb{Q}}_l$.

If $I \subset J$ and A_2 is a character sheaf of L_J, the complex $\mathrm{res}_{P_I \cap L_J}^{L_J}(A_2)$ will be denoted $r_I^J(A_2)$; it is a direct sum of character sheaves of L_I; see 6.9(a). By linearity, this extends to a homomorphism $r_I^J\colon \mathscr{K}_0(L_J) \to \mathscr{K}_0(L_I)$ and to a linear map $r_I^J\colon \mathscr{K}_0(L_J) \otimes \bar{\mathbb{Q}}_l \to \mathscr{K}_0(L_1) \otimes \bar{\mathbb{Q}}_l$. By 4.2, we have the transitivity formula

$$i_J^K i_I^K = i_I^K, \qquad \text{for} \quad I \subset J \subset K. \tag{15.3.1}$$

From 4.4(d) and the semisimplicity of $\mathrm{ind}(A_1)$, $\mathrm{res}(A_2)$, it follows that

$$(r_I^J(A_2)\colon A_1) = (A_2\colon i_I^J(A_1)) \tag{15.3.2}$$

for any A_1, A_2 as above. The same formula is then automatically true if A_1, A_2 are replaced by any elements of $\mathscr{K}_0(L_I) \otimes \bar{\mathbb{Q}}_l$, $\mathscr{K}_0(L_J) \otimes \bar{\mathbb{Q}}_l$, respectively. Here $(:)$ is defined by (14.10.4) for L_I, L_J instead of G. From (15.3.1), (15.3.2), and the non-degeneracy of $(:)$ it follows that

$$r_I^J r_J^K = r_I^K \qquad \text{for} \quad I \subset J \subset K. \tag{15.3.3}$$

We can restate (15.2.1) as

$$r_J^S i_I^S = \sum_x i_{x^{-1}Jx \cap J}^J \gamma_x r_{I \cap xJx^{-1}}^I \qquad (I, J \subset S), \tag{15.3.4}$$

sum over all elements $x \in W$ which have minimal length in $W_i x W_J$. (Here $\gamma_x\colon \mathscr{K}_0(L_{I \cap xJx^{-1}}) \to \mathscr{K}_0(L_{x^{-1}Ix \cap J})$ is induced by the isomorphism $L_{I \cap xJx^{-1}} \xrightarrow{\sim} L_{x^{-1}Ix \cap J}$ defined by conjugation by a representative of x^{-1} in $N(T)$.)

15.4. We now define a homomorphism

$$d = d_G = \sum_{I \subset S} (-1)^{|I|}\, i_I^S r_I^S\colon \mathscr{K}_0(G) \to \mathscr{K}_0(G)$$

$$(\text{or } \mathscr{K}_0(G) \otimes \bar{\mathbb{Q}}_l \to \mathscr{K}_0(G) \otimes \bar{\mathbb{Q}}_l). \tag{15.4.1}$$

This is entirely analogous to the well-known duality operation on the class functions on a reductive group over a finite field, which is defined replacing ind and res in (5.4.1) by <u>ind</u>, <u>res</u> (see the proof of 15.2). Here are some properties of d:

$$d_G i_I^S = i_I^S d_{L_I} \tag{15.4.2}$$

$$d^2 = \text{identity} \tag{15.4.3}$$

$$(dA_1 : dA_2) = (A_1 : A_2) \qquad (A_1, A_2 \in \mathscr{K}_0(G)) \tag{15.4.4}$$

$$(dA_1 : A_2) = (A_1 : dA_2) \qquad (A_1, A_2 \in \mathscr{K}_0(G)), \tag{15.4.5}$$

which are analogous to the known properties of the duality operation for class functions (Curtis, Alvis, and Kawanaka). (See [8].) They are formal consequences of the identities (15.3.1)–(15.3.4), and of the following identity (see [9, 2.5]):

$$\sum_{J \subset S} (-1)^{|J|} \#\{x \in W \mid I \cap xJx^{-1} = K, x \text{ has minimal length in } W_I x W_J\}$$
$$= (-1)^{|K|} \qquad (\text{for } I, K \subset S).$$

We shall call d the duality operation (on character sheaves). It should not be confused with the Verdier duality D.

From (15.4.4), it follows that

$$\text{If } A \in \hat{G}, \text{ then } \pm dA \in \hat{G}. \tag{15.4.6}$$

(More precisely, dA or $-dA$ is the class in $\mathscr{K}_0(G)$ of a character sheaf.)

If $A \in \hat{G}$ is cuspidal, then $r_I^S A = 0$ for all ($I \subsetneq S$); from (15.4.1) it follows that

$$dA = (-1)^{|S|} A \qquad (A \in \hat{G}, \text{ cuspidal}). \tag{15.4.7}$$

We now prove

PROPOSITION 15.5. *For any character sheaf A of G we have $dA = (-1)^{\delta} A'$, where $\delta = \operatorname{codim}_G \operatorname{supp} A$, and A' is a character sheaf with the same support as A.*

Proof. By 4.4(a), we can find $I \subset S$ and a cuspidal character sheaf A_1 of $L = L_I$ such that $(A : i_I^S A_1) > 0$. From (15.4.2) and (15.4.7) it follows that $d(i_I^S A_1) = (-1)^{|I|} i_I^S A_1$. Since $i_I^S A_1$ is a linear combination with >0 coefficients of character sheaves, we deduce that $dA = (-1)^{|I|} A'$, where A' is a character sheaf such that $(A' : i_I^S A_1) > 0$ and hence such that $\operatorname{supp} A' = \operatorname{supp} A = \bar{Y}_{(L,\Sigma)}$ (see (4.3.1)); here $\Sigma \subset L$ is as in 4.3 and its closure is equal to $\operatorname{supp} A_1$. It remains to show that $\operatorname{codim} \bar{Y}_{(L,\Sigma)} \equiv |I| \pmod 2$. By (8.2.2) we have

$$\operatorname{codim} \bar{Y}_{(L,\Sigma)} = \dim L - \dim \Sigma = \dim(L/\mathscr{Z}_L^0) - \dim(\Sigma/\mathscr{Z}_L^0).$$

But $\Sigma/\mathscr{Z}_L^0$ is a single conjugacy class in $L/\mathscr{Z}_L^0$, hence it has even dimension. Hence codim $\bar{Y}_{(L,\Sigma)} \equiv \dim(L/\mathscr{Z}_L^0) \equiv |I|$ (mod 2). The proposition is proved.

15.6. We shall now investigate the behavior of the elements $R_E^{\mathscr{L}}$ (see (14.10.3)), with respect to induction, restriction, and the duality operation d. We first introduce some notation. Let I be a subset of S and let $\mathscr{L} \in \mathscr{S}(T)$. We define $W'_{\mathscr{L},I}$, $K_w^{\mathscr{L},I}$, $\bar{K}_w^{\mathscr{L},I} \in \mathscr{D}L_I$ ($w \in W'_{\mathscr{L},I}$) in terms of $L_I, \mathscr{L}$ exactly as $W'_{\mathscr{L}}, K_w^{\mathscr{L}}, \bar{K}_w^{\mathscr{L}}$ were defined in terms of $G, \mathscr{L}$. Then $W'_{\mathscr{L},I} = W_I \cap W'_{\mathscr{L}}$. For any virtual representation E_1 of $W'_{\mathscr{L},I}$, we define $R_{E_1}^{\mathscr{L},I} \in \mathscr{K}_0(L_I) \otimes \bar{\mathbb{Q}}_l$ in terms of L_I, $\mathscr{L}$ just as $R_E^{\mathscr{L}}$ was defined in terms of G, $\mathscr{L}$. Let $\mathrm{ind}(E_1)$ or $\mathrm{ind}_I^S(E_1)$ be the virtual representation of $W'_{\mathscr{L}}$ obtained by inducing E_1 from $W'_{\mathscr{L},I}$ to $W'_{\mathscr{L}}$. For a virtual representation E of $W'_{\mathscr{L}}$ we denote $\mathrm{res}(E)$ or $\mathrm{res}_I^S(E)$ the restriction of E to $W'_{\mathscr{L},I}$ (a virtual representation of $W'_{\mathscr{L},I}$).

For $x \in W$, we denote by ${}^xE$ the virtual representation of $W'_{{}^x\mathscr{L}}$ (${}^x\mathscr{L} = (x^{-1})^* \mathscr{L}$), obtained from E by composing with the isomorphism $W'_{\mathscr{L}} \xrightarrow{\sim} W'_{{}^x\mathscr{L}}$ given by conjugation by x; $\mathrm{res}({}^xE)$ is the restriction of ${}^xE$ to $W'_{{}^x\mathscr{L},I}$. We can now state:

PROPOSITION 15.7. (a) $i_I^S(R_{E_1}^{\mathscr{L},I}) = R_{\mathrm{ind}(E_1)}^{\mathscr{L}}$, *for any virtual representation* E_1 *of* $W'_{\mathscr{L},I}$.

(b) $r_I^S(R_E^{\mathscr{L}}) = \sum_{x \in W} |W_I x W'_{\mathscr{L}}|^{-1} R_{\mathrm{res}({}^xE)}^{{}^x\mathscr{L},I}$, *for any virtual representation* E *of* $W'_{\mathscr{L}}$.

(c) $(A: r_I^S R_E^{\mathscr{L}}) = (A: R_{\mathrm{res}\,E}^{\mathscr{L},I})$, *for any virtual representation* E *of* $W'_{\mathscr{L}}$ *and any* $A \in (\hat{L}_I)_{\mathscr{L}}$.

(d) $i_I^S({}^pH^i \bar{K}_w^{\mathscr{L},I}) = {}^pH^{i+\dim G - \dim L} \bar{K}_w^{\mathscr{L}}$ *for any* $w \in W'_{\mathscr{L},I}$ *and any integer* i.

Proof. (a) From 4.8(a) and 6.5, it follows that

$$i_I^S\left(\sum_i (-1)^i\, {}^pH^i(K_w^{\mathscr{L},I})\right)$$
$$= (-1)^{\dim G - \dim L_I}\left(\sum_i (-1)^i\, {}^pH^i(K_w^{\mathscr{L}})\right) \in \mathscr{K}_0(G) \qquad (15.7.1)$$

for any $w \in W'_{\mathscr{L},I}$. Using the definition (14.10.3) we have

$$i_I^S(R_{E_1}^{\mathscr{L},I}) = |W'_{\mathscr{L},I}|^{-1} \sum_{w \in W'_{\mathscr{L},I}} \mathrm{Tr}(w^{-1}, E_1)\, i_I^S\left(\sum_i (-1)^{i+\dim L_I}\, {}^pH^i(K_w^{\mathscr{L},I})\right)$$
$$= |W'_{\mathscr{L},I}|^{-1} \sum_{w \in W'_{\mathscr{L},I}} \mathrm{Tr}(w^{-1}, E_1)\left(\sum_i (-1)^{i+\dim G}\, {}^pH^i(K_w^{\mathscr{L}})\right)$$

and

$$\begin{aligned}
R^{\mathscr{L}}_{\mathrm{ind}(E_1)} &= |W'_{\mathscr{L}}|^{-1} \sum_{w\in W'_{\mathscr{L}}} \mathrm{Tr}(w^{-1}, \mathrm{ind}(E_1)) \sum_i (-1)^{i+\dim G}\, {}^pH^i(K^{\mathscr{L}}_w) \\
&= |W'_{\mathscr{L},I}|^{-1}\, |W'_{\mathscr{L}}|^{-1} \sum_{\substack{w\in W'_{\mathscr{L}}\\ z\in W'_{\mathscr{L}}\\ zw^{-1}z^{-1}\in W'_{\mathscr{L},I}}} \mathrm{Tr}(zw^{-1}z^{-1}, E_1) \\
&\quad \times \sum_i (-1)^{i+\dim G}\, {}^pH^i(K^{\mathscr{L}}_w) \\
&= |W'_{\mathscr{L},I}|^{-1} \sum_{u\in W'_{\mathscr{L},I}} \mathrm{Tr}(u^{-1}, E_1) \sum_i (-1)^{i+\dim G}\, {}^pH^i(K^{\mathscr{L}}_u).
\end{aligned}$$

(We have used the fact that for $w\in W'_{\mathscr{L}}$, $\sum_i (-1)^i\, {}^pH^i(K^{\mathscr{L}}_w)\in \mathscr{K}_0(G)$ depends only on the conjugacy class of w in $W'_{\mathscr{L}}$; see 6.5.) This proves (a).

(b) From (14.10.3) and 6.5, we see that, for fixed $\mathscr{L}$, the following four $\bar{\mathbb{Q}}_l$-subspaces of $\mathscr{K}_0(G)\otimes\bar{\mathbb{Q}}_l$ coincide:

—the subspace spanned by all $\chi(\bar{K}^{\mathscr{L}}_{\mathbf{s}})$ (see 6.5), where $\mathbf{s}$ is any sequence $(s_1, s_2,..., s_r)$ in S such that $s_1 s_2\cdots s_r\in W'_{\mathscr{L}}$;

—the subspace spanned by all $\chi(K^{\mathscr{L}}_{\mathbf{s}})$ (see 6.5), where $\mathbf{s}$ is as above;

—the subspace spanned by all $\chi(K^{\mathscr{L}}_w)$, $w\in W'_{\mathscr{L}}$ (see 6.5);

—the subspace spanned by all $R^{\mathscr{L}}_E$ $(E\in \hat{W}'_{\mathscr{L}})$.

From (3.8.2) and 6.9(a) we see that

$$r^S_I(\chi(\bar{K}^{\mathscr{L}}_{\mathbf{s}})) = \sum_i (-1)^i\, {}^pH^i(\mathrm{res}\, \bar{K}^{\mathscr{L}}_{\mathbf{s}})$$

and the last sum has been expressed in (6.7.1) (with $u=1$) as a $\mathbb{Z}$-linear combination of elements $\chi(K^{{}^x\mathscr{L},I}_{\mathbf{t}})$, where x are various elements of W and $\mathbf{t}$ are various sequences in I whose product is $W'_{{}^x\mathscr{L},I}$. (Here $K^{{}^x\mathscr{L},I}_{\mathbf{t}}$ is defined like $K^{\mathscr{L}}_{\mathbf{s}}$ for L_I, ${}^x\mathscr{L}$ instead of G, $\mathscr{L}$.)

It follows that $r^S_I(R^{\mathscr{L}}_E)$ is a $\bar{\mathbb{Q}}_l$-linear combination of elements $R^{{}^x\mathscr{L},I}_{E'}\in\mathscr{K}_0(L_I)\otimes\bar{\mathbb{Q}}_l$ for various $x\in W$ and various $E'\in\hat{W}'_{{}^x\mathscr{L},I}$. Hence in order to prove (b) it is enough to show that the two sides of (b) have the same inner product (:) with any $R^{{}^y\mathscr{L},I}_{{}^yE'}$ $(y\in W,\ E'\in\hat{W}'_{\mathscr{L},I})$. We compute

$$\begin{aligned}
(r^S_I(R^{\mathscr{L}}_E) : R^{{}^y\mathscr{L},I}_{{}^yE'}) &= (R^{\mathscr{L}}_E : i^S_I(R^{{}^y\mathscr{L},I}_{{}^yE'})), && \text{by (15.3.2)}\\
&= (R^{\mathscr{L}}_E : R^{{}^y\mathscr{L}}_{\mathrm{ind}\,{}^yE'}), && \text{by (a)}\\
&= (R^{\mathscr{L}}_E : R^{\mathscr{L}}_{\mathrm{ind}\,E'}), && \text{by 11.2(b)}\\
&= \text{multiplicity of } E' \text{ in restriction of } E\\
&\qquad \text{to } W'_{\mathscr{L},I}, && \text{by 14.13.}
\end{aligned}$$

The inner product $(R^{x\mathscr{L},I}_{\mathrm{res}\,{}^xE}: R^{y\mathscr{L},I}_{{}^yE'})$ is zero unless ${}^x\mathscr{L} = {}^{zy}\mathscr{L}$ for some $z \in W_I$ (see 11.2(e)), i.e., unless $x \in W_I y W'_{\mathscr{L}}$. If the condition $x = zyu$, $z \in W_I$, $u \in W'_{\mathscr{L}}$ is satisfied, then by 11.2(b) we have $R^{x\mathscr{L},I}_{\mathrm{res}\,{}^xE} = R^{y\mathscr{L},I}_{\mathrm{res}\,{}^yE}$. Hence the inner product of the right side of (b) with $R^{y\mathscr{L},I}_{{}^yE'}$ is equal to

$$(R^{y\mathscr{L},I}_{\mathrm{res}\,{}^yE}: R^{y\mathscr{L},I}_{{}^yE'}).$$

By 14.13, this is equal to the multiplicity of ${}^yE'$ in the restriction of ${}^yE$ to $W'_{{}^y\mathscr{L},I}$, hence to the multiplicity of E' in the restriction of E to $W'_{\mathscr{L},I}$. This proves (b).

(c) From (b) and its proof we see that

$$\begin{aligned}(A: r_I^S R_E^{\mathscr{L}}) &= \sum_{x \in W_I W'_{\mathscr{L}}} |W_I W'_{\mathscr{L}}|^{-1} (A: R^{x\mathscr{L},I}_{\mathrm{res}({}^xE)}) && \text{(see 11.2(c))}\\ &= (A: R^{\mathscr{L},I}_{\mathrm{res}\,E}) && \text{(see 11.2(b)).}\end{aligned}$$

Property (d) follows from (a) and (14.11.1). The proposition is proved.

COROLLARY 15.8. (a) *If $w \in W'_{\mathscr{L}}$ and no W-conjugate of w is contained in W_I, then $r_I^S(\sum_i (-1)^i\, {}^pH^iK_w^{\mathscr{L}}) = 0$.*

(b) *For any $w \in W'_{\mathscr{L}}$, we have*

$$d\left(\sum (-1)^i\, {}^pH^iK_w^{\mathscr{L}}\right) = (-1)^{l(w)} \sum_i (-1)^i\, {}^pH^iK_w^{\mathscr{L}}.$$

(c) *For any virtual representation E of $W'_{\mathscr{L}}$ we have*

$$d(R_E^{\mathscr{L}}) = R^{\mathscr{L}}_{E \otimes \varepsilon},$$

where ε denotes the representation $w \to (-1)^{l(w)}$ of $W'_{\mathscr{L}}$.

Proof. (a) From (14.10.3) we get

$$\sum_i (-1)^i\, {}^pH^iK_w^{\mathscr{L}} = (-1)^{\dim G} \sum_{E \in \hat{W}'_{\mathscr{L}}} \mathrm{Tr}(w, E)\, R_E^{\mathscr{L}}.$$

We now apply 15.7(b):

$$\begin{aligned}r_I^S\left(\sum_i (-1)^i\, {}^pH^iK_w^{\mathscr{L}}\right) &= (-1)^{\dim G} \sum_E \mathrm{Tr}(w, E)\, r_I^S(R_E^{\mathscr{L}})\\ &= (-1)^{\dim G} \sum_E \sum_x |W_I x W'_{\mathscr{L}}|^{-1}\, \mathrm{Tr}(w, E)\, R^{x\mathscr{L},I}_{\mathrm{res}\,{}^xE}\end{aligned}$$

$$= (-1)^{\dim G + \dim L_I} \sum_E \sum_{x \in W} \sum_{y \in \hat{W}'_{x\mathscr{L},I}} |W'^x\mathscr{L}, I|^{-1}$$
$$\times |W_I x W'_{\mathscr{L}}|^{-1} \operatorname{Tr}(w, E) \operatorname{Tr}(y^{-1}, {}^xE)$$
$$\times \sum_i (-1)^i\, {}^pH^i(K_y^{x\mathscr{L},I}).$$

To show that this is zero, it is enough to show that

$$\sum_{E \in \hat{W}'_{\mathscr{L}}} \operatorname{Tr}(w, E) \operatorname{Tr}(y^{-1}, {}^xE) = 0$$

for any $y \in W_I$ and any $x \in W$. But this is equivalent to

$$\sum_{E \in \hat{W}'_{\mathscr{L}}} \operatorname{Tr}(w, E) \operatorname{Tr}(xy^{-1}x^{-1}, E) = 0,$$

which follows from the fact that w, xyx^{-1} are not conjugate in $W'_{\mathscr{L}}$ (they are not conjugate even in W).

(b) If w is as in (a), the desired formula follows from (a), the definition of d, and the well-known congruence $l(w) \equiv |S|$ (mod 2) (see, for example, [6, p. 193]).

Assume now that $xwx^{-1} \in W_I$ for some $x \in W$ and some $I \subsetneq S$. To prove our formula, we may assume by 11.2(b) that $w \in W_I$. In this case, using (15.7.1) and (15.4.2) we are reduced to the case where G is replaced by L_I, for which our formula may be assumed to be already known.

Property (c) clearly follows from (b). The corollary is proved.

COROLLARY 15.9. *If $A \in \hat{G}_{\mathscr{L}}$, then $\pm dA \in \hat{G}_{\mathscr{L}}$.*

Proof. Let $E \in \hat{W}'_{\mathscr{L}}$ be such that $(A: R_E^{\mathscr{L}}) \neq 0$ (see 14.12). Then $(dA: dR_E^{\mathscr{L}}) \neq 0$ hence $(dA: R_{E\otimes\varepsilon}^{\mathscr{L}}) \neq 0$ (see 15.8(c)). It follows that $\pm dA \in \hat{G}_{\mathscr{L}}$.

COROLLARY 15.10.

$$\sum_i (-1)^i (A: d\,{}^pH^i(\bar{K}_w^{\mathscr{L}}))\, u^{i/2}$$
$$= \sum_{E \in \hat{W}'_{\mathscr{L}}} (-1)^{\dim G + l(w)} u^{(1/2)(\dim G + l(w))} (A: R_E^{\mathscr{L}}) \operatorname{Tr}(C_w, E(u)) \quad (15.10.1)$$

for all $w \in W'_{\mathscr{L}}$, $A \in \hat{G}_{\mathscr{L}}$. (Identity in $\mathbb{Q}_l[u^{1/2}, u^{-1/2}]$; C_w is as in (12.9.2).)

Proof. Using (14.11.1) and (15.4.5) we see that the left-hand side of (15.10.1) is equal to

$$\sum_E (-1)^{\dim G} u^{(1/2)(\dim G + l(w))}(dA\colon R_E^{\mathscr{L}}) \operatorname{Tr}(C'_w, E(u))$$
$$= \sum_E (-1)^{\dim G} u^{(1/2)(\dim G + l(w))}(A\colon R_{E\otimes\varepsilon}^{\mathscr{L}}) \operatorname{Tr}(C'_w, E(u))$$
(by (15.4.5) and 15.8(c))
$$= \sum_E (-1)^{\dim G} u^{(1/2)(\dim G + l(w))}(A\colon R_E^{\mathscr{L}}) \operatorname{Tr}(C'_w, (E\otimes\varepsilon)(u))$$

and it remains to use the identity

$$\operatorname{Tr}(C'_w, (E\otimes\varepsilon)(u)) = (-1)^{l(w)} \operatorname{Tr}(C_w, E(u)),$$

which follows from (12.9.7). (Compare with [6, 6.9].)

15.11. Let A be a character sheaf of G. Then A is a component of ${}^pH^i(\bar{K}_w^{\mathscr{L}})$ for some integer i, some $\mathscr{L} \in \mathscr{S}(T)$, and some $w \in W'_{\mathscr{L}}$. We define

$$\varepsilon_A = (-1)^{i + \dim G} \tag{15.11.1}$$

(cf. 13.10(a)) and

$$\hat{\varepsilon}_A = (-1)^{\operatorname{codim} \operatorname{supp} A}. \tag{15.11.2}$$

It is clear that $\hat{\varepsilon}_A$ is an invariant of A. We now show that ε_A *is also an invariant of* A. It is enough to show that

$$(A\colon {}^pH^i\bar{K}_w^{\mathscr{L}}) \neq 0, \qquad (A\colon {}^pH^{i'}\bar{K}_{w'}^{\mathscr{L}'}) \neq 0 \Rightarrow i \equiv i' \ (\text{mod } 2). \tag{15.11.3}$$

By 11.2(c), we have $\mathscr{L}' = (x^{-1})^* \mathscr{L}$ for some $x \in W$.

Writing x as a product of simple reflections and using 14.15 repeatedly, we see that $(A\colon {}^pH^{i'}\bar{K}_{w'}^{\mathscr{L}'}) \neq 0 \Rightarrow (A\colon {}^pH^{i''}\bar{K}_{w''}^{\mathscr{L}}) \neq 0$ for some i'', $w'' \in W'_{\mathscr{L}}$ such that $i'' \equiv i'$ (mod 2).

Thus, to prove (15.11.3), we may assume that $\mathscr{L}' = \mathscr{L}$. In that case, we have $i \equiv i'$ (mod 2) by 13.10(a). Thus, (15.11.3) is proved.

15.12. The invariants $\varepsilon_A, \hat{\varepsilon}_A$ are conserved by induction, in the following sense. Let A_1 be a character sheaf of L_I (see 5.3) and let A be any irreducible component of $\operatorname{ind}_{P_I}^G(A_1)$. Then

$$\varepsilon_{A_1} = \varepsilon_A, \qquad \hat{\varepsilon}_{A_1} = \hat{\varepsilon}_A. \tag{15.12.1}$$

The formula $\varepsilon_{A_1} = \varepsilon_A$ follows from 15.7(d); the formula $\hat{\varepsilon}_{A_1} = \hat{\varepsilon}_A$ follows from (4.3.1) and (8.2.2).

15.13. We say that $(G, \mathscr{L})$ satisfies the *parity condition* if

$$\varepsilon_A = \hat{\varepsilon}_A \tag{15.13.1}$$

for all $A \in \hat{G}_{\mathscr{L}}$. If this condition is satisfied, then

$(-1)^{i+\dim(G)}\ (A: d^pH^i\bar{K}_w^{\mathscr{L}})$ is an integer $\geqslant 0$ for all $A \in \hat{G}_{\mathscr{L}}$, all $w \in W'_{\mathscr{L}}$, and all i. (15.13.2)

Indeed, the expression (15.13.2) is

$$(-1)^{i+\dim G}(dA: {}^pH^i\bar{K}_w^{\mathscr{L}}).$$

If this is non-zero, then it is equal to

$$\begin{aligned}\varepsilon_{A'}(dA: {}^pH^i\bar{K}_w^{\mathscr{L}}) &= \hat{\varepsilon}_{A'}(dA: {}^pH^i\bar{K}_w^{\mathscr{L}})\\ &= (A': {}^pH^i\bar{K}_w^{\mathscr{L}}) \geqslant 0,\end{aligned}$$

where $A' \in \hat{G}_{\mathscr{L}}$ is defined by $dA = \hat{\varepsilon}_A A'$ (see 15.5); we have $\hat{\varepsilon}_A = \hat{\varepsilon}_{A'}$, again by 15.5.

16. The Two-Sided Cell Attached to a Character Sheaf

16.1. In this chapter we shall fix $\mathscr{L} \in \mathscr{S}(T)$. We shall define a partition of $W'_{\mathscr{L}}$ into "two-sided cells" and we shall define (under certain assumptions) a map of $\hat{G}_{\mathscr{L}}$ into the set of two-sided cells of $W'_{\mathscr{L}}$.

16.2. The group $W_{\mathscr{L}}$ is a Coxeter group.

We refer to [6, p. 139] for the definition of the relations $E_1 \leqslant_{LR} x$, $E_1 \sim_{LR} x$, $E_1 <_{LR} x$ ($E_1 \in \hat{W}_{\mathscr{L}}$, $x \in W_{\mathscr{L}}$) and to [6, p. 160] for the definition of the relation $E_1 \sim_{LR} E'_1$ ($E_1, E'_1 \in \hat{W}_{\mathscr{L}}$). We refer to [6, p. 76] for the definition of the invariants $a_{E_1}, A_{E_1}, f_{E_1}$ of $E_1 \in \hat{W}_{\mathscr{L}}$, in terms of the formal dimension polynomial $D_{E_1}(u)$.

We shall extend these definitions to $W'_{\mathscr{L}} = \Omega_{\mathscr{L}} \cdot W_{\mathscr{L}}$ which in general is not a Coxeter group.

If $E \in \hat{W}'_{\mathscr{L}}$ and $x \in W'_{\mathscr{L}}$ we say that $E \leqslant_{LR} x$ (resp. $E \sim_{LR} x$, $E <_{LR} x$) if there exists an irreducible $W_{\mathscr{L}}$-submodule E_1 of the restriction $E \mid W_{\mathscr{L}}$ and an element $\tilde{x} \in W_{\mathscr{L}} \cap \Omega_{\mathscr{L}} x \Omega_{\mathscr{L}}$ such that $E_1 \leqslant_{LR} \tilde{x}$ (resp. $E_1 \sim_{LR} \tilde{x}$, $E_1 <_{LR} \tilde{x}$).

If $E, E' \in \hat{W}'_{\mathscr{L}}$ we say that $E \sim_{LR} E'$ if there exist irreducible

$W_{\mathscr{L}}$-submodules $E_1 \subset E \mid W_{\mathscr{L}}$, $E'_1 \subset E' \mid W_{\mathscr{L}}$ such that $E_1 \sim_{LR} E'_1$. If $E_2 \in \hat{W}'_{\mathscr{L}}$, we define $a_E = a_{E_1}$, $A_E = A_{E_1}$, where E_1 is any irreducible $W_{\mathscr{L}}$-submodule of $E \mid W_{\mathscr{L}}$. For any $E \in \hat{W}'_{\mathscr{L}}$ any $h \in H'_{\mathscr{L}}$ and any integer i, we define $\mathrm{Tr}(h, E(u); i/2) \in \bar{\mathbb{Q}}_l$ by

$$\mathrm{Tr}(h, E(u)) = \sum_i \mathrm{Tr}(h, E(u); i/2)\, u^{i/2}$$

(see (12.9.3)).

For any $x \in W'_{\mathscr{L}}$ and any $E \in \hat{W}'_{\mathscr{L}}$ we define

$$\begin{aligned} c_{x,E} &= (-1)^{l(x)}\, \mathrm{Tr}(u^{-\bar{l}(x)/2} T_x, E(u); -a_E/2) \\ &= (-1)^{l(x)}\, \mathrm{Tr}(C_x, E(u); -a_E/2) \end{aligned} \tag{16.2.1}$$

(compare [6, (5.1.21), (5.2.1)]).

$$\begin{aligned} c'_{x,E} &= \mathrm{Tr}(u^{-\bar{l}(x)/2} T_x, E(u); (\nu - A_E)/2) \\ &= \mathrm{Tr}(C'_x, E(u); (\nu - A_E)/2) \end{aligned} \tag{16.2.2}$$

(compare [6, (5.1.23), (5.11.1)]); ν is the number of reflections in $W_{\mathscr{L}}$.

From (12.9.3) it follows that $c_{x,E}$, $c'_{x,E}$ are integers times roots of 1. From (12.9.5) it follows that

$$c_{x^{-1},E} = c_{x,E^*}. \tag{16.2.3}$$

From (12.9.7) it follows that

$$c'_{x,E} = c_{x,E \otimes \varepsilon} \tag{16.2.4}$$

where $\varepsilon\colon W'_{\mathscr{L}} \to \pm 1$ is as in (12.9.7).

By considering the coefficient of $u^{-(a_E + a_{E'})/2}$ in the two sides of (12.9.6) we obtain

$$\sum_{x \in W'_{\mathscr{L}}} c_{x,E} c_{x^{-1},E} = \begin{cases} |\Omega_{\mathscr{L}}|\, f_{E_1} \dim E_1, & \text{if } E \approx E' \\ 0, & \text{otherwise,} \end{cases} \tag{16.2.5}$$

where E_1 is an irreducible $W_{\mathscr{L}}$-submodule of $E \mid W_{\mathscr{L}}$.

$$\text{If } c_{x,E} \neq 0 \text{ then } E \sim_{LR} x \tag{16.2.6}$$

(compare [6, 5.2(ii)]).

We now define, for any $x \in W'_{\mathscr{L}}$,

$$\alpha_x = \sum c_{x,E} E, \qquad \mathscr{A}_x = \sum_E c'_{x,E} E \tag{16.2.7}$$

(both sums are taken over all $E \in \hat{W}'_{\mathscr{L}}$). These are elements of

$\mathscr{R}(W'_{\mathscr{L}}) \otimes \bar{\mathbb{Q}}_l$, where $\mathscr{R}(W'_{\mathscr{L}})$ is the Grothendieck group of virtual representations of $W'_{\mathscr{L}}$.

From (16.2.4) it follows that

$$\mathscr{A}_x = \alpha_x \otimes \varepsilon. \tag{16.2.8}$$

From (16.2.5) it follows that

$$\sum_{x \in W'_{\mathscr{L}}} c_{x^{-1},E}\alpha_x = |\Omega_{\mathscr{L}}|\ (f_{E_1} \dim E_1) \cdot E, \tag{16.2.9}$$

where E_1 is an irreducible $W_{\mathscr{L}}$-submodule of $E \mid W_{\mathscr{L}}$.

16.3. The two-sided cells of the Coxeter group $W_{\mathscr{L}}$ are defined as in [12]. A subset of $W'_{\mathscr{L}}$ is said to be a two-sided cell if it is of form $\Omega_{\mathscr{L}} c_1 \Omega_{\mathscr{L}}$ for some two-sided cell c_1 of $W_{\mathscr{L}}$. The two-sided cells form a partition of $W'_{\mathscr{L}}$. They are in 1–1 correspondence with the $\Omega_{\mathscr{L}}$-orbits on the set of two-sided cells of $W_{\mathscr{L}}$ ($\Omega_{\mathscr{L}}$ acts on that set by conjugation). Each two-sided cell of $W'_{\mathscr{L}}$ is stable under the map $x \to x^{-1}$ (see [6, 5.2(iii)]). If c is a two-sided cell of $W'_{\mathscr{L}}$ then $w_0 \cdot c = c \cdot w_0$ is again a two-sided cell of $W'_{\mathscr{L}}$, where w_0 is the longest element of the Coxeter group $W_{\mathscr{L}}$.

If $E \in \hat{W}'_{\mathscr{L}}$ and $x \in W'_{\mathscr{L}}$ then the following two conditions are equivalent: $E \sim_{LR} x$, $E \otimes \varepsilon \sim_{LR} w_0 x$. (16.3.1)

(Compare [6, 5.14(ii)].)

We now prove the following result.

LEMMA 16.4. *Let c be a two-sided cell of $W'_{\mathscr{L}}$. The following three $\bar{\mathbb{Q}}_l$-subspaces of $\mathscr{R}(W'_{\mathscr{L}}) \otimes \bar{\mathbb{Q}}_l$ coincide*:

(a) *the subspace spanned by all E ($E \in \hat{W}'_{\mathscr{L}}$) such that $E \sim_{LR} x$ for some $x \in c$*;

(b) *the subspace spanned by all $\alpha_x (x \in c)$*;

(c) *the subspace spanned by all $\mathscr{A}_{w_0 x}$ $(x \in c)$.*

Proof. The subspace (b) is contained in the subspace (a) by (16.2.6). The subspace (a) is contained in the subspace (b) by (16.2.9) and by the invariance of c under $x \to x^{-1}$. By (16.2.8) the subspace (c) coincides with the subspace spanned by all $\alpha_x \otimes \varepsilon$ ($x \in w_0 c$), hence (by the first part of the argument) with the subspace spanned by all $E \otimes \varepsilon$ ($E \in \hat{W}'_{\mathscr{L}}$), such that $E \sim_{LR} x$ for some $x \in w_0 c$. By (16.3.1) this also coincides with the subspace (a).

16.5. We refer to [15, Sect. 2] for the definition of the function a: $W_{\mathscr{L}} \to \mathbb{N}$.

It has the following property: if $E_1 \in \hat{W}_{\mathscr{L}}$, $x \in W_{\mathscr{L}}$ satisfy $E_1 \sim_{LR} x$ then $a_{E_1} = a(x)$. (See [15, 6.4] and [6, 5.27].)

We extend this to a function $a\colon W'_{\mathscr{L}} \to \mathbb{N}$ by setting $a(x \cdot y) = a(y)$ $(x \in \Omega_{\mathscr{L}}, y \in W_{\mathscr{L}})$. Note that we also have $a(y \cdot x) = a(y)$ $(x \in \Omega_{\mathscr{L}}, y \in W_{\mathscr{L}})$. Then a is constant on the two-sided cells of $W'_{\mathscr{L}}$.

$$\text{If } E \in \hat{W}'_{\mathscr{L}},\, x \in W'_{\mathscr{L}} \text{ satisfy } E \sim_{LR} x \text{ then } a_E = a(x). \tag{16.5.1}$$

This follows from the corresponding property of $W_{\mathscr{L}}$.

We can now state the following result.

THEOREM 16.6. *Assume that G is clean and that $(G, \mathscr{L})$ satisfies the parity condition* (15.13).

(a) *Let $w \in W'_{\mathscr{L}}$. The elements $R^{\mathscr{L}}_{\alpha_w}$, and $(-1)^{l(w)-a(w)} R^{\mathscr{L}}_{\mathscr{A}_w}$ of $\mathscr{K}_0(G) \otimes \bar{\mathbb{Q}}_l$ (see* (14.10.3), (16.2.7)) *have the property*

$$(A\colon R^{\mathscr{L}}_{\alpha_w}) \text{ is an integer} \geqslant 0$$

$$(A\colon (-1)^{l(w)-a(w)} R^{\mathscr{L}}_{\mathscr{A}_w}) \text{ is an integer} \geqslant 0$$

for any $A \in \hat{G}$.

(b) *Let $A \in \hat{G}_{\mathscr{L}}$ and let $E, E' \in \hat{W}'_{\mathscr{L}}$ be such that $(A\colon R^{\mathscr{L}}_E) \neq 0$, $(A\colon R^{\mathscr{L}}_{E'}) \neq 0$. Then $E \underset{LR}{\sim} E'$.*

COROLLARY 16.7. *There is a unique (surjective) map $\hat{G}_{\mathscr{L}} \to \{$two-sided cells of $W'_{\mathscr{L}}\}$ with the following property: If $A \in \hat{G}_{\mathscr{L}}$ is mapped to the two-sided cell c, and if $(A\colon R^E_{\mathscr{L}}) \neq 0$ $(E \in \hat{W}'_{\mathscr{L}})$, then $E \underset{LR}{\sim} x$ for some $x \in c$.*

16.8. For the proof of Theorem 16.6 we shall need the following result.

Let V be a $\bar{\mathbb{Q}}_l$-vector space with a given basis $\{e_i\}_{1 \leqslant i \leqslant n}$, and with a bilinear form $(\ ,\ )\colon V \times V \to \bar{\mathbb{Q}}_l$ such that $(\rho_i, \rho_j) = \delta_{i,j}$ for all i, j. Given $v \in V$ we shall say that v satisfies (P) if all coordinates of v in the $\{e_i\}$-basis are integers $\geqslant 0$. Let I be a finite set with a preorder relation $x \leqslant y$ and let $\sim$ be the associated equivalence relation on I; we write $x < y$ for "$x \leqslant y$ and $x \not\sim y$." Assume given two families of elements $r_x \in V$, $\tilde{r}_x \in V$ $(x \in I)$ such that

(a) $(r_x, r_{x'}) = 0$ whenever $x \not\sim x'$.

(b) When x runs through a fixed equivalence class for $\sim$, the r_x span the same subspace of V as the $\tilde{r}_x$.

(c) For any $x \in I$, there exists a linear combination $r_x + \sum_{x' < x} d_{x',x} r_{x'}$ $(d_{x',x} \in \bar{\mathbb{Q}}_l)$, which satisfies (P).

(d) For any $x \in I$, there exists a linear combination $\tilde{r}_x + \sum_{x < x'} \tilde{d}_{x',x} \tilde{r}_{x'}$ ($d_{x',x} \in \bar{\mathbb{Q}}_l$), which satisfies (P).

The assumptions imply that for any $x \in I$, both r_x and $\tilde{r}_x$ satisfy (P).

This is proved by repeating, essentially word for word, the proof in [6, 6.16].

16.9. *Proof of Theorem* 16.6 (compare with [6, 6.17]). In 16.8 we shall take V to be the $\bar{\mathbb{Q}}_l$-subspace of $\mathscr{K}_0(G) \otimes \bar{\mathbb{Q}}_l$ spanned by all $A \in \hat{G}_{\mathscr{L}}$. These A define the basis of V which was denoted $\{e_i\}$ in 16.8. The form (,) is (:) of (14.10.4). We take I in 16.8 to be the set $W'_{\mathscr{L}}$ with the preorder relation: "$x \leqslant x'$ if there exists $E \in \hat{W}'_{\mathscr{L}}$ such that $E \sim_{LR} x$, $E \leqslant_{LR} x'$." (The corresponding equivalence classes are just the two-sided cells of $W'_{\mathscr{L}}$.) For $x \in W'_{\mathscr{L}}$, we take $r_x = R^{\mathscr{L}}_{\alpha_x} \in V$, $\tilde{r}_x \in (-1)^{l(w_0 x) - a(w_0 x)} R^{\mathscr{L}}_{\mathscr{A}_{w_0 x}} \in V$, where w_0 is the longest element in $W_{\mathscr{L}}$.

We must verify that the elements $r_x, \tilde{r}_x \in V$ satisfy conditions (a)–(d) in 16.8. Conditions (a), (b) in 16.8 follow from 16.4 and 14.13.

If $A \in \hat{G}_{\mathscr{L}}$ and $x \in W'_{\mathscr{L}}$, we have

$$(-1)^{l(x)} \sum_{E \in \hat{W}_{\mathscr{L}}} \mathrm{Tr}(C_x, E(u); -a(x)/2)(A: R^{\mathscr{L}}_E)$$
$$= (-1)^{l(x) - a(x)}(A: d({}^pH^{\dim G + l(x) - a(x)}(\bar{K}^{\mathscr{L}}_x))), \qquad \text{by (15.10.1)}. \tag{16.9.1}$$

The part of the sum (16.9.1) corresponding to those E for which $E \sim_{LR} x$ is equal to

$$(-1)^{l(x)} \sum_{\substack{E \\ E \sim_{LR} x}} \mathrm{Tr}(C_x, E(u); -a_E/2)(A: R^{\mathscr{L}}_E)$$
$$= \sum_{\substack{x \\ E \sim_{LR} x}} c_{x,E}(A: R^{\mathscr{L}}_E) = (A: R^{\mathscr{L}}_{\alpha_x})$$

(cf. (16.5.1) and (16.2.6)). The part of the sum (16.9.1) corresponding to those E for which $E \not\leqslant_{LR} x$ is zero. (See the proof of [6, 5.2].) The part of the sum (16.9.1) corresponding to those E for which $E <_{LR} x$ is a $\bar{\mathbb{Q}}_l$-linear combination of terms $(A: R_{\alpha_{x'}})$, for x' in two-sided cells strictly lower than that of x (by 16.4). Hence (16.9.1) can be written as

$$(A: R_{\alpha_x}) + \sum d_{x',x}(A: R_{\alpha_{x'}})$$
$$= (-1)^{l(x) - a(x)}(A: d({}^pH^{\dim G + l(x) - a(x)}(\bar{K}^{\mathscr{L}}_x)))$$
$$= \text{integer} \geqslant 0 \qquad \text{by (15.13.2)}, \tag{16.9.2}$$

($d_{x',x} \in \bar{\mathbb{Q}}_l$), where x' runs over elements in two-sided cells strictly lower than that of x. Hence the condition (c) in 16.8 is verified. Replacing x by w_0x in (16.9.2) and using the identity

$$R_{\mathscr{A}_{w_0 x}} = dR_{\alpha_{w_0 x}}$$

which follows from (16.2.8) and 15.8(c) we get

$$\begin{aligned}
&(A: (-1)^{l(w_0x)-a(w_0x)} R_{\mathscr{A}_{w_0x}}) \\
&\quad + \sum d_{w_0x', w_0x} (-1)^{l(w_0x)-a(w_0x)+l(w_0x')-a(w_0x')} \\
&\quad \times (A: (-1)^{l(w_0x')-a(w_0x')} R^{\mathscr{L}}_{\mathscr{A}_{w_0x'}}) \\
&= (A: {}^pH^{\dim G + l(w_0x)-a(w_0x)}(\bar{K}^{\mathscr{L}}_{w_0x})) \\
&= \text{integer} \geqslant 0. \qquad (16.9.3)
\end{aligned}$$

(Here w_0x' runs over elements in two-sided cells strictly lower than that of w_0x; or equivalently, x' runs over elements in two-sided cells strictly higher than that of x.) Hence the condition (d) in 16.8 is verified.

From 16.8 it now follows that part (a) of Theorem 16.6 holds. We now prove (b). With the assumption of (b), we see from 16.4 that there exist $x, x' \in W'_{\mathscr{L}}$ such that $E \sim_{LR} x$, $E' \sim_{LR} x'$, $(A: R^{\mathscr{L}}_{\alpha_x}) \neq 0$, $(A: R^{\mathscr{L}}_{\alpha_{x'}}) \neq 0$. By (a), the last two inner products are $\geqslant 0$ hence they are >0. It follows that

$$(R^{\mathscr{L}}_{\alpha_x}: R^{\mathscr{L}}_{\alpha_{x'}}) > 0. \qquad (16.9.4)$$

(By (a), it is a sum of $\geqslant 0$ terms, one of which is >0.) Using 16.4, we can write

$$R^{\mathscr{L}}_{\alpha_x} = \sum_{\substack{E'' \\ E'' \sim_{LR} x}} \phi_{E''} R^{\mathscr{L}}_{E''}, \qquad R^{\mathscr{L}}_{\alpha_{x'}} = \sum_{\substack{E''' \\ E''' \sim_{LR} x'}} \psi_{E'''} R^{\mathscr{L}}_{E'''}$$

$(\phi_{E''}, \phi_{E'''} \in \bar{\mathbb{Q}}_l)$, and (16.9.4) implies

$$0 \neq \sum_{\substack{E'' \sim_{LR} x \\ E''' \sim_{LR} x'}} \phi_{E''} \psi_{E'''} (R^{\mathscr{L}}_{E''}: R^{\mathscr{L}}_{E'''}).$$

Using now 14.13, we see that there exists $E'' \in \hat{W}'_{\mathscr{L}}$ such that $E'' \sim_{LR} x$ and $E'' \sim_{LR} x'$. It follows that x, x' are in the same two-sided cell of $W'_{\mathscr{L}}$, so that $E \sim_{LR} E'$. The theorem is proved.

16.10. Corollary (of the proof). (a) *For any $x \in W'_{\mathscr{L}}$, the element $R_{\alpha_x} \in \mathscr{K}_0(G)$ is the class of a semisimple perverse sheaf on G which is a direct summand of $(-1)^{l(x)-a(x)} d({}^pH^{\dim G + l(x)-a(x)}(\bar{K}^{\mathscr{L}}_x))$ (which is itself realizable as a semisimple perverse sheaf on G, by (15.13.2)).*

(b) *For any* $x \in W'_{\mathscr{L}}$, *the element* $(-1)^{l(x)-a(x)} R_{\mathscr{A}_x} \in \mathscr{K}_0(G)$ *is the class of a semisimple perverse sheaf on G which is a direct summand of* ${}^pH^{\dim G + l(x) - a(x)}(\bar{K}_x^{\mathscr{L}})$.

References

Part I

1. A. A. Beilinson, J. Bernstein, and P. Deligne, Faisceaux pervers, *Astérisque* **100** (1982).
2. P. Deligne, La conjecture de Weil, II, *Publ. Math. IHES* **52** (1979), 138–252.
3. P. Deligne and G. Lusztig, Representations of reductive groups over finite fields, *Ann. of Math.* **103** (1976), 103–161.
4. G. Lusztig, Intersection cohomology complexes on a reductive group, *Invent. Math.* **75** (1984), 205–272.

Part II

5. G. Lusztig, Character sheaves, I, *Adv. in Math.* **56** (1985), 193–237.
6. G. Lusztig, Characters of reductive groups over a finite field, Ann. of Math. Studies 107, Princeton Univ. Press, Princeton, N.J., 1984.
7. T. A. Springer, Trigonometric sums, Green functions of finite groups and representations of Weyl groups, *Invent. Math.* **36** (1976), 173–207.

Part III

8. D. Alvis, The duality operation in the character ring of a finite Chevalley group, *Bull. Amer. Math. Soc.* **1** (1979), 907–911.
9. C. W. Curtis, Truncation and duality in the character ring of a finite group of Lie type, *J. Algebra* **62** (2), (1980), 320–332.
10. F. Digne and J. Michel, Descente de Shintani des caractères d'un groupe de Chevalley fini, *C. R. Acad. Sci. Paris Sér. A* **291** (1980), 571–574.
11. R. B. Howlett and R. W. Kilmoyer, Principal series representations of finite groups with split *BN* pairs, *Comm. Algebra* **8** (6)(1980), 543–583.
12. D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.
13. G. Lusztig, Character sheaves, II, *Adv. in Math.* **57** (1985), 226–265.
14. G. Lusztig, Unipotent characters of the symplectic and odd orthogonal groups over a finite field, *Invent. Math.* **64** (1981), 263–296.
15. G. Lusztig, Cells in affine Weyl groups, *in* "Algebraic groups and related topics," (R. Hotta, ed.) Advanced Studies in Pure Math., vol. 6, 1985, North Holland and Kinokunia.

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Character Sheaves, IV

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Contents. 17. Parametrization of $\hat{G}_{\mathscr{L}}$ (statement and first reductions). 18. Groups of type A. 19. Classical groups of low rank. 20. Groups of type E_6, E_7, and G_2. 21. Groups of type E_8 and F_4.

This paper is part of a series [5, 13, 17] devoted to the study of a class $\hat{G}$ of irreducible perverse sheaves (called character sheaves) on a connected reductive algebraic group G. (The numbering of chapters, sections and references continues that of [5, 13, 17].) This paper contains a classification of the character sheaves of G assuming that G is almost simple of type A or an exceptional group (in good characteristic). It is proved that such G are clean (in the sense of (13.9.2)), that they satisfy the parity condition (15.13), and that the class of character sheaves coincides with the class of admissible complexes defined in [4]. We also prove (for the groups in question) a multiplicity formula rather analogous to the main theorem (4.23) in [6]. The case of classical groups will be considered in part, V.

17. PARAMETRIZATION OF $\hat{G}_{\mathscr{L}}$ (STATEMENT AND FIRST REDUCTIONS)

17.1. In this chapter, $\mathbf{W}$ denotes the Weyl group of a root system with a fixed set $\mathbf{S}$ of simple reflections. An isomorphism of Weyl groups is always assumed to come from an isomorphism of root systems and to map simple reflections to simple reflections. This applies in particular to the group of automorphisms Aut $\mathbf{W}$ of $\mathbf{W}$.

17.2. Let $\sigma \in \operatorname{Aut} \mathbf{W}$ be an automorphism of order c and let $C_c\mathbf{W}$ be the semidirect product of the cyclic group C_c of order c with generator σ and $\mathbf{W}$ (with $\mathbf{W}$ normal and $\sigma w \sigma^{-1} = \sigma(w)$, $w \in \mathbf{W}$).

Let E be an irreducible $\mathbf{W}$-module (over $\bar{\mathbf{Q}}_l$) which is extendable to a $C_c\mathbf{W}$-module. Then E can be extended in c different ways to a $C_c\mathbf{W}$-

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module. We shall single out one particular extension $\tilde{E}$ and call it the *preferred extension*; this is done separately in the various cases.

(a) $\mathbf{W}$ irreducible, $\sigma = 1$. Take $\tilde{E} = E$.

(b) $\mathbf{W}$ of type A_n $(n \geqslant 2)$ or E_6 and $c = 2$. Define $\tilde{E}$ by the condition that $\sigma\colon E \to E$ acts as $(-1)^{a_E} \cdot w_0$ where w_0 is the longest element in $\mathbf{W}$ and a_E is as in 16.2.

(c) $\mathbf{W}$ of type D_4 and $c = 3$. Define $\tilde{E}$ to be the unique extension of E which is defined over Q [6, 3.2].

(d) $\mathbf{W}$ of type D_n $(n \geqslant 4)$ and $c = 2$. The irreducible representations of $C_2\mathbf{W}$ which remain irreducible upon restriction to $\mathbf{W}$ have been parametrized in [6, 4.18] by certain symbols with two rows (an upper row and a lower row) and an even number of entries; the two representations of $C_2\mathbf{W}$ which extend a given irreducible representation of $\mathbf{W}$ correspond to symbols which differ one from another only by interchanging the two rows. We say that $\tilde{E}$ is preferred if the corresponding symbol has the following property: the smallest entry which appears in only one row appears in the lower row. For example, the preferred extension of the unit representation of $\mathbf{W}$ is the unit representation of $C_2\mathbf{W}$; its symbol is $\binom{n}{0}$.

(e) Assume that $\mathbf{W} = \mathbf{W}_1 \times \mathbf{W}_2 \times \cdots \times \mathbf{W}_r$ with $\mathbf{W}_i$ irreducible Weyl groups and that σ permutes cyclically the factors $\mathbf{W}_i$: $\sigma(w_1, w_2, \ldots, w_r) = (\phi_r(w_r), \phi_1(w_1), \ldots, \phi_{r-1}(w_{r-1}))$ where $\phi_1\colon \mathbf{W}_1 \to \mathbf{W}_2$, $\phi_2\colon \mathbf{W}_2 \to \mathbf{W}_3, \ldots, \phi_{r-1}\colon \mathbf{W}_{r-1} \to \mathbf{W}_r$, and $\phi_r\colon \mathbf{W}_r \to \mathbf{W}_1$ are isomorphisms of Weyl groups (see 17.1). Then E can be written as an external tensor product $E = E_1 \boxtimes E_2 \boxtimes \cdots \boxtimes E_r$ where E_i are irreducible $\mathbf{W}_i$-modules $(1 \leqslant i \leqslant r)$. Since E is extendable to $C_c\mathbf{W}$, there exist isomorphisms of $\bar{\mathbf{Q}}_l$-vector spaces:

$$h_1\colon E_1 \to E_2,\ h_2\colon E_2 \to E_3, \ldots, h_{r-1}\colon E_{r-1} \to E_r,\ h_r\colon E_r \to E_1$$

such that

$$h_i(w_i e_i) = \phi_i(w_i)\, h_i(e_i) \qquad (\forall w_i \in \mathbf{W}_i,\ e_i \in E_i).$$

for $1 \leqslant i \leqslant r$. (These isomorphisms are unique up to non-zero scalars.) We normalize them in such a way that $h_{r-1} \ldots h_2 h_1 h_r\colon E_r \to E_r$ defines a preferred extension (see (a)–(d) above) of the W_r-module E_r to a $C_{c/r}\mathbf{W}_r$-module where the generator of $C_{c/r}$ acts on $\mathbf{W}_r$ as $\phi_{r-1} \ldots \phi_2 \phi_1 \phi_r\colon \mathbf{W}_r \to \mathbf{W}_r$. (Note that E_r is extendable to $C_{c/r}\mathbf{W}_r$ since E is extendable to $C_c\mathbf{W}$.) We then define $\tilde{E}$ to be the extension of E to a $C_c\mathbf{W}$ module for which $\sigma\colon E \to E$ is given by

$$\sigma(e_1 \boxtimes e_2 \boxtimes \cdots \boxtimes e_r) = h_r(e_r) \boxtimes h_1(e_1) \boxtimes \cdots \boxtimes h_{r-1}(e_{r-1}) \qquad (e_i \in E_i).$$

This is independent of the choices of h_i, by the normalization condition for $h_{r-1} \dots h_2 h_1 h_r$.

(f) In the general case, we can write uniquely $\mathbf{W} = \mathbf{W}^1 \times \mathbf{W}^2 \times \cdots \times \mathbf{W}^t$ and $E = E^1 \boxtimes E^2 \boxtimes \cdots \boxtimes E^t$ where $\mathbf{W}^i$ are σ-stable Weyl subgroups , E^i are irreducible $\mathbf{W}^i$-modules and $\mathbf{W}^i$ and E^i satisfy the conditions of (e) for all i. The preferred extension $\tilde{E}$ is given by the map σ: $E \to E$ which is the external tensor product of the maps $\sigma^{(i)}: E^i \to E^i$ defined as in (e).

(g) When $\mathbf{W}$ is of type D_2 or D_3 and $c = 2$ then the method of (d) still gives a preferred extension of E to $C_2\mathbf{W}$. On the other hand, in the case D_2, the method of (e) gives a preferred extension and in the case $D_3 = A_3$ the method of (b) gives a preferred extension of E to $C_2\mathbf{W}$. It is easy to check that these definitions of the preferred extension coincide.

17.3. Now let $\mathbf{\Omega}$ be a finite abelian group with a given homomorphism $\mathbf{\Omega} \to \mathrm{Aut}\, \mathbf{W}$. This gives rise to the semidirect product $\mathbf{\Omega W}$, with $\mathbf{W}$ normal and $\sigma w \sigma^{-1} = \sigma(w)$ for $\sigma \in \mathbf{\Omega}$, $w \in \mathbf{W}$. The irreducible representations of $\mathbf{\Omega W}$ can be described as follows. Start with an irreducible representation E of $\mathbf{W}$. Let $\mathbf{\Omega}_E$ be the set of all $\sigma \in \mathbf{\Omega}$ such that E can be extended to a $C_c\mathbf{W}$-module. where C_c is the cyclic subgroup of Aut $\mathbf{W}$ generated by the image of σ in Aut $\mathbf{W}$. For $\sigma \in \mathbf{\Omega}_E$, there is a well defined map σ: $E \to E$ which gives rise to the preferred extension (17.2) of E to the group $C_c\mathbf{W}$ just considered. The maps σ: $E \to E$ for the various $\sigma \in \mathbf{\Omega}_E$ define an extension of E to a representation $\tilde{E}$ of the semidirect product $\mathbf{\Omega}_E\mathbf{W}$. Now let θ be an one-dimensional representation $\mathbf{\Omega}_E \to \bar{\mathbf{Q}}_l^*$; we regard it as a representation of $\mathbf{\Omega}_E\mathbf{W}$, trivial on $\mathbf{W}$, and we consider the induced representation $\tilde{E}_\theta = \mathrm{ind}_{\mathbf{\Omega}_E\mathbf{W}}^{\mathbf{\Omega W}} (\theta \boxtimes \tilde{E})$ of $\mathbf{\Omega W}$. It follows from Mackey's theorem that $\tilde{E}_\theta$ is irreducible and that the existence of an isomorphism $\tilde{E}_\theta \approx \tilde{E}'_{\theta'}$ (where E, E' are irreducible representations of $\mathbf{W}$) implies that E, E' are in the same $\mathbf{\Omega}$-orbit and that $\theta = \theta'$ as characters of $\mathbf{\Omega}_E = \mathbf{\Omega}_{E'}$.

17.4. The set $\hat{\mathbf{W}}$ of irreducible representations of $\mathbf{W}$ (up to isomorphism) is partitioned into families (see [6, 4.2]). By a result of Barbasch and Vogan [6, 5.25], $E, E' \in \hat{\mathbf{W}}$ are in the same family if and only if $E \sim_{LR} E'$ (see 16.2). More generally, the set $\widehat{\mathbf{\Omega W}}$ of irreducible representations of $\mathbf{\Omega W}$ (see 17.3) can be partitioned into families as follows. We say that two irreducible representations of $\mathbf{\Omega W}$ are in the same family if their restrictions to $\mathbf{W}$ each contain some irreducible component which is in the same family (of $\mathbf{W}$).

Thus we have a 1–1 correspondence between the set of families of $\mathbf{\Omega W}$ and the set of families of $\mathbf{W}$ modulo the obvious action of $\mathbf{\Omega}$: if $\mathscr{F}$ is a family of $\mathbf{W}$ and $\mathscr{F}'$ is the corresponding family of $\mathbf{\Omega W}$, then $\mathscr{F}'$ consists of the representations $\tilde{E}_\theta$, where E runs over the representations in $\mathscr{F}$ and θ runs over the characters of $\mathbf{\Omega}_E$ (see 17.3). Moreover, the set of families of $\mathbf{\Omega W}$ is in 1–1 correspondence with the set of two-sided cells $\mathbf{\Omega W}$ defined as in 16.3 (for $\mathbf{\Omega W}$ instead of $\Omega_{\mathscr{L}} W_{\mathscr{L}}$). It is characterized by the property: if $E \in \mathscr{F}'$ and $E_{\widetilde{LR}} x \in \mathbf{\Omega W}$ ($E_{\widetilde{LR}}$ defined as in 16.2) then $\mathscr{F}'$ corresponds to the two-sided cell containing x.

17.5. In [6, 4.4–4.13] we have attached to each family $\mathscr{F}$ of $\mathbf{W}$ (assumed irreducible) a finite group $\mathscr{G}_{\mathscr{F}}$, isomorphic to a symmetric group $\mathfrak{S}_n$ ($n \leqslant 5$) or to a product of cyclic groups of order 2. Moreover, we have defined an imbedding $\mathscr{F} \hookrightarrow \mathscr{M}(\mathscr{G}_{\mathscr{F}})$, where for any finite group $\mathscr{G}$, the set $\mathscr{M}(\mathscr{G})$ is defined as follows. $\mathscr{M}(\mathscr{G})$ consists of all pairs (x, τ) where x is an element of $\mathscr{G}$ and τ is an irreducible representation over $\bar{\mathbf{Q}}_l$ (up to isomorphism) of $Z_{\mathscr{G}}(x)$ modulo the equivalence relation $(x, \tau) \sim (gxg^{-1}, \tau^g)$ for any $g \in \mathscr{G}$, where τ^g is the irreducible representation of $Z_{\mathscr{G}}(gxg^{-1}) = gZ_{\mathscr{G}}(x)g^{-1}$ defined by composing τ with conjugation by g^{-1}.

This can be extended to the case where $\mathbf{W}$ is no longer assumed to be irreducible. Write $\mathbf{W} = \mathbf{W}_1 \times \mathbf{W}_2 \times \cdots \times \mathbf{W}_n$ with $\mathbf{W}_i$ irreducible. A family $\mathscr{F}$ of $\mathbf{W}$ is of the form $\mathscr{F}_1 \boxtimes \mathscr{F}_2 \boxtimes \cdots \boxtimes \mathscr{F}_n$ where $\mathscr{F}_i$ are families of $\mathbf{W}_i$. We define $\mathscr{G}_{\mathscr{F}}$ to be $\mathscr{G}_{\mathscr{F}_1} \times \mathscr{G}_{\mathscr{F}_2} \times \cdots \times \mathscr{G}_{\mathscr{F}_n}$. Then we have a natural bijection $\mathscr{M}(\mathscr{G}_{\mathscr{F}}) = \mathscr{M}(\mathscr{G}_{\mathscr{F}_1}) \times \mathscr{M}(\mathscr{G}_{\mathscr{F}_2}) \times \cdots \times \mathscr{M}(\mathscr{G}_{\mathscr{F}_n})$ (see [6, (4.3.1)]) and the product of the embeddings $\mathscr{F}_i \hookrightarrow \mathscr{M}(\mathscr{G}_{\mathscr{F}_i})$ gives the required imbedding $\mathscr{F} \hookrightarrow \mathscr{M}(\mathscr{G}_{\mathscr{F}})$.

The group $\mathscr{G}_{\mathscr{F}}$ is functorial in the following sense: an isomorphism of Weyl groups $\mathbf{W}_1$, $\mathbf{W}_2$ which takes a family $\mathscr{F}_1$ to a family $\mathscr{F}_2$ induces an isomorphism $\mathscr{G}_{\mathscr{F}_1} \to \mathscr{G}_{\mathscr{F}_2}$. We require that this isomorphism be compatible with the decomposition of $\mathbf{W}$ into a product of irreducible Weyl groups and the corresponding decomposition of $\mathscr{G}_{\mathscr{F}}$. Hence to define it we may assume that $\mathbf{W}_1 = \mathbf{W}_2$ is irreducible. If $\mathscr{F}_1 \neq \mathscr{F}_2$ then we have necessarily $\mathscr{G}_{\mathscr{F}_1} = \mathscr{G}_{\mathscr{F}_2} = \{e\}$ (and $\mathbf{W}_1$ if of type D_{2n}) so there is a unique isomorphism $\mathscr{G}_{\mathscr{F}_1} \xrightarrow{\sim} \mathscr{G}_{\mathscr{F}_2}$; if $\mathscr{F}_1 = \mathscr{F}_2$ we define $\mathscr{G}_{\mathscr{F}_1} \to \mathscr{G}_{\mathscr{F}_2}$ to be the identity map.

17.6. Now let $\mathscr{F}'$ be a family of $\mathbf{\Omega W}$, and let $\mathscr{F}$ be the corresponding family of $\mathbf{W}$ (defined up to the action of $\mathbf{\Omega}$). Let $\mathbf{\Omega}_{\mathscr{F}}$ be the stabilizer of $\mathscr{F}$ in $\mathbf{\Omega}$. Then $\mathbf{\Omega}_{\mathscr{F}}$ acts naturally on the group $\mathscr{G}_{\mathscr{F}}$ (by the functoriality of $\mathscr{G}_{\mathscr{F}}$). Using this action we construct the semidirect product $\mathbf{\Omega}_{\mathscr{F}} \mathscr{G}_{\mathscr{F}}$ (with $\mathscr{G}_{\mathscr{F}}$ normal). We define

$$(17.6.1) \quad \mathscr{G}_{\mathscr{F}'} = \mathbf{\Omega}_{\mathscr{F}} \mathscr{G}_{\mathscr{F}}.$$

Note that $\mathscr{F}$ is not uniquely determined by $\mathscr{F}'$ (only its $\mathbf{\Omega}$-orbit is). If $\mathscr{F}_1$ is another family of $\mathbf{W}$ in the Ω-orbit of $\mathscr{F}$, then any element $\sigma \in \mathbf{\Omega}$ which takes $\mathscr{F}$ to $\mathscr{F}_1$ defines an isomorphism $\mathscr{G}_{\mathscr{F}} \xrightarrow{\sim} \mathscr{G}_{\mathscr{F}_1}$ (by the functoriality of $\mathscr{G}_{\mathscr{F}}$) and hence an isomorphism i_σ: $\mathbf{\Omega}_{\mathscr{F}}\mathscr{G}_{\mathscr{F}} \to \mathbf{\Omega}_{\mathscr{F}_1}\mathscr{G}_{\mathscr{F}_1}$, which is the identity on $\mathbf{\Omega}_{\mathscr{F}}$; note that $\mathbf{\Omega}_{\mathscr{F}} = \mathbf{\Omega}_{\mathscr{F}_1}$ since $\mathbf{\Omega}$ is abelian. If we replace $\sigma \in \mathbf{\Omega}$ by $\sigma\sigma_0$ where $\sigma_0 \in \mathbf{\Omega}_{\mathscr{F}}$ then $i_{\sigma\sigma_0} = i_\sigma \circ i_{\sigma_0}$ where i_{σ_0}: $\mathbf{\Omega}_{\mathscr{F}}\mathscr{G}_{\mathscr{F}} \to \mathbf{\Omega}_{\mathscr{F}}\mathscr{G}_{\mathscr{F}}$ is conjugation by σ_0, hence an inner automorphism. Now i_σ, $i_{\sigma\sigma_0}$ induce isomorphisms j_σ, $j_{\sigma\sigma_0}$: $\mathscr{M}(\mathbf{\Omega}_{\mathscr{F}}\mathscr{G}_{\mathscr{F}}) \to \mathscr{M}(\mathbf{\Omega}_{\mathscr{F}_1}\mathscr{G}_{\mathscr{F}_1})$. W have $j_{\sigma\sigma_0} = j_\sigma j_{\sigma_0}$ where j_{σ_0}: $\mathscr{M}(\mathbf{\Omega}_{\mathscr{F}}\mathscr{G}_{\mathscr{F}}) \to \mathscr{M}(\mathbf{\Omega}_{\mathscr{F}}\mathscr{G}_{\mathscr{F}})$ is induced by an inner automorphism of $\mathbf{\Omega}_{\mathscr{F}}\mathscr{G}_{\mathscr{F}}$ and hence $j_{\sigma_0} =$ identity. Thus $j_{\sigma\sigma_0} = j_\sigma$. It follows that

(17.6.2) $\mathscr{G}_{\mathscr{F}}$ is well defined up to isomorphism and $\mathscr{M}(\mathscr{G}_{\mathscr{F}})$ is well defined up to unique isomorphism.

17.7. For any family $\mathscr{F}'$ of $\mathbf{\Omega}\mathbf{W}$, we consider the subset

$$\mathscr{M}_0(\mathscr{G}_{\mathscr{F}'}) = \{(x', \tau') \in \mathscr{M}(\mathscr{G}_{\mathscr{F}'}) \mid x' \in \mathscr{G}_{\mathscr{F}}\}$$

of $\mathscr{M}(\mathscr{G}_{\mathscr{F}'})$, with notations in 17.6. We shall define an imbedding

(17.7.1) $\mathscr{F}' \hookrightarrow \mathscr{M}_0(\mathscr{G}_{\mathscr{F}'})$.

We choose a family $\mathscr{F}$ of $\mathbf{W}$ as in 17.6. For simplicity, we shall write $\mathscr{G}$ instead of $\mathscr{G}_{\mathscr{F}}$ and $\mathscr{G}'$ instead of $\mathscr{G}_{\mathscr{F}'}$. Let $E \in \mathscr{F}$ and let θ be a one-dimensional representation of $\mathbf{\Omega}_E$ (see 17.3). We want to associate to $\tilde{E}_\theta$ (see 17.3) an element of $\mathscr{M}_0(\mathscr{G}')$. The imbedding $\mathscr{F} \hookrightarrow \mathscr{M}(\mathscr{G})$ (see 17.5) associates to E an element $(x, \tau) \in \mathscr{M}(\mathscr{G}_{\mathscr{F}})$.

We write $\mathbf{W} = \mathbf{W}_1 \times \cdots \times \mathbf{W}_n$ with $\mathbf{W}_i$ irreducible. Accordingly, we have $\mathscr{G} = \mathscr{G}_1 \times \cdots \times \mathscr{G}_n$ where $\mathscr{G}_i = \mathscr{G}_{\mathscr{F}_i}$ and $\mathscr{F}_i$ is a family in $\mathbf{W}_i$. The group $\mathbf{\Omega}$ acts on $\mathbf{W}$ and induces a permutation of the set of indices $[1, n]$. By functoriality of $\mathscr{G}_{\mathscr{F}}$, we may identify canonically $\mathscr{G}_i$, $\mathscr{G}_j$ for i, j in the same orbit of $\Omega_{\mathscr{F}}$ on $[1, n]$. Then the action of $\mathbf{\Omega}_{\mathscr{F}}$ on $\mathscr{G}$ is simply by permutations of the n coordinates. Let $\bar{\mathscr{G}}_i$ be a set of representatives for the conjugacy classes in $\mathscr{G}_i$; we may assume that $\bar{\mathscr{G}}_i = \bar{\mathscr{G}}_j$ whenever $\mathscr{G}_i = \mathscr{G}_j$. We may assume that $x = (x_1, \ldots, x_n) \in \mathscr{G}$ satisfies $x_i \in \bar{\mathscr{G}}_i$.

Let $\mathbf{\Omega}_x$ be the centralizer of x in $\mathbf{\Omega}_{\mathscr{F}}$. Then $\mathbf{\Omega}_x$ normalizes $Z_{\mathscr{G}}(x)$. Moreover, $Z_{\mathscr{G}'}(x)$ is the semidirect product $\mathbf{\Omega}_x Z_{\mathscr{G}}(x)$. Indeed, let $\sigma\gamma \in Z_{\mathscr{G}'}(x)$, $\sigma \in \mathbf{\Omega}_{\mathscr{F}}$, $\gamma \in \mathscr{G}$. Then $\sigma^{-1}x\sigma = \gamma x \gamma^{-1}$. We have $\sigma^{-1}x\sigma = (x_{a(1)}, x_{a(2)}, \ldots, x_{a(n)})$ where a is a permutation of $[1, n]$. It follows that $x_{a(i)} = \gamma_i x_i \gamma_i^{-1}$ ($\gamma_i \in \mathscr{G}_i$) for all i. Since $x_{a(i)}, x_i \in \bar{\mathscr{G}}_i$, it follows that $x_{a(i)} = x_i$ for all i. Thus, we have $\sigma^{-1}x\sigma = x = \gamma x \gamma^{-1}$, hence $Z_{\mathscr{G}'}(x) \subset \mathbf{\Omega}_x Z_{\mathscr{G}}(x)$. The reverse inclusion is trivial. The group $\mathbf{\Omega}_x$ normalizes $Z_{\mathscr{G}}(x)$, hence it acts on the set of irreducible representations of $Z_{\mathscr{G}}(x)$. We now show that

(17.7.2) The stabilizer of τ in $\mathbf{\Omega}_x$ is equal to $\mathbf{\Omega}_E$.

Let $\sigma \in \mathbf{\Omega}_{\mathscr{F}}$. Since the imbedding $\mathscr{F} \subsetneq \mathscr{M}(\mathscr{G})$ is functorial, it maps E^σ to $(\sigma(x), \tau^\sigma)$, where E^σ, τ^σ have an obvious meaning. If $E^\sigma = E$ then $(\sigma(x), \tau^\sigma) = (x, \tau)$ in $\mathscr{M}(\mathscr{G})$. In particular, $\sigma(x)$ is conjugate to x in $\mathscr{G}$. As we have seen earlier this implies $\sigma(x) = x$, i.e., $\sigma \in \mathbf{\Omega}_x$. We also have $\tau^\sigma = \tau$, hence σ stablizes τ. Conversely, if $\sigma \in \mathbf{\Omega}_x$ stablizes τ, then we see that E and E^σ have the same image under $\mathscr{F} \to \mathscr{M}(\mathscr{G})$. Since this map is injective, we have $E = E^\sigma$, hence $\sigma \in \mathbf{\Omega}_E$ and (17.7.2) is proved.

We can write $Z_G(x) = Z_{\mathscr{G}_1}(x_1) \times \cdots \times Z_{\mathscr{G}_n}(x_n)$, and $\mathbf{\Omega}_E$ acts on $Z_{\mathscr{G}}(x)$ by permuting the coordinates. We have $\tau = \tau_1 \boxtimes \cdots \boxtimes \tau_n$ where τ_i are irreducible $Z_{\mathscr{G}_i}(x_i)$-modules and we may assume that $\tau_i = \tau_j$ for i, j in the same $\mathbf{\Omega}_E$-orbit on $[1, n]$. Hence τ extends naturally to a $\mathbf{\Omega}_E Z_{\mathscr{G}}(x)$-module $\tilde{\tau}$; an element of $\mathbf{\Omega}_E$ acts on τ by permutations of components of a tensor. We now consider the $Z_{\mathscr{G}'}(x)$-module $\tilde{\tau}_\theta = \operatorname{Ind}_{\mathbf{\Omega}_E Z_{\mathscr{G}}(x)}^{Z_{\mathscr{G}'}(x)} (\theta \boxtimes \tilde{\tau})$. (Recall that $Z_{\mathscr{G}'}(x) = \mathbf{\Omega}_x Z_{\mathscr{G}}(x) \supset \mathbf{\Omega}_E Z_{\mathscr{G}}(x)$.) Here θ is regarded as a character of $\mathbf{\Omega}_E Z_{\mathscr{G}}(x)$, trivial on $Z_{\mathscr{G}}(x)$. Fom (17.7.2) and Mackey's theorem it follows that $\tilde{\tau}_\theta$ is irreducible. We now define the map (17.7.1) by $\tilde{E}_\theta \mapsto (x, \tilde{\tau}_\theta)$. It is easy to see that it is well defined and injective.

17.8. Let us fix $\mathscr{L} \in \mathscr{S}(T)$; recall that T is a maximal torus of G. In the discussion of 17.1–17.7, we take $\mathbf{W} = W_{\mathscr{L}}$, $\mathbf{\Omega} = \Omega_{\mathscr{L}}$ (see 5.1) so that $\Omega_{\mathscr{L}} W_{\mathscr{L}} = W'_{\mathscr{L}}$. The imbeddings (17.7.1) give rise together to an imbedding

$$(17.8.1) \quad \hat{W}'_{\mathscr{L}} \subsetneq \coprod_{\mathscr{F}'} \mathscr{M}_0(\mathscr{G}_{\mathscr{F}'}) \subsetneq \coprod_{\mathscr{F}'} \mathscr{M}(\mathscr{G}_{\mathscr{F}'})$$

(disjoint union over all families $\mathscr{F}'$ of $W'_{\mathscr{L}}$); the restriction of this imbedding to $\mathscr{F}'$ is just (17.7.1). We denote by m_E the image of $E \in \hat{W}'_{\mathscr{L}}$ under (17.8.1); it is an element of $\mathscr{M}(\mathscr{G}_{\mathscr{F}'})$ for some $\mathscr{F}'$.

Consider the pairing $\{\ ,\ \}$ on $\mathscr{M}(\mathscr{G}_{\mathscr{F}'})$ defined by

$$(17.8.2) \quad \{(x, \tau), (x', \tau')\} = \sum_{\substack{g \in \mathscr{G}_{\mathscr{F}'} \\ xgx'g^{-1} = gx'g^{-1}x}} |Z(x)|^{-1} |Z(x')|^{-1} \operatorname{Tr}(g^{-1}x^{-1}g, \tau') \operatorname{Tr}(gx'g^{-1}, \tau)$$

(see [6, (4.14.3)]).

We extend it to a pairing $\{\ ,\ \}$ on $\coprod_{\mathscr{F}'} \mathscr{M}(\mathscr{G}_{\mathscr{F}'})$ as follows. If m, m' are in the same piece $\mathscr{M}(\mathscr{G}_{\mathscr{F}'})$ then $\{m, m'\}$ is given by (17.8.2); otherwise, $\{m, m'\}$ is defined to be zero.

We consider the following statements for $(\mathscr{G}, \mathscr{L})$.

(17.8.3) *There exists a bijection* $\hat{G}_{\mathscr{L}} \leftrightarrow \coprod_{\mathscr{F}'} \mathscr{M}(\mathscr{G}_{\mathscr{F}'})$, $(A \leftrightarrow \bar{m}_A)$, *($\mathscr{F}'$ runs over all families in $W'_{\mathscr{L}}$) such that*

$$(A : R_E^{\mathscr{L}}) = \hat{\varepsilon}_A \{\bar{m}_A : m_E\}$$

for all $A \in \hat{G}_{\mathscr{L}}$ *and all* $E \in \hat{W}'_{\mathscr{L}}$. *(Here* $\hat{\varepsilon}_A = \pm 1$ *is defined as in* (15.11.2) *and* $R_E^{\mathscr{L}}$ *is defined in* (14.10.3).)

(17.8.4) *G is clean and any character sheaf* $A \in \hat{G}_{\mathscr{L}}$ *satisfies the condition* $\varepsilon_A = \hat{\varepsilon}_A$ *(see* (15.13.1)).

We shall also consider the following statement for G.

(17.8.5) *Any irreducible cuspidal perverse sheaf on G is a character sheaf (see* (7.1.1)).

We would like to prove that the statements (17.8.3)–(17.8.5) are always true. In this paper we shall verify them in the case where G is of type A or an almost simple exceptional group (with some restrictions on the characteristic on k).

In Sections 17.9–17.16, we shall give some reductions of the statements (17.8.3)–(17.8.5).

17.9. Let $\mathscr{E}$ be a local system of rank 1 on G which is the inverse image under $G \to G/G_{\mathrm{der}}$ of a local system $\mathscr{E}_0 \in \mathscr{S}(G/G_{\mathrm{der}})$. (Here G_{der} is the derived group of G, hence G/G_{der} is a torus. The class of local systems $\mathscr{S}(G/G_{\mathrm{der}})$ is defined just as $\mathscr{S}(T)$ in 2.2.) For each $\mathscr{L} \in \mathscr{S}(T)$ we define $\mathscr{L} \otimes \mathscr{E}$ as the tensor product of $\mathscr{L}$ and the restriction $\mathscr{E} \mid T$; then $\mathscr{L} \otimes \mathscr{E} \in \mathscr{S}(T)$. It is clear that $W'_{\mathscr{L}} = W'_{\mathscr{L} \otimes \mathscr{E}}$ and that $\bar{K}_w^{\mathscr{L} \otimes \mathscr{E}} = \bar{K}_w^{\mathscr{L}} \otimes \mathscr{E}$ for all $w \in W'_{\mathscr{L}}$. It follows that

(17.9.1) $A \to A \otimes \mathscr{E}$ is a bijection $\hat{G}_{\mathscr{L}} \xrightarrow{\sim} \hat{G}_{\mathscr{L} \otimes \mathscr{E}}$,

(17.9.2) $R_E^{\mathscr{L} \otimes \mathscr{E}} = R_E^{\mathscr{L}} \otimes \mathscr{E}$, for all $E \in \hat{W}'_{\mathscr{L}}$,

(17.9.3) $(A : R_E^{\mathscr{L}}) = (A \otimes \mathscr{E} : R_E^{\mathscr{L} \otimes \mathscr{E}})$, for all $A \in \hat{G}_{\mathscr{L}}$ and all $E \in \hat{W}'_{\mathscr{L}}$,

(17.9.4) ${}^pH^i(\bar{K}_w^{\mathscr{L} \otimes \mathscr{E}}) = {}^pH^i(\bar{K}_w^{\mathscr{L}}) \otimes \mathscr{E}$, for all $w \in W'_{\mathscr{L}}$ and all i.

It follows that the statements (17.8.3), (17.8.4) hold for $(G, \mathscr{L})$ if and only if they hold for $(G, \mathscr{L} \otimes \mathscr{E})$.

17.10. Let $G' = G/Z_G^0$, $T' = T/Z_G^0$, $\mathscr{L}' \in \mathscr{S}(T')$, $\mathscr{L} =$ inverse image of $\mathscr{L}'$ under the canonical map $T \to T'$. Then $\mathscr{L} \in \mathscr{S}(T)$. Let π: $G \to G'$ be the canonical map. It is smooth, with connected fibres, hence $\tilde{\pi}$ takes irreducible perverse sheaves on G' to irreducible perverse sheaves on G (see 1.8). From the definitions it follows immediately that $\pi^*(\bar{K}_w^{\mathscr{L}'}) = \bar{K}_w^{\mathscr{L}}$ for all

$w \in W'_{\mathscr{L}} = W'_{\mathscr{L}'}$, where $\bar{K}_w^{\mathscr{L}'}$ is defined with respect to G'. From (1.8.1) it then follows that for all i, we have $\tilde{\pi}({}^pH^{i-\dim G'}\bar{K}_w^{\mathscr{L}'}) = {}^pH^{i-\dim G}(\bar{K}_w^{\mathscr{L}})$. From this and (1.8.1)–(1.8.4) we deduce

(17.10.1) $A' \to \tilde{\pi}A'$ is a bijection $\hat{G}'_{\mathscr{L}'} \xrightarrow{\sim} \hat{G}_{\mathscr{L}}$,

(17.10.2) $\tilde{\pi}R_{E'}^{\mathscr{L}'} = R_{E'}^{\mathscr{L}}$, for any $E' \in \hat{W}'_{\mathscr{L}'} = \hat{W}'_{\mathscr{L}}$,

(17.10.3) $(A' : R_{E'}^{\mathscr{L}'}) = (\tilde{\pi}A' : R_{E'}^{\mathscr{L}})$, for any $A' \in \hat{G}'_{\mathscr{L}}$, and any $E' \in \hat{W}'_{\mathscr{L}'} = \hat{W}'_{\mathscr{L}}$,

(17.10.4) $\varepsilon_{A'} = \varepsilon_{\tilde{\pi}A'}$, $\hat{\varepsilon}_{A'} = \hat{\varepsilon}_{\tilde{\pi}A'}$, for any $A' \in \hat{G}'_{\mathscr{L}'}$.

It follows that the statements (17.8.3), (17.8.4) hold for $(G', \mathscr{L}')$ if and only if they hold for $(G, \mathscr{L})$.

Since any local system in $\mathscr{S}(T)$ is of form $\mathscr{L} \otimes \mathscr{E}$ with $\mathscr{L}$ as above (coming from $\mathscr{L}'$) and $\mathscr{E}$ as in 17.9, we see, using 17.9, that the statements (17.8.3), (17.8.4) hold for G' and any $\mathscr{L}' \in \mathscr{S}(T')$ if and only if they hold for G and any $\mathscr{L}_1 \in \mathscr{S}(\mathrm{T})$. In the same way we see that (17.8.5) holds for G' if and only if it holds for G.

17.11. Assume now that G is a product $G_1 \times \cdots \times G_m$ where G_i are reductive connected groups over k. The character sheaves of G are precisely the complexes of form $A_1 \boxtimes \cdots \boxtimes A_m$ where A_i is a character sheaf on G_i for each i. If the statements (17.8.3)–(17.8.5) hold for each G_i then they hold also for G. (The proof is left to the reader.)

17.12. Let $\mathscr{L} \in \mathscr{S}(\mathrm{T})$ and let I be a subset of the set S of simple reflections in W such that $W'_{\mathscr{L}} = W'_{\mathscr{L},I}$ (notation of 15.6). Let L_I, $R_E^{\mathscr{L},I}$ be as in 15.6 and let i_I^S be as in 15.3. Assume that G is clean. We shall prove that

(17.12.1) i_I^S defines a bijection $(\hat{L}_I)_{\mathscr{L}} \xrightarrow{\sim} \hat{G}_{\mathscr{L}}$,

(17.12.2) $i_I^S(R_E^{\mathscr{L},I}) = R_E^{\mathscr{L}}$ for any $E \in \hat{W}'_{\mathscr{L}} = \hat{W}'_{\mathscr{L},I}$,

(17.12.3) $(i_I^S A : R_E^{\mathscr{L}}) = (A : R_E^{\mathscr{L},I})$ for any $A \in (\hat{L}_I)_{\mathscr{L}}$ and any $E \in \hat{W}'_{\mathscr{L}}$,

(17.12.4) If $I \neq S$, then $\hat{G}_{\mathscr{L}}$ contains no cuspidal character sheaves.

From (15.3.4) and (15.3.2) we see that

(17.12.5) $(i_I^S A : i_I^S A') = \sum_x (r^I_{x^{-1}Ix \cap I} A : \gamma_x r^I_{I \cap xIx^{-1}} A')$

where $A, A' \in (\hat{L}_I)_{\mathscr{L}}$ and the notations are as in (15.3.4) with $I = J$. Consider the term in the last sum corresponding to a fixed x; assume that it is non-zero. From the proof of 15.7, we see that

$$r^I_{x^{-1}Ix \cap I} A = \text{ sum of character sheaves in } (\hat{L}_{x^{-1}Ix \cap I})_{\mathscr{L}}$$

for various $y \in W_{x^{-1}Ix \cap I}$,

$$\gamma_x(r^I_{I \cap xIx^{-1}} A') = \text{ sum of character sheaves in } (\hat{L}_{x^{-1}Ix \cap I})_{x^{-1}z\mathscr{L}}$$

for various $z \in W_{I \cap xIx^{-1}}$. Since our term is non-zero, it follows that there exist y, z as above such that $(\hat{L}_J)_{\mathscr{L}_1}$, $(\hat{L}_J)_{\mathscr{L}_2}$ are not disjoint, where $J = x^{-1}Ix \cap I$, $\mathscr{L}_1 = {}^y\mathscr{L}$, $\mathscr{L}_2 = {}^{x^{-1}z}\mathscr{L}$. Using now 11.2(c), it follows that $\mathscr{L}_1$, $\mathscr{L}_2$ are in the same W_J-orbit. It follows that $uy = x^{-1}zv$ for some $u \in W_{x^{-1}Ix \cap I}$, $v \in W'_{\mathscr{L}}$. We have y, z, $u \in W_I$ and $v \in W'_{\mathscr{L}} = W'_{\mathscr{L},I}$ (by assumption), hence $v \in W_I$ and $x \in W_I$. Since x has minimal length in $W_I x W_I$, we must have $x = 1$. Hence (17.12.5) simplifies to

(17.12.6) $(i^S_A A : i^S_I A') = (A : A')$.

This implies that i^S_I: $(\hat{L}_I)_{\mathscr{L}} \to \hat{G}_{\mathscr{L}}$ is well defined and injective. Now (17.12.2) follows from 15.7(a) and our assumption $W'_{\mathscr{L}} = W'_{\mathscr{L},I}$. Consider $\tilde{A} \in \tilde{G}_{\mathscr{L}}$. Then $(\tilde{A} : R^{\mathscr{L}}_{E}) \neq 0$ for some $E \in \hat{W}'_{\mathscr{L}}$ (see 14.12). Using (17.12.2), we then have $(\tilde{A} : i^S_I R^{\mathscr{L},I}_{E}) \neq 0$. Hence there exists $A_1 \in (\hat{L}_I)_{\mathscr{L}}$ such that $(\tilde{A} : i^S_I A_1) \neq 0$. Since $i^S_I A_1$ is a character sheaf, we must have $\tilde{A} = i^S_I A_1$, and (17.12.1) is proved. Now (17.12.3) follows from (17.12.2) and (17.12.6). Finally, (17.12.4) clearly follows from (17.12.1). From (17.12.1), (17.2.3), and 15.12 we deduce that

(17.12.7) *if the statements* (17.8.3), (17.8.4) *hold for* $(L_I, \mathscr{L})$ *then they also hold for* $(G, \mathscr{L})$ *(assuming that G is clean and that* $W'_{\mathscr{L}} = W'_{\mathscr{L},I}$*).*

17.13. We preserve the notations in 17.12, but we drop the assumption $W'_{\mathscr{L}} = W'_{\mathscr{L},I}$. For each $E_1 \in \widehat{W'_{\mathscr{L},I}}$ we define a $W'_{\mathscr{L}}$-module $J(E_1) = J^{W'_{\mathscr{L}}}_{W'_{\mathscr{L},I}}(E_1)$ as the $W'_{\mathscr{L}}$-submodule of $\operatorname{ind}^{W'_{\mathscr{L}}}_{W'_{\mathscr{L},I}}(E_1)$ generated by all irreducible $W'_{\mathscr{L}}$-submodules E satisfying $a_E = a_{E_1}$; for any irreducible submodule E of $\operatorname{ind}^{W'_{\mathscr{L}}}_{W'_{\mathscr{L},I}}(E_1)$, we have $a_E \geqslant a_{E_1}$ [6, (4.1.5)]. (Here a_E is defined with respect to $W'_{\mathscr{L}}$ and a_{E_1} is defined with respect to $W'_{\mathscr{L},I}$ (see 16.2).) We extend J by linearity to a homomorphism J: $\mathscr{R}(W'_{\mathscr{L},I}) \to \mathscr{R}(W'_{\mathscr{L}})$ or J: $\mathscr{R}(W'_{\mathscr{L},I}) \otimes \bar{\mathbf{Q}}_l \to \mathscr{R}(W'_{\mathscr{L}}) \otimes \bar{\mathbf{Q}}_l$ (notation of (14.10.3)). From the definition of families in [6, 4.2] and 17.4, it follows that given a family $\mathscr{F}_0$ of $W'_{\mathscr{L},I}$ there is a unique family $\mathscr{F}$ of $W'_{\mathscr{L}}$ with the following property: for any $E_1 \in \mathscr{F}_0$, any irreducible W'-submodule of $J(E_1)$ is in $\mathscr{F}$. We say that $\mathscr{F}$ is the family of $W'_{\mathscr{L}}$ *induced* by $\mathscr{F}_0$.

We shall make the following assumption:

(17.13.1) the statement (17.8.4) holds both for $(G, \mathscr{L})$ and for $(L_I, \mathscr{L})$.

We then have a partition

(17.13.2) $\hat{G}_{\mathscr{L}} = \coprod_{\mathscr{F}} \hat{G}_{\mathscr{L},\mathscr{F}}$

($\mathscr{F}$ runs over the families in $\hat{W}'_{\mathscr{L}}$): we say that $A \in \hat{G}_{\mathscr{L}}$ is in $\hat{G}_{\mathscr{L},\mathscr{F}}$ if the two-sided cell in $W'_{\mathscr{L}}$ attached in 16.7 to A corresponds by 17.4 to $\mathscr{F}$. We define similarly a subset $(\hat{L}_I)_{\mathscr{L},\mathscr{F}_0}$ for any family $\mathscr{F}_0$ in $\hat{W}'_{\mathscr{L},I}$.

We now define for any character sheaf $A_1 \in (\hat{L}_I)_{\mathscr{L},\mathscr{F}_0}$, ($\mathscr{F}_0$ as above) an element $j_I^S(A_1) \in \mathscr{K}_0(G)$ as follows. Let $\mathscr{F}$ be the family in $\hat{W}'_{\mathscr{L}}$ induced by $\mathscr{F}_0$. Consider $i_I^S(A_1)$ (see 15.3); it is a linear combination $\sum_A m_A A$ of character sheaves $A \in \hat{G}_{\mathscr{L}}$ with integral, $\geqslant 0$ coefficients m_A (see 4.8(b)).

We set, by definition, $j_I^S(A_1) = \sum m_A A$, sum over all $A \in \hat{G}_{\mathscr{L},\mathscr{F}}$. We also define by linearity $j_I^S(x)$ for any element $x \in \mathscr{K}_0(L_I) \otimes \bar{\mathbf{Q}}_l$ which is a $\bar{\mathbf{Q}}_l$-linear combination of character sheaves in $(\hat{L}_I)_{\mathscr{L},\mathscr{F}_0}$. From 16.7 and 15.7(a) it follows immediately that

(17.13.3) $j_I^S(R_{E_1}^{\mathscr{L},I}) = R_{J(E_1)}^{\mathscr{L}}$,

for any $E_1 \in \mathscr{R}(W'_{\mathscr{L},I}) \otimes \bar{\mathbf{Q}}_l$ which is a $\bar{\mathbf{Q}}_l$-linear combination of representations in $\mathscr{F}_0$. (Here $R_{E_1}^{\mathscr{L},I} \in \mathscr{K}_0(L_I) \otimes \bar{\mathbf{Q}}_l$, $R_{J(E_1)}^{\mathscr{L}} \in \mathscr{K}_0(G) \otimes \bar{\mathbf{Q}}_l$, and J: $\mathscr{R}(W'_{\mathscr{L},I}) \otimes \bar{\mathbf{Q}}_l \to \mathscr{R}(W'_{\mathscr{L}}) \otimes \bar{\mathbf{Q}}_l$ is defined as above.)

It is easy to see that for any $w \in W'_{\mathscr{L},I}$ in the two-sided cell corresponding (17.4) to $\mathscr{F}_0$, we have

(17.13.4) $J(\alpha_{w,W'_{\mathscr{L},I}}) = \alpha_{w,W'_{\mathscr{L}}}$

where $\alpha_{w,W'_{\mathscr{L},I}}$ (resp. $\alpha_{w,W'_{\mathscr{L}}}$) is the element α_w of (16.2.7) defined with respect to $W'_{\mathscr{L},I}$(resp. $W'_{\mathscr{L}}$). (Compare [6, [5.10.5)].)

Introducing this into (17.13.3) we get

(17.13.5) $j_I^S(R^{\mathscr{L},I}_{\alpha_{w,W'_{\mathscr{L},I}}}) = R^{\mathscr{L}}_{\alpha_{w,W'_{\mathscr{L}}}}$.

Next, we note that if $E \in R(W'_{\mathscr{L}}) \otimes \bar{\mathbf{Q}}_l$ is a $\bar{\mathbf{Q}}_l$-linear combination of representations in $\mathscr{F}$ and $A_1 \in (\hat{L}_I)_{\mathscr{L},\mathscr{F}_0}$ then

(17.13.6) $(j_I^S(A_1) : R_E^{\mathscr{L}}) = (A_1 : R^{\mathscr{L},I}_{{}^tJ(E)})$.

(Here ${}^tJ$: $\mathscr{R}(W'_{\mathscr{L}}) \otimes \bar{\mathbf{Q}}_l \to \mathscr{R}(W'_{\mathscr{L},I}) \otimes \bar{\mathbf{Q}}_l$ is the linear map defined by: coefficient of E_1 in ${}^tJ(E)$ = coefficient of E in $J(E_1)$, ($E_1 \in \hat{W}'_{\mathscr{L},I}$, $E \in \hat{W}'_{\mathscr{L}}$).) Indeed, by the definition of j_I^S, the left-hand side of (17.13.6) is equal to $(i_I^S(A_1) : R_E^{\mathscr{L}})$. Similarly, the right-hand side of (17.13.6) is equal to $(A_1$: $R^{\mathscr{L},I}_{\mathrm{res}\,E}) = (A_1 : r_I^S R_E^{\mathscr{L}})$ (see 15.7(c)). It remains to use (15.3.2).

In the remainder of this section we shall assume (in addition to (17.13.1)) that $\mathscr{F}$, $\mathscr{F}_0$ have the following property. The map $E_1 \to J(E_1)$ is a bijection $J: \mathscr{F}_0 \xrightarrow{\sim} \mathscr{F}$; moreover, there is an isomorphism $\mathscr{G}_{\mathscr{F}_0} \xrightarrow{\sim} \mathscr{G}_{\mathscr{F}}$ (see 17.5) such that the diagram

$$\begin{tikzcd} \mathscr{F}_0 \arrow[r, "\sim", "J"'] \arrow[d, hook, "(17.7.1)"'] & \mathscr{F} \arrow[d, hook, "(17.7.1)"] \\ \mathscr{M}_0(\mathscr{G}_{\mathscr{F}_0}) \arrow[r, "\sim"] & \mathscr{M}_0(\mathscr{G}_{\mathscr{F}}) \end{tikzcd}$$

is commutative. (The bottom arrow is induced by $\mathscr{G}_{\mathscr{F}_0} \xrightarrow{\sim} \mathscr{G}_{\mathscr{F}}$.) In this case we shall say that $\mathscr{F}$ is *smoothly induced* by $\mathscr{F}_0$.

Under these assumptions we shall prove the following.

(17.13.7) *If* (17.8.3) *holds for* $(L_I, \mathscr{L})$, *then* (17.8.3) *also holds for* $(G, \mathscr{L})$ *as far as* $\hat{G}_{\mathscr{L},\mathscr{F}}$ *is concerned; in other words, there exists a bijection* $\hat{G}_{\mathscr{L},\mathscr{F}} \leftrightarrow \mathscr{M}(\mathscr{G}_{\mathscr{F}})$, $(A \leftrightarrow \bar{m}_A)$ *such that the equality in* (17.8.3) *for* $(G, \mathscr{L})$ *holds for all* $A \in \hat{G}_{\mathscr{L},\mathscr{F}}$ *and all* $E \in \mathscr{F}$.

More precisely, we shall prove

(17.13.8) j_I^S *defines a bijection* $(\hat{L}_I)_{\mathscr{L},\mathscr{F}_0} \xrightarrow{\sim} \hat{G}_{\mathscr{L},\mathscr{F}}$.

We can then define the bijection in (17.13.7) in such a way that the diagram

$$\begin{tikzcd} (\hat{L}_I)_{\mathscr{L},\mathscr{F}_0} \arrow[r, "j_I^S", "\sim"'] \arrow[d, "\wr"] & \hat{G}_{\mathscr{L},\mathscr{F}} \arrow[d, "\wr"] \\ \mathscr{M}(\mathscr{G}_{\mathscr{F}_0}) \arrow[r, "\sim"] & \mathscr{M}(\mathscr{G}_{\mathscr{F}}) \end{tikzcd}$$

is commutative. (The left vertical arrow is provided by (17.8.3) for $(L_I, \mathscr{L})$.) This bijection has the required property, by (17.13.6).

It remains to prove (17.13.8). First, we show that $j_I^S(A_1) \neq 0$ for all $A_1 \in (\hat{L}_I)_{\mathscr{L},\mathscr{F}_0}$. If we had $j_I^S(A_1) = 0$ then, from (17.13.6), it would follows that $(A_1 : R_{t_{J(E)}}^{\mathscr{L},I}) = 0$ for all $E \in \hat{W}'_{\mathscr{L}}$. From our assumption $J: \mathscr{F}_0 \xrightarrow{\sim} \mathscr{F}$ it follows that $(A_1 : R_{E_1}^{\mathscr{L},I}) = 0$ for all $E_1 \in \mathscr{F}_0$, contradicting 14.12. Thus, $j_I^S(A_1) \neq 0$ for all $A_1 \in (\hat{L}_I)_{\mathscr{L},\mathscr{F}_0}$.

We now fix $A_1 \in (\hat{L}_I)_{\mathscr{L},\mathscr{F}_0}$. From 14.12 and 16.4 it follows that there exists $w \in W'_{\mathscr{L},I}$ such that $(A_1 : R_{\alpha_w}^{\mathscr{L},I}) \neq 0$, where α_w is defined as in (16.2.7) with respect to $W'_{\mathscr{L},I}$. By 16.6 we have

(17.13.9) $R_{\alpha_w}^{\mathscr{L},I} = n_1 A_1 + n_2 A_2 + \cdots + n_r A_r$

where $A_i \in (\hat{L}_I)_{\mathscr{L},\mathscr{F}_0}$ are distinct $(1 \leqslant i \leqslant r)$ and $n_1, n_2, \ldots, n_r$ are integers > 0. Let $\tilde{\alpha}_w$ be the element α_w of (16.2.7) defined with respect to $W'_{\mathscr{L}}$. We have $J(\alpha_w) = \tilde{\alpha}_w$ (17.13.4); since $J: \mathscr{F}_0 \xrightarrow{\sim} \mathscr{F}$ it follows that

(17.13.10) ${}'J(\tilde{\alpha}_w) = \alpha_w +$ a linear combination of representations in $\hat{W}'_{\mathscr{L},I}$ not in $\mathscr{F}_0$.

We have

$$\begin{aligned}
\sum_{i,j=1}^{r} n_i n_j (j_I^S A_i : j_I^S A_j) &= (j_I^S(R_{\alpha_w}^{\mathscr{L},I}) : j_I^S(R_{\alpha_w}^{\mathscr{L},I})) && \text{(by (17.13.9))}\\
&= (j_I^S(R_{\alpha_w}^{\mathscr{L},I}) : R_{\tilde{\alpha}_w}^{\mathscr{L}}) && \text{(by (17.13.5))}\\
&= (R_{\alpha_w}^{\mathscr{L},I} : R_{{}'J(\tilde{\alpha}_w)}^{\mathscr{L},I}) && \text{(by (17.13.6))}\\
&= (R_{\alpha_w}^{\mathscr{L},I} : R_{\alpha_w}^{\mathscr{L},I}) && \text{(by (17.13.10))}\\
&= \sum_{i=1}^{r} n_i^2 && \text{(by (17.13.9)).}
\end{aligned}$$

On the other hand, $j_I^S A_i$ is a non-zero linear combination of character sheaves with integral $\geqslant 0$ coefficients. Hence $(j_I^S A_i : j_I^S A_j)$ is $\geqslant 1$ for $i = j$ and is $\geqslant 0$ for $i \neq j$. Hence from the equality

$$\sum_{i=1}^{r} n_i^2 = \sum_{i,j=1}^{r} n_i n_j (j_I^S A_i : j_I^S A_j)$$

it follows that $(j_I^S A_i : j_I^S A_j)$ is 1 if $i = j$ and is 0 if $i \neq j$. In particular, $j_I^S A_1$ is a character sheaf in $\hat{G}_{\mathscr{L},\mathscr{F}}$. Hence j_I^S defines a map $(\hat{L}_I)_{\mathscr{L},\mathscr{F}_0} \to \hat{G}_{\mathscr{L},\mathscr{F}}$. We show that this map is surjective. Let $A \in \hat{G}_{\mathscr{L},\mathscr{F}}$. By 14.12, there exists $E \in \mathscr{F}$ such that $(A : R_E^{\mathscr{L}}) \neq 0$. We have $E = J(E_1)$ for some $E_1 \in \mathscr{F}_0$, hence $0 \neq (A : R_{J(E_1)}^{\mathscr{L}}) = (A : j_I^S(R_{E_1}^{\mathscr{L},I}))$ (by (17.13.3)). Hence there exists $A_1 \in (\hat{L}_I)_{\mathscr{L},\mathscr{F}_0}$ such that $(A_1 : R_{E_1}^{\mathscr{L},I}) \neq 0$ and $(A : j_I^S(A_1)) \neq 0$. This implies $A = j_I^S(A_1)$; thus j_I^S is surjective. We now show it is injective. Assume that $A = j_I^S(A_1) = j_I^S(A_2)$, $(A_1, A_2 \in (\hat{L}_I)_{\mathscr{L},\mathscr{F}_0})$. Let $E \in \mathscr{F}$ and let $E_1 \in \mathscr{F}_0$ be such that $J(E_1) = E$. Then ${}'J(E) = E_1 +$ a linear combination of representations in $\hat{W}'_{\mathscr{L},I}$ not in $\mathscr{F}_0$. We have for $i = 1$ or 2:

$$\begin{aligned}
(A : R_E^{\mathscr{L}}) = (j_I^S(A_i) : R_E^{\mathscr{L}}) &= (A_i : R_{{}'J(E)}^{\mathscr{L},I}) && \text{(by (17.13.6))}\\
&= (A_i : R_{E_1}^{\mathscr{L},I}).
\end{aligned}$$

Hence $(A_1 : R_{E_1}^{\mathscr{L},I}) = (A_2 : R_{E_1}^{\mathscr{L},I})$ for all $E_1 \in \mathscr{F}_0$. This implies that $(A_1 : R_{\alpha_w}^{\mathscr{L},I}) = (A_2 : R_{\alpha_w}^{\mathscr{L},I})$ for all $w \in W'_{\mathscr{L},I}$ where α_w has the same meaning as in (17.13.9). We now choose w such that $(A_1 : R_{\alpha_w}^{\mathscr{L},I}) \neq 0$. Then $(A_2 : R_{\alpha_w}^{\mathscr{L},I}) \neq 0$ and we may assume that A_1, A_2 are the first two terms in the right-hand side of (17.13.9). But we have shown earlier for these A_i that $(j_I^S A_1 : j_I^S A_2) = 0$. This contradicts the assumption $j_I^S A_1 = j_I^S A_2$, and completes the proof of (17.13.8) and of (17.13.7).

We also note the following.

(17.13.11) *In the setup of* (17.13.7), *assume also that* $L_I \neq G$. *Then* $\hat{G}_{\mathscr{L},\mathscr{F}}$ *contains no cuspidal character sheaves.*

Indeed, (17.13.8) shows that any $A \in \hat{G}_{\mathscr{L},\mathscr{F}}$ is of form $j_I^S(A_1)$ $(A_1 \in (\hat{L}_I)_{\mathscr{L},\mathscr{F}_0})$, hence it is a direct summand of $\mathrm{ind}_{P_I}^G (A_1)$, with P_I as in 15.3. Hence A is not cuspidal.

17.14. We now assume that we are given $\mathscr{L} \in \mathscr{S}(T)$ and a family $\mathscr{F} \subset \hat{W}'_{\mathscr{L}}$ such that (17.8.4) holds for $(G, \mathscr{L})$ and such that (17.8.3) holds for $(G, \mathscr{L})$ as far as $\hat{G}_{\mathscr{L},\mathscr{F}}$ is concerned (see (17.13.7)). Then $\mathscr{F} \otimes \varepsilon$ (with ε as in (12.9.7)) is again a family in $W'_{\mathscr{L}}$.

Assume that we are given a bijection ϕ: $\mathscr{M}(\mathscr{G}_{\mathscr{F}}) \to \mathscr{M}(\mathscr{G}_{\mathscr{F}\otimes\varepsilon})$ such that

(17.14.1) $\{\phi(m), \phi(m')\} = \varepsilon_A\{m, m'\}$ for all $m, m' \in \mathscr{M}(\mathscr{G}_{\mathscr{F}})$,

where m corresponds to $A \in \hat{G}_{\mathscr{L},\mathscr{F}}$ under (17.8.3) and m' corresponds to $E \in \mathscr{F}$ under (17.7.1). Recall that $\{\ ,\ \}$ is given by (17.8.2).

We assume also that the following diagram is commutative.

(17.14.2)
$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\sim", "\otimes\varepsilon"'] \arrow[d, hook, "(17.7.1)"'] & \mathscr{F}\otimes\varepsilon \arrow[d, hook, "(17.7.1)"] \\
\mathscr{M}(\mathscr{G}_{\mathscr{F}}) \arrow[r, "\sim", "\phi"'] & \mathscr{M}(\mathscr{G}_{\mathscr{F}\otimes\varepsilon})
\end{tikzcd}$$

We show that under these assumptions, the statement (17.8.3) holds for $(G, \mathscr{L})$ as far as $\hat{G}_{\mathscr{L},\mathscr{F}\otimes\varepsilon}$ is concerned.

Indeed, we define a bijection

(17.14.3) $\hat{G}_{\mathscr{L},\mathscr{F}\otimes\varepsilon} \xrightarrow{\sim} \mathscr{M}(\mathscr{G}_{\mathscr{F}\otimes\varepsilon})$

by the requirement that the diagram

$$\begin{tikzcd}
G_{\mathscr{L},\mathscr{F}} \arrow[r, "\sim"'] \arrow[d, "\pm d"', "\wr"] & \mathscr{M}(\mathscr{G}_{\mathscr{F}}) \arrow[d, "\phi", "\wr"'] \\
\hat{G}_{\mathscr{L},\mathscr{F}\otimes\varepsilon} \arrow[r, dashed] & \mathscr{M}(\mathscr{G}_{\mathscr{F}\otimes\varepsilon})
\end{tikzcd}$$

be commutative. Here, the top arrow is the bijection of (17.8.3) for $\hat{G}_{\mathscr{L},\mathscr{F}}$ and $\pm d$ is defined by $A \mapsto \hat{\varepsilon}_A d(A)$ (see 15.5). The fact that $\pm d$ is well defined and bijective follows from 15.8(c), (15.4.5), and (15.4.3). It is then clear that the bijection (17.14.3) has the property required in (17.8.3).

17.15. Let $\mathscr{L} \in \mathscr{S}(T)$ be such that (17.8.3) (resp. (17.8.4)) holds for $(G, \mathscr{L})$. Let $w \in W$; we set $\mathscr{L}' = w^*\mathscr{L}$. Then (17.8.3) (resp. (17.8.4)) holds for $(G, \mathscr{L}')$. This follows easily from 14.15. (We may assume that w is a simple reflection of W.)

17.16. Let G' be a connected reductive group over k and let π: $G' \to G$ be a surjective homomorphism with finite kernel. Let $\mathscr{L} \in \mathscr{S}(T)$ and let $\mathscr{L}'$ be the inverse image of $\mathscr{L}$ under π: $\pi^{-1}(T) \to T$. We have $W_{\mathscr{L}} = W_{\mathscr{L}'} \subset W'_{\mathscr{L}} \subset W'_{\mathscr{L}'}$ in a natural way. If $w \in W'_{\mathscr{L}}$ and $i \in \mathbb{Z}$ then it follows from definitions that

$$(17.16.1) \quad \pi^* \bar{K}_w^{\mathscr{L}} = \bar{K}_w^{\mathscr{L}'}, \qquad \pi^* {}^pH^i \bar{K}_w^{\mathscr{L}} = {}^pH^i \bar{K}_w^{\mathscr{L}'},$$

and

$$(17.16.2) \quad \pi_* \bar{K}_w^{\mathscr{L}'} = \bigoplus_{\mathscr{E}} \bar{K}_w^{\mathscr{L} \otimes \mathscr{E}}, \qquad \pi_* {}^pH^i \bar{K}_w^{\mathscr{L}'} = \bigoplus_{\mathscr{E}} {}^pH^i \bar{K}_w^{\mathscr{L} \otimes \mathscr{E}}.$$

(Here $\mathscr{E}$ runs over the one-dimensional local systems on T which are direct summands of the direct image of $\bar{\mathbf{Q}}_l$ under π: $\pi^{-1}(T) \to T$. They are in $\mathscr{S}(T)$.)

It follows that if A is a character sheaf of G then $\pi^* A$ is a direct sum of character sheaves of G' such that the associated action (11.5) of $\ker \pi$ is trivial. Conversely if A' is a character sheaf of G' such that the associated action (11.5) of $\ker \pi$ is trivial, then $\pi_* A$ is a direct sum of character sheaves of G. From this we can deduce:

(17.16.3) If (17.8.5) holds for G' then it also holds for G.

Indeed, let A be an irreducible cuspidal perverse sheaf on G. Then A is a direct summand of $\pi_* \pi^* A$. It is clear that $\pi^* A$ is a direct sum of irreducible cuspidal perverse sheaves on G' with trivial action of $\ker \pi$. By (17.8.5) for G', it is a direct sum of character sheaves of G' with trivial action of $\ker \pi$. Hence $\pi_* \pi^* A$ is a direct sum of character sheaves of G, and therefore A is a character sheaf of G.

We also see that:

(17.16.4) If (17.8.4) holds for G' then it also holds for G.

Indeed, let A be a cuspidal character sheaf, and let A' be a direct summand of $\pi^* A$. Then A' is a cuspidal character sheaf of G', hence it is clean, by assumption. It follows that $\pi_* A'$ is clean. Since A is a direct summand of $\pi_* A'$, it is also clean. Applying this argument to the Levi subgroups of parabolic subgroups of G, we see that G is clean.

If A is a character sheaf on G and A' is a direct summand of $\pi^* A$, then by (17.16.1), we have $\varepsilon_A = \varepsilon_{A'}$. Since A, A' have supports of the same dimension, we also have $\hat{\varepsilon}_A = \hat{\varepsilon}_{A'}$. Thus, the equality $\varepsilon_{A'} = \hat{\varepsilon}_{A'}$ implies $\varepsilon_A = \hat{\varepsilon}_A$, and (17.16.4) follows.

17.17. For each element $z \in Z_G =$ centre of G, we denote $t_z\colon G \to G$ the map $g \to zg$. It follows easily from definitions that

(17.17.1) $t_z^* \bar{K}_w^{\mathscr{L}} = \bar{K}_w^{\mathscr{L}}$

for any $\mathscr{L} \in \mathscr{S}(T)$ and any $w \in W'_{\mathscr{L}}$.

This implies that:

(17.17.2) If $A \in \hat{G}_{\mathscr{L}}$ then $t_z^* A \in \hat{G}_{\mathscr{L}}$. (But $t_z^* A$ is not necessarily isomorphic to A.)

Hence t_z^* defines a homomorphism $t_z^*\colon \mathscr{K}_0(G) \otimes \bar{\mathbf{Q}}_l \to \mathscr{K}_0(G) \otimes \bar{\mathbf{Q}}_l$. From (17.17.1) it follows that

(17.17.3) $t_z^* R_E^{\mathscr{L}} = R_E^{\mathscr{L}}$

for any $\mathscr{L} \in \mathscr{S}(T)$ and any $E \in \hat{W}'_{\mathscr{L}}$.

PROPOSITION 17.18. *Assume that G is clean and let $\mathscr{L} \in \mathscr{S}(T)$.*

(a) *For any $w \in W'_{\mathscr{L}}$, the elements $R_{\alpha_w}^{\mathscr{L}}$, $R_{\mathscr{A}_w}^{\mathscr{L}}$ of $\mathscr{K}_0(G) \otimes \bar{\mathbf{Q}}_l$ are $\mathbb{Z}$-linear combinations of character sheaves $A \in \hat{G}_{\mathscr{L}}$ such that $\varepsilon_A = (-1)^{l(w)-a(w)}$. (Here ε_A is given by* (15.1.1) *and $l(w)$, $a(w)$ are as in* 16.6(a).)

(b) *If $A \in \hat{G}_{\mathscr{L}}$ satisfies $(A\colon R_{\alpha_w}^{\mathscr{L}}) \neq 0$ then the map* 11.9 *attaches to A the coset $wW_{\mathscr{L}}$.*

(Note that this is variant of Theorem 16.6(a) in which the parity condition (15.13) is not part of assumption; part (a) is similar to the integrality theorem [6, 6.14(i)].)

Proof. In the proof of [6, 6.14] it is shown that (with notations in 16.2 and 16.5)

$$(17.18.1) \quad \alpha_w = (-1)^{l(w)} \sum_E \operatorname{Tr}(C_w - \sum_y a_{y,w} C_y, E(u); -a(w)/2) E$$

where E runs over $\hat{W}'_{\mathscr{L}}$ and y runs over the set of elements in $wW_{\mathscr{L}}$ such that $y \underset{LR}{\sim} w$; we have

$$a_{y,w} = \sum_{\substack{i \in \mathbb{Z} \\ i > 0 \\ i \equiv l(w) + l(y) \pmod 2}} a_{y,w}^{(i)} u^{i/2}.$$

Strictly speaking, the proof in [6] applies in the case $w \in W_{\mathscr{L}}$; however, in the general case, the proof is the same. From (17.18.1) and (15.10.1) it follows that

(17.18.2)

$$\begin{aligned}(A: R^{\mathscr{L}}_{\alpha_w}) &= (-1)^{l(w)} \sum_E \operatorname{Tr}(C_w, E(u); -a(w)/2)(A: R^{\mathscr{L}}_E)\\ &\quad - \sum_E \sum_y \sum_{i>0} (-1)^{l(w)} a^{(i)}_{y,w} \operatorname{Tr}(C_y, E(u); -a(w)/2 - i/2)(A: R^{\mathscr{L}}_E)\\ &= (-1)^{l(w)-a(w)}(A: d({}^pH^{l(w)-a(w)+\dim G}\bar{K}^{\mathscr{L}}_w))\\ &\quad - \sum_y \sum_{i>0} (-1)^{l(w)-a(w)-i} a^{(i)}_{y,w}\ (A: d({}^pH^{l(y)-a(w)+\dim G-i}\bar{K}^{\mathscr{L}}_y)).\end{aligned}$$

This shows that $(A: R^{\mathscr{L}}_{\alpha_w})$ is an integer. Since in the last sum over y, i we have the restriction $i \equiv l(w) + l(y) \pmod 2$, it follows that $(A: R^{\mathscr{L}}_{\alpha_w}) = 0$ unless $\varepsilon_{\pm dA} = (-1)^{l(w)-a(w)}$. (Here, the sign is taken so that $\pm dA$ is a character sheaf.) From the definition of d (15.4.1) and from the conservation of ε_A by induction (5.12) it follows that dA is a $\mathbb{Z}$-linear combination of character sheaves A' such that $\varepsilon_{A'} = \varepsilon_A$. Hence, we have

(17.18.3) $\varepsilon_{\pm dA} = \varepsilon_A$.

Thus $(A: R^{\mathscr{L}}_{\alpha_w}) = 0$ unless $\varepsilon_A = (-1)^{l(w)-a(w)}$. The analogous result for $R^{\mathscr{L}}_{\mathscr{A}_w}$ follows from the formula

(17.18.4) $R^{\mathscr{L}}_{\mathscr{A}_w} = dR^{\mathscr{L}}_{\alpha_w}$

(see 15.8(c) and (16.2.8)).

We now prove (b). In the sum over y in (17.18.2), y runs over elements in $wW_{\mathscr{L}}$. Hence if $(A: R^{\mathscr{L}}_{\alpha_w}) \neq 0$, then we have $(dA:\ {}^pH^i\bar{K}^{\mathscr{L}}_y) = (A: d{}^pH^i\bar{K}^{\mathscr{L}}_y) \neq 0$ for some $y \in wW_{\mathscr{L}}$. From 13.10 it follows that $(dA: \sum_i (-1)^i\, {}^pH^i\bar{K}^{\mathscr{L}}_y) \neq 0$ and from 12.6 it follows that $(dA: \sum (-1)^i\, {}^pH^iK^{\mathscr{L}}_{y'}) \neq 0$ for some $y' \in wW_{\mathscr{L}}$. Now using 15.8(b), we deduce $(A: \sum_i (-1)^{i\,p}H^iK^{\mathscr{L}}_{y'}) \neq 0$ hence $(A: {}^pH^jK^{\mathscr{L}}_{y'}) \neq 0$ for some j. From 11.10, we see that under the map 11.9, A is sent to $y'W_{\mathscr{L}} = wW_{\mathscr{L}}$, as required.

17.19. Let $\mathscr{L} \in \mathscr{S}(T)$. In (a)–(f) below, we consider the following pairs (E_1, E_2) of irreducible representations of $W'_{\mathscr{L}}$.

(a) If W_L is of type E_7, $\Omega_{\mathscr{L}} = \{e\}$, take $E_1 = 512'_a$, $E_2 = 512_a$ (notation of [6, 4.12]).

(b) If $W_{\mathscr{L}}$ is of type $E_7 \times A_1$, $\Omega_{\mathscr{L}} = \{e\}$, take $E_1 = 512'_a \boxtimes 1$, $E_2 = 512_a \boxtimes 1$ or $E_1 = 512'_a \boxtimes \varepsilon$, $E_2 = 512_a \boxtimes \varepsilon$ ($\varepsilon =$ non-trivial character of A_1 – factor).

(c) If $W_{\mathscr{L}}$ is of type E_8, $\Omega_{\mathscr{L}} = \{e\}$, take $E_1 = 4096_z$, $E_2 = 4096_x$ or $E_1 = 4096'_x$, $E_2 = 4096'_z$ (notation of [6, 4.13]).

In the cases (d), (e), and (f), assume $\Omega_{\mathscr{L}} = \mathbb{Z}/2\mathbb{Z}$ acts non-trivially on $W_{\mathscr{L}}$, E_1 is the preferred extension (7.12) of $E \in \hat{W}_{\mathscr{L}}$ to $W'_{\mathscr{L}}$ and E_2 is the other extension.

(d) If $W_{\mathscr{L}}$ is of type A_5, E is the unique 16-dimensional representation in $\hat{W}_{\mathscr{L}}$.

(e) If $W_{\mathscr{L}}$ is of type A_7, E is one of the two 64-dimensional representations in $\hat{W}_{\mathscr{L}}$.

(f) If $W_{\mathscr{L}}$ is of type E_6, E is one of the two 64-dimensional representations in $\hat{W}_{\mathscr{L}}$.

In all cases, there exist an element $x \in W'_{\mathscr{L}}$ such that $(-1)^{l(x)-\bar{l}(x)-1}\alpha_x = E_1 - E_2$, $\bar{l}(x) \equiv a(x)+1 \pmod 2$ and an element $x' \in W_{\mathscr{L}}$ such that

$$\alpha_{x'} = E_1 + E_2, \qquad \bar{l}(x) \equiv a(x) \qquad \pmod 2.$$

(See [6, 5.21, 5.22, 5.23, (7.6.2), 7.10].)

17.20. Let $\mathscr{L} \in \mathscr{S}(T)$. Assume that $W_{\mathscr{L}}$ has no irreducible factors of type E_7 or E_8 and that for any irreducible factor W' of $W_{\mathscr{L}}$ of type A_n or E_6, the following condition is satisfied: if $w \in \Omega_{\mathscr{L}}$ normalizes W', then it centralizes W'. Then: for any $x \in W'_{\mathscr{L}}$ such that $\alpha_x \neq 0$ we have $\bar{l}(x) \equiv a(x) \pmod 2$. The same conclusion holds if $W'_{\mathscr{L}}$ is as in 17.19 and if $x \in W'_{\mathscr{L}}$ is assumed to be outside the two-sided cell corresponding to E_1 and E_2 where (E_1, E_2) is one of the pairs in 17.19. (See [6, (6.18.10) and pp. 231, 232].)

PROPOSITION 17.21. *Assume that G is clean and let $\mathscr{L} \in \mathscr{S}(T)$ be such that $W'_{\mathscr{L}}$ is as in 17.19(e) or (f). Assume also that the generator of $\Omega_{\mathscr{L}}$ has odd length in W. Let $A \in \hat{G}_{\mathscr{L}}$ be such that $\varepsilon_A = 1$ and such that $dA = \hat{\varepsilon}_A A$. Assume that under the map 11.9, A is mapped to the non-trivial coset in $W'_{\mathscr{L}}/W_{\mathscr{L}}$. Then $t_z^* A = A$ for all $z \in Z_G$, (see 17.17).*

Proof. Let $E \in \hat{W}'_{\mathscr{L}}$ be such that $(A : R_E^{\mathscr{L}}) \neq 0$. By 16.4, we have $(A : R_{\alpha_x}^{\mathscr{L}}) \neq 0$ for some x in the two-sided cell corresponding to E and from 17.18(a) it follows that $l(x) \equiv a(x) \pmod 2$. From 17.18(b) it follows that $x \in W'_{\mathscr{L}} - W_{\mathscr{L}}$, hence $l(x) \equiv \bar{l}(x) + 1 \pmod 2$, since the generator of $\Omega_{\mathscr{L}}$ has odd length in W. Thus, $\bar{l}(x) \equiv a(x)+1 \pmod 2$.

From 17.20, it now follows that $E = E_1$ or E_2, where (E_1, E_2) is a pair as in 17.19 for $W'_{\mathscr{L}}$. In the same way, we see that $(A : R_{E_1}^{\mathscr{L}} + R_{E_2}^{\mathscr{L}}) = 0$. Since $(A : R_{E_1}^{\mathscr{L}}) \neq 0$, it follows that $(A : R_{E_1}^{\mathscr{L}} - R_{E_2}^{\mathscr{L}}) \neq 0$. By 17.18 and 17.19, $R_{E_1}^{\mathscr{L}} - R_{E_2}^{\mathscr{L}}$ is a $\mathbb{Z}$-linear combination of character sheaves and by 14.13, it has inner product (:) with itself equal to 2. It follows that $R_{E_1}^{\mathscr{L}} - R_{E_2}^{\mathscr{L}} = vA + v'A'$ where $v, v' \in \{\pm 1\}$ and $A' \in \hat{G}_{\mathscr{L}}$ is different from A. Applying d and using 15.8(e) we see that $R_{E_1 \otimes \varepsilon}^{\mathscr{L}} - R_{E_2 \otimes \varepsilon}^{\mathscr{L}} = v\, dA + v'\, dA'$. We have

$\{E_1, E_2\} \cap \{E_1 \otimes \varepsilon, E_2 \otimes \varepsilon\} = \varnothing$ hence, by 14.13, we have $(R^{\mathscr{L}}_{E_1} - R^{\mathscr{L}}_{E_2}: R^{\mathscr{L}}_{E_1 \otimes \varepsilon} - R^{\mathscr{L}}_{E_2 \otimes \varepsilon}) = 0$. Thus, we have $(vA + v'A': v\, dA + v'\, dA') = 0$. By our assumption we have $dA = \hat{\varepsilon}_A A$. It follows that $(A': dA') + \hat{\varepsilon}_A = 0$ hence $dA' = -\hat{\varepsilon}_A A'$, and $\hat{\varepsilon}_{A'} = -\hat{\varepsilon}_A$. Since the operation t^*_z (17.17) clearly preserves the dimension of support and hence $\hat{\varepsilon}$, it follows that $A' \neq t^*_z A$ for any $z \in \mathscr{Z}_G$.
By (17.17.3), $R^{\mathscr{L}}_{E_1} - R^{\mathscr{L}}_{E_2}$ is invariant under t^*_z. It follows that $vA + v'A' = vt^*_z A + v't^*_z A'$. Hence we have either $t^*_z A = A$, $t^*_z A' = A'$ or $t^*_z A = A'$, $t^*_z A' = A$. We have just seen that the second alternative cannot hold. It follows that $t^*_z A = A$ and the proposition is proved.

18. Groups of Type A

18.1. The main result of this chapter is Proposition 18.5, which states that the statements (17.8.3)–(17.8.5) hold for groups of type A. We shall begin with a result characterizing the cuspidal character sheaves for any clean G.

PROPOSITION 18.2. *Assume that G is clean and let $A \in \hat{G}_{\mathscr{L}}$. Then A is cuspidal if and only if the following condition is satisfied: if $(A: \sum_i (-1)^i\, {}^pH^i(K^{\mathscr{L}}_w)) \neq 0$, $(w \in W'_{\mathscr{L}})$, then w: $T/Z^0_G \to T/Z^0_G$ has only finitely many fixed points.*

Proof. Assume first that A is not cuspidal. Then there exists $I \subsetneqq S$ such that $r^S_I A \neq 0$. Since $r^S_I A$ is a combination with $\geqslant 0$ integral coefficients of character sheaves of L_I, there exists $\mathscr{L}' \in \mathscr{S}(T)$, $w' \in W'_{\mathscr{L}',I}$ and $i \in \mathbb{Z}$ such that $(r^S_I A: {}^pH^i\, \bar{K}^{\mathscr{L}',I}_{w'}) > 0$ (notations of 15.6) Applying (14.11.1) with G, A, $\mathscr{L}$, w replaced by L_I, $r^S_I A$, $\mathscr{L}'$, w', we see that $(r^S_I A: R^{\mathscr{L}',I}_{E_1}) \neq 0$ for some $E_1 \in \hat{W}_{\mathscr{L}',I}$. It follows that $(r^S_I A: \sum_i (-1)^i\, {}^pH^i K^{\mathscr{L}',I}_{w''}) \neq 0$ for some $w'' \in W'_{\mathscr{L}',I}$. Now using (15.3.2) and (15.7.1), we deduce that $(A: \sum_i (-1)^i\, {}^pH^i K^{\mathscr{L}'}_{w''}) \neq 0$. (Here, $K^{\mathscr{L}'}_{w''}$ is defined with respect to G.) From 11.2(c) we see that $\mathscr{L}'$ must be in the W-orbit of $\mathscr{L}$ and from 11.2(b) we see that $(A: \sum_i (-1)^i\, {}^pH^i K^{\mathscr{L}}_w) \neq 0$ for some element $w \in W$ which is W-conjugate to $w'' \in W_I$. The fixed point set of w'': $T/\mathscr{Z}^0_G \to T/\mathscr{Z}^0_G$ contains $\mathscr{Z}^0_{L_I}/\mathscr{Z}^0_G$ hence it has dimension $\geqslant 1$. Since w is conjugate to w'', the fixed point set of w: $T/\mathscr{Z}^0_G \to T/\mathscr{Z}^0_G$ also has dimension $\geqslant 1$. This proves one-half of the lemma. Conversely, assume that $A \in \hat{G}_{\mathscr{L}}$ satisfies $(A: \sum_i (-1)^i\, {}^pH^i(K^{\mathscr{L}}_w)) \neq 0$ for some $w \in W'_{\mathscr{L}}$ such that w: $T/\mathscr{Z}^0_G \to T/\mathscr{Z}^0_G$ has a fixed point set of dimension $\geqslant 1$. Replacing w by a conjugate, we may assume that $w \in W'_{\mathscr{L},I}$, $(I \subsetneqq S)$. Using (15.7.1) we see that $(A: i^S_I(\sum (-1)^i\, {}^pH^i(K^{\mathscr{L},I}_w))) \neq 0$, and using (15.3.2) we see that $r^S_I A \neq 0$ so that A is not cuspidal. The lemma is proved.

18.3. *Assume that* G *is clean and let* $A \in \hat{G}_{\mathscr{L}}$ *be cuspidal. Let* xW $(x \in \Omega_{\mathscr{L}})$ *be the coset in* $W'_{\mathscr{L}}/W_{\mathscr{L}}$ *attached to* A *in* 11.9 *and let* N *be the number of orbits of the permutation of* $S_{\mathscr{L}}$ (*see* 2.3) *defined by* $s \mapsto xsx^{-1}$. *Assume that for any* $E \in \hat{W}'_{\mathscr{L}}$ *such that* $(A : R_E^{\mathscr{L}}) \neq 0$, *we have*

$$(18.3.1) \quad \mathrm{Tr}(T_w, E(u)) \in u^{N/2}\bar{\mathbf{Q}}_l[u, u^{-1}] \textit{ for all } w \in xW_{\mathscr{L}}$$

(*notation of* (12.9.3)). *Then* $\varepsilon_A = \hat{\varepsilon}_A$ (*see* (15.13.1)).

Proof. Let $w \in W'_{\mathscr{L}}$ be an element of minimal possible length in W such that $(A: {}^pH^iK_w^{\mathscr{L}}) \neq 0$ for some i. Then, by (12.7.1), we have $(A: {}^pH^j\bar{K}_w^{\mathscr{L}}) = (A: {}^pH^jK_w^{\mathscr{L}})$ for any j and, in particular, $(A: {}^pH^i\bar{K}_w^{\mathscr{L}}) \neq 0$. It follows that $\sum_j(-1)^j(A: {}^pH^j\bar{K}_w^{\mathscr{L}}) \neq 0$. (By 13.10(a), only the terms corresponding to $j \equiv i$ (mod 2) can be non-zero, hence they all have the same sign.) It also follows that $\sum_j(-1)^j(A: {}^pH^jK_w^{\mathscr{L}}) \neq 0$. Using 10.2, we see that w acts on the vector space V spanned by the roots (in the character group of T) without eigenvalue 1. Since w: $V \to V$ is of finite order and defined over Q, its determinant on V must be equal to $(-1)^{\dim V}$. Hence $l(w) \equiv \dim V$ (mod 2). On the other hand, we have $\dim V \equiv \dim(G/\mathscr{Z}_G^0)$ (mod 2) hence

$$(18.3.2) \quad l(w) \equiv \dim(G/\mathscr{Z}_G^0) \qquad (\mathrm{mod}\ 2).$$

We now write the identity (14.11.1) for our A and w. The left-hand side is a non-zero element in $u^{\dim G/2}Q[u, u^{-1}]$ if $\varepsilon_A = 1$ and in $u^{(\dim G + 1)/2}Q[u, u^{-1}]$ if $\varepsilon_A = -1$. The right-hand side is, by (18.3.1), in $u^{(N + l(w) + \dim G - \bar{l}(w))/2}\bar{Q}_l[u, u^{-1}]$. It follows that

$$(18.3.3) \quad \varepsilon_A = (-1)^{N + l(w) - \bar{l}(w)};$$

on the other hand, we have

$$(18.3.4) \quad \hat{\varepsilon}_A = (-1)^{\dim(G/\mathscr{Z}_G^0) - \dim(\operatorname{supp} A/\mathscr{Z}_G^0)} = (-1)^{\dim(G/\mathscr{Z}_G^0)}$$

since, by 3.12, $(\operatorname{supp} A)/\mathscr{Z}_G^0$ is the closure of a single conjugacy class in $G/\mathscr{Z}_G^0$ and hence, it has even dimension. The identity $\varepsilon_A = \hat{\varepsilon}_A$ is therefore equivalent to the congruence

$$N + l(w) + \bar{l}(w) + \dim(G/\mathscr{Z}_G^0) \equiv 0 \qquad (\mathrm{mod}\ 2).$$

By (18.3.2), this is equivalent to the congruence

$$(18.3.5) \quad \bar{l}(w) \equiv N \qquad (\mathrm{mod}\ 2).$$

This is proved as follows. Let V_1 be the subspace of V spanned by the roots in $R_{\mathscr{L}}$ (see 2.3) and let $\pi_{\mathscr{L}}$ be the basis of V_1 formed by the simple roots of $R_{\mathscr{L}}$. Then w: $V \to V$ leaves V_1 stable, and being of finite order, defined over Q, without eigenvalue 1, it satisfies $\det(w, V_1) = (-1)^{\dim V_1}$. We have $w = xw_1$, where $x \in \Omega_{\mathscr{L}}$, $w_1 \in W_{\mathscr{L}}$, and $\det(w, V_1) =$

$\det(x, V_1)\det(v_1, V_1) = (-1)^{\bar{l}(w_1)}\det(x, V_1) = (-1)^{\bar{l}(w)}\det(x, V_1)$. Since x: $V_1 \to V_1$ permutes the elements in the basis $\pi_{\mathscr{L}}$ of V_1, we have $\det(x, V_1) = (-1)^{\dim V_1 - N}$ where N is the number of orbits of the permutation defined by x. It follows that $(-1)^{\dim V_1} = (-1)^{\bar{l}(w)}(-1)^{\dim V_1 - N}$ and (18.3.5) follows. The corollary is proved.

LEMMA 18.4. *Assume that $G/\mathscr{Z}_G$ is a product of projective general linear groups and let $\mathscr{L} \in \mathscr{S}(T)$ be such that $W_{\mathscr{L}} = \{e\}$. Assume that* (17.8.4) *holds for $(G, \mathscr{L})$ and that there exists $x \in \Omega_{\mathscr{L}}$ such that x is a Coxeter element of W. Let α_x be the element defined in* (16.2.7) *in terms of G, $\mathscr{L}$, x. Then*

(a) $|\Omega_{\mathscr{L}}| = |\mathscr{Z}_G/\mathscr{Z}_G^0|$.

(b) $R_{\alpha_x}^{\mathscr{L}} = \bigoplus_z A_{x,z}$ *(sum over all $z \in \mathscr{Z}_G/\mathscr{Z}_G^0$), where $A_{x,z}$ is a cuspidal character sheaf of G with support contained in $z\mathscr{Z}_G^0$ {unipotent variety of G}. In particular, the $A_{x,z}$ $(z \in \mathscr{Z}_G/\mathscr{Z}_G^0)$ are distinct.*

Proof. We may assume that G is semisimple. The following statement is a reformulation of (a) in terms of the dual group G'.

(18.4.1) Let s be a regular semisimple element of G' such that there exists a Coxeter element w of the Weyl group of G' with respect to the maximal torus $Z^0(s)$, such that $w \in Z(s)$. Then $|Z(s)/Z^0(s)|$ is equal to the order of the kernel Γ of the simply connected covering π: $\tilde{G} \to G'$ of G'.

Let $\tilde{s} \in \tilde{\pi}^{-1}(s)$. We have $Z_{G'}(s)/Z_{G'}^0(s) = \{x \in \tilde{G}' \mid x\tilde{s}x^{-1} \in \tilde{s}\Gamma\}/Z_{\tilde{G}'}(\tilde{s})$. This shows that $|Z_{G'}(s)/Z_{G'}^0(s)| \leqslant |\Gamma|$ and that to prove equality is equivalent to showing that for any $y \in \Gamma$, $\tilde{s}$ is conjugate to $\tilde{s}y$. This statement clearly follows from the statement (18.4.1) in the case where G' is adjoint. We can further reduce ourselves to the case where G' is adjoint, simple, hence, $G' = PGL_n(k)$. In this case our assumption on s implies that n is invertible in k and that s can be represented by the image in $PGL_n(k)$ of the diagonal matrix $(1, \zeta, \zeta^2, ..., \zeta^{n-1})$ where ζ is a primitive nth root of 1 in k. In this case it is clear that $|Z_{G'}(s)/Z_{G'}^0(s)| = n$ and (a) is proved.

We now prove (b). Since $W'_{\mathscr{L}} = \Omega_{\mathscr{L}}$, each character θ: $\Omega_{\mathscr{L}} \to \bar{\mathbf{Q}}_l^*$ may also be regarded as a representation of $W'_{\mathscr{L}}$. For any $x' \in \Omega_{\mathscr{L}}$, we have $\alpha_{x'} = (-1)^{l(x')}\sum_\theta \theta(x')\theta$ and from 16.6(a) we see that

(18.4.2) $R_{\alpha_{x'}}^{\mathscr{L}} = (-1)^{l(x')}\sum_\theta \theta(x')R_\theta^{\mathscr{L}} = (-1)^{l(x')}\sum_i (-1)^{i+\dim G}$ ${}^pH^i(K_{x'}^{\mathscr{L}})$ is a combination with integral $\geqslant 0$ coefficients of character sheaves. By (14.13), we have

$$(18.4.3)\quad (R_{\alpha_{x'}}^{\mathscr{L}} : R_{\alpha_{x''}}^{\mathscr{L}}) = \begin{cases} |\Omega_{\mathscr{L}}|, & \text{if } x' = x'' \\ 0, & \text{if } x' \neq x'' \end{cases} \qquad (x', x'' \in \Omega_{\mathscr{L}}).$$

Hence, if A is a character sheaf such that $(A: R_{\alpha_x}^{\mathscr{L}}) \neq 0$ then $(A: R_{\alpha_{x'}}^{\mathscr{L}}) = 0$ for $x' \neq x$. Using (18.4.2) and 18.2, it follows that A is cuspidal. From 3.12 it follows that the support of A is contained in the set $z\mathscr{Z}_G^0 \cdot \{$unipotent variety of $G\}$ for some $z \in \mathscr{Z}_G$. Then $\operatorname{supp}(t_z^* A) \subset \mathscr{Z}_G^0 \cdot \{$unipotent variety$\}$ (see 17.7). Note that $t_z^* A$ is again a character sheaf (17.17.2) and it is a component of $R_{\alpha_x}^{\mathscr{L}}$, by (17.17.3). When z runs over a set of representatives for $\mathscr{Z}_G/\mathscr{Z}_G^0$, then the $t_z^* A$ have distinct supports hence they are distinct. This gives at least $|\mathscr{Z}_G/\mathscr{Z}_G^0|$ distinct character sheaves which are components of $R_{\alpha_x}^{\mathscr{L}}$. By (18.4.3) and (a), we have $(R_{\alpha_x}^{\mathscr{L}} : R_{\alpha_x}^{\mathscr{L}}) = |\Omega_{\mathscr{L}}| = |\mathscr{Z}_G/\mathscr{Z}_G^0|$, so that all components of $R_{\alpha_x}^{\mathscr{L}}$ must be obtained as described, and they all have multiplicity one. The lemma is proved.

PROPOSITION 18.5. *Assume that $G/\mathscr{Z}_G$ is a product of adjoint groups of type A. Then*

(a) *G satisfies* (17.8.5).

(b) *$(G, \mathscr{L})$ satisfies* (17.8.4) *for any $\mathscr{L} \in \mathscr{S}(T)$.*

(c) *Let $\mathscr{L} \in \mathscr{S}(T)$, let $\mathscr{F} = \{E\}$ be a family in $\hat{W}_{\mathscr{L}}$ and let $\mathscr{F}' = \{\tilde{E}_\theta \mid \theta$ character of $\mathscr{G}_{\mathscr{F}'} = \Omega_{\mathscr{F}}\}$ be the corresponding family in $\hat{W}'_{\mathscr{L}}$ (see* (17.3)). *Recall that the imbedding $\mathscr{F}' \subsetneq \mathscr{M}(\mathscr{G}_{\mathscr{F}'})$ (see* (17.7.1)) *is $\tilde{E}_\theta \mapsto (1, \theta)$.*

Let $\hat{G}_{\mathscr{L},\mathscr{F}'}$ be defined as in (17.13.2). *There exists a bijection $\hat{G}_{\mathscr{L},\mathscr{F}'} \leftrightarrow \mathscr{M}(\mathscr{G}_{\mathscr{F}'})$, $A \leftrightarrow (x_A, \theta_A) \in \mathscr{M}(\mathscr{G}_{\mathscr{F}'})$, such that*

$$(A: R_{\tilde{E}_\theta}^{\mathscr{L}}) = (1/|\Omega_{\mathscr{F}}|)\, \varepsilon_A \theta(x_A)^{-1}, \qquad \varepsilon_A = (-1)^{l(x_A)}, \ (A \in \hat{G}_{\mathscr{L},\mathscr{F}'}, \tilde{E}_\theta \in \mathscr{F}').$$

(l is the restriction of the length function of W to $\Omega_{\mathscr{F}}$; we have $x_A \in G_{\mathscr{F}'} = \Omega_{\mathscr{F}}$.) In particular, (17.8.3) *holds for $(G, \mathscr{L})$.*

Proof. The proposition is trivial when G has a single element. Assume now $\dim G \geqslant 1$ and that the proposition is true for groups of the same type as G, of strictly smaller dimension than G; we shall prove that it is also true for G. If G is not semisimple, then $\dim(G/\mathscr{Z}_G^0) < \dim G$ and the method of 17.10 reduces us to the case of $G/\mathscr{Z}_G^0$ to which the induction hypothesis applies. Hence, we may assume that G is semisimple.

We now prove (b). To prove (b), we may assume by (17.16.4) that G is simply connected and, by 17.11, that $G = SL_n(k)$. Using the induction hypothesis, we see that it is enough to show that any cuspidal character sheaf A on G is clean and satisfies $\varepsilon_A = \hat{\varepsilon}_A$.

From 3.12 it follows that the support of A is contained in the set $z\{$unipotent variety of $G\}$, for some $z \in \mathscr{Z}_G$. As in the proof of 18.4, by replacing A by $t_z^* A$ ($z \in \mathscr{Z}_G$; see 17.7) we see that we are reduced to the case where the support of A consists of unipotent elements. We shall apply 7.9 to A. The hypothesis of 7.9 are verified. Indeed, since A is a character sheaf, it is strongly cuspidal (7.1.13). From 3.12 it follows that for any cuspidal pair $(\Sigma, \mathscr{E})$ for G with Σ a unipotent class we have $\Sigma=$ regular unipotent class. Finally, if $\mathscr{L}$ is a Levi subgroup of a proper parabolic subgroup of G, then any irreducible cuspidal perverse sheaf on L is a character sheaf (by the induction hypothesis) hence it is strongly cuspidal. Thus, 7.9 is applicable to A and shows that A is clean.

Let $j \in \mathbb{Z}$, $\mathscr{L} \in \mathscr{S}(T)$, and $w \in W'_{\mathscr{L}}$ be such that $(A: {}^pH^jK_w^{\mathscr{L}}) \neq 0$. Since A is an irreducible cuspidal perverse sheaf on $SL_n(k)$ with support in the unipotent variety, we see from [4, 10.3] that n is invertible in k and that the action (11.5) of $\mathscr{Z}_G$ on A is through a character of order n. From 11.10 we see that the image of the coset $wW_{\mathscr{L}} \subset W'_{\mathscr{L}}/W_{\mathscr{L}}$ under the homomorphism (11.8.1) has order n. Using (11.8.2) it follows that this coset has order n in the group $W'_{\mathscr{L}}/W_{\mathscr{L}}$.

It is easy to check the following statement: if $\mathscr{L}_1 \in \mathscr{S}(T)$ and $w_1 \in W'_{\mathscr{L}_1}$ is such that $w_1 W'_{\mathscr{L}_1}$ has order n in $W'_{\mathscr{L}_1}/W_{\mathscr{L}_1}$ then w_1 is a Coxeter element in W and $W_{\mathscr{L}_1}=\{e\}$. (An equivalent statement is: if s is a semisimple element in $PGL_n(k)$ such that the group of components of its centralizer has some element ρ of order n, then s is regular (hence contained in a unique maximal torus) and ρ represents a Coxeter element in the Weyl group of that torus.)

In our case it follows that w is a Coxeter element and that $W_{\mathscr{L}}=\{e\}$. Then 18.3 is applicable and shows that $\varepsilon_A=\hat{\varepsilon}_A$. This completes the proof of (b), assuming the induction hypothesis.

We now prove (c). As we have seen earlier, we may assume that G is semisimple. Assume first that either E is not the sign representation of $W_{\mathscr{L}}$ or that $W_{\mathscr{L}}=\{e\}$ and no element of $\Omega_{\mathscr{L}}$ is a Coxeter element of W. Then it is easy to see that (after replacing if necessary $\mathscr{L}$ by $w^*\mathscr{L}$ for some $w \in W$, see 17.15), there exists a proper subset I of the set of simple reflections in W and a family $\mathscr{F}'_0$ of $W'_{\mathscr{L},I}$ such that $\mathscr{F}'$ is smoothly induced by $\mathscr{F}'_0$, as in 17.13. By the induction hypothesis, (17.8.3) (or, more precisely, (c)) holds for $(L_I, \mathscr{L})$ and (17.8.4) holds for both $(G, \mathscr{L})$ and $(L_I, \mathscr{L})$. Using (17.13.8) we see (just as in the proof of (17.13.7)) that (c) holds for our family $\mathscr{F}'$. Next, assume that E is the sign representation of $W_{\mathscr{L}}$ and that $W_{\mathscr{L}} \neq \{e\}$. We have $\mathscr{G}_{\mathscr{F}'}=\mathscr{G}_{\mathscr{F}'\otimes\varepsilon}=\Omega_{\mathscr{L}}$, where ε: $W'_{\mathscr{L}} \to \{\pm 1\}$ is given by $\varepsilon(w)=(-1)^{l(w)}$. Define a bijection ϕ: $\mathscr{M}(\mathscr{G}_{\mathscr{F}'}) \to \mathscr{M}(\mathscr{G}_{\mathscr{F}'\otimes\varepsilon})$ by $\phi(x, \theta)=(x, \theta \otimes (\varepsilon \mid \Omega_{\mathscr{L}}))$, $(x \in \Omega_{\mathscr{L}}$, $\theta \in \operatorname{Hom}(\Omega_{\mathscr{L}}, \bar{\mathbf{Q}}_l^*))$. Then the diagram (17.14.2) with this ϕ (and $\mathscr{F}'$ instead of $\mathscr{F}$) is commutative. It is also clear that $\{\phi(x, \theta), \phi(1, \theta')\}=(-1)^{l(x)}\{(x, \theta),(1, \theta')\}$ for all $x \in \Omega_{\mathscr{L}}$, and all characters θ, θ' of $\Omega_{\mathscr{L}}$. (Both sides are $(-1)^{l(x)}\theta'(x)^{-1}$.) Hence the identity (17.14.1) is satisfied in our case. (Since (c) is already established for $\mathscr{F}'$, we have $\varepsilon_A=(-1)^{l(x)}$ for $A \leftrightarrow m=(x, \theta)$ in (17.14.1).) The proof in 17.14 then shows that (c) holds for $\mathscr{F}' \otimes \theta$. (Note that $\varepsilon_{\pm dA}=\varepsilon_A$, as a consequence of 15.12 and (15.4.1).)

We can therefore assume that $W=\{e\}$ and that $\Omega_{\mathfrak{L}}$ contains some Coxeter element of W. We now show that

(18.5.1) for each $x\in\Omega_{\mathscr{L}}$, the element $R_{\alpha_x}^{\mathscr{L}}$ of (18.4.2) is a sum of character sheaves, each with multiplicity 1.

When x is a Coxeter element of W, this follows from 18.4. Assume now that $x\in\Omega_{\mathscr{L}}$ is not a Coxeter element. Replacing $\mathscr{L}$ by a W-conjugate, we may assume that x is a Coxeter element of W_I where $I\subsetneqq S$. From (17.13.5) we see that $R_{\alpha_x}^{\mathscr{L}}$ is obtained by applying j_I^S (see 17.13) to the analogous element $R_{\alpha_x}^{\mathscr{L},I}$ defined in terms of L_I, $\mathscr{L}$, x. We can apply 18.4 to $R_{\alpha_x}^{\mathscr{L},I}$ (instead of $R_{\alpha_x}^{\mathscr{L}}$) and we see that $R_{\alpha_x}^{\mathscr{L},I}=A_1\oplus A_2\oplus\cdots\oplus A_r$ where $A_1,\dots,A_r$ are distinct cuspidal character sheaves on L_I such that $\operatorname{supp} A_i=z_i\mathscr{Z}^0_{L_I}$ $\{$unipotent variety of $L_I\}$ and $z_1,\dots,z_r$ is a system of representatives for the cosets $\mathscr{Z}_{L_I}/\mathscr{Z}^0_{L_I}$. We shall assume, as we may, that each $z_i\in\mathscr{Z}_G$. We have $j_I^S A_i=i_I^S A_i$ since $W_{\mathscr{L}}=\{e\}$. Hence it is enough to prove the following two statements.

(18.5.2) For any i, $i_I^S A_i$ is a sum of character sheaves, each with multiplicity 1.

(18.5.3) $(i_I^S A_i : i_I^S A_j)=0$ for $i\neq j$.

To prove (18.5.2) we argue as follows. From the definition of induction we see that $\operatorname{ind}(A_i)$ (the perverse sheaf on G induced by A_i) has the property that its restriction to $z_i\times\{$regular unipotent class of $G\}$ is, up to shift, a local system of rank 1. This forces $\operatorname{ind} A_i$ (which is semisimple) to have at least one irreducible summand with multiplicity one. According to [4, 3.5, (4.1.1)], the endomorphism algebra of $\operatorname{ind}(A_i)$ is a twisted group algebra of a certain finite group: the isotropy group Γ_i in $N_G(L_I)/L_I$ of A_i. Any element γ of Γ_i can be represented by an element in $N_W(W_I)$. Since it keeps A_i fixed, and $A_i\in(\widehat{L_I})_{\mathscr{L}}$, it must map $\mathscr{L}$ to a local system in the W_I-orbit of $\mathscr{L}$ (by 11.2(c)). Replacing γ by an element in the same W_I-coset, we see that γ can be represented by an element in $W'_{\mathscr{L}}\cap N_W(W_I)=\Omega_{\mathscr{L}}\cap N_W(W_I)$. It follows that Γ_i is abelian. Our twisted group algebra has some one-dimensional representation (since $\operatorname{ind}(A_i)$ has some irreducible component with multiplicity one); hence it is an ordinary (untwisted) group algebra. Thus, the endomorphism algebra of $\operatorname{ind}(A_i)$ is abelian and (18.5.2) follows.

Next, we prove (18.5.3). From (4.3.1) we see that $\operatorname{ind}(A_i)$ is a direct sum of irreducible perverse sheaves on G with support equal to the closure of

$$(18.5.4)\quad X_i=\bigcup_{x\in G} x(z_i(\mathscr{Z}^0_{L_I})_{\mathrm{reg}}U_{L_I})x^{-1}$$

where U_{L_I} is the set of regular unipotent element in L_I and $(\mathscr{Z}^0_{L_I})_{\mathrm{reg}}$ is the set of all $g \in \mathscr{Z}^0_{L_I}$ such that $\mathscr{Z}^0_G(g) = L_I$. It is enough to show that

$$(18.5.5) \quad X_i \cap X_j = \varnothing \qquad \text{for} \quad i \neq j.$$

Assume that we have $X_i \cap X_j \neq \varnothing$. Then there exist $g, g' \in (\mathscr{Z}^0_{L_I})_{\mathrm{reg}}$, $u, u' \in U_{L_I}$, $x, x' \in G$ such that $xz_i gux^{-1} = x'z_j g'u'x'^{-1}$; we want to deduce that $i = j$. We may assume that $x' = 1$. By uniqueness of Jordan decomposition, we have $xz_i gx^{-1} = z_j g'$. We have $\mathscr{Z}^0_G(xz_i gx^{-1}) = xz_i \mathscr{Z}^0_G(g) x^{-1} = xL_I x^{-1} = \mathscr{Z}^0_G(z_j g') = L_I$. Hence $x \in N_G(L_I)$. It follows that $x\mathscr{Z}^0_{L_I} x^{-1} = \mathscr{Z}^0_{L_I}$. Hence from $xz_i gx^{-1} = z_j g'$ we deduce $z_i^{-1} z_j = xgx^{-1} \cdot g'^{-1} \in \mathscr{Z}^0_{L_I}$. But the z_i are representatives for the cosets $\mathscr{Z}_{L_I}/\mathscr{Z}^0_{L_I}$, hence $z_i^{-1} z_j \in \mathscr{Z}^0_{L_I}$ implies $i = j$. Thus (18.4.3) is proved. At the same time, (18.5.1) is proved.

From (18.5.1) and (18.4.3) it follows that, for any $x \in \Omega_{\mathscr{L}}$, $R^{\mathscr{L}}_{\alpha_x}$ has exactly $|\Omega_{\mathscr{L}}|$ irreducible components (with multiplicity one) which can be put in 1–1 correspondence with the various characters θ of $\Omega_{\mathscr{L}}$; we shall denote them $A_{x,\theta} \in \hat{G}_{\mathscr{L}}$; thus $\hat{G}_{\mathscr{L}}$ consists of $|\Omega_{\mathscr{L}}|^2$ character sheaves $A_{x,\theta}$ ($x \in \Omega_{\mathscr{L}}$, θ: $\Omega_{\mathscr{L}} \to \bar{\mathbf{Q}}_l^*$). We have

$$\begin{aligned}(A_{x,\theta} : R^{\mathscr{L}}_{\alpha_{x'}}) &= 1, \qquad \text{if} \quad x = x' \\ &= 0, \qquad \text{if} \quad x \neq x'.\end{aligned}$$

Let $R^{\mathscr{L}}_\theta$ be as in the proof of 18.4. Then

$$R^{\mathscr{L}}_\theta = |\Omega_{\mathscr{L}}|^{-1} \sum_{x \in \Omega_{\mathscr{L}}} (-1)^{l(x)} \theta(x)^{-1} R^{\mathscr{L}}_{\alpha_x},$$

see (18.4.2), hence,

$$(A_{x,\theta} : R^{\mathscr{L}}_{\theta'}) = (-1)^{l(x)} |\Omega_{\mathscr{L}}|^{-1} \theta'(x)^{-1}.$$

By (17.18(a), we have $\varepsilon_{A_{x,\theta}} = (-1)^{l(x)}$. This completes the proof of (c), assuming the induction hypothesis.

We now prove (a). Using (17.16.3) we see that we may assume that G is semisimple, simply connected and using 17.11, we are reduced to the case $G = SL_n(k)$. From [4, 10.3, 2.10] we see that the number of irreducible cuspidal perverse sheaves on G is $n\phi(n)$ (if n is invertible in k) and is zero otherwise. Here $\phi(n)$ is the Euler function. Hence to prove (a) we may assume that n is invertible in k. Let w be a Coxeter element in W. We can find $\mathscr{L} \in \mathscr{S}(T)$ such that $w \in W'_{\mathscr{L}}$, $W_{\mathscr{L}} = \{e\}$. The group $W'_{\mathscr{L}} = \Omega_{\mathscr{L}}$ is cyclic of order n and each generator of it is a Coxeter element in W. By 18.4, applied to $\mathscr{L}$ and any generator of $\Omega_{\mathscr{L}}$, we see that there are at least $n\phi(n)$ cuspidal character sheaves in $\hat{G}_{\mathscr{L}}$; they are indexed by pairs (x, z) where x

is a generator of $\Omega_{\mathscr{L}}$ and $z \in \mathscr{Z}_G$. It follows that each of the $n\phi(n)$ irreducible cuspidal perverse sheaves on G belongs to $\hat{G}_{\mathscr{L}}$. This proves (a) and completes the proof of the proposition.

19. Classical Groups of Low Rank

19.1. The main results in this chapter are 19.3, 19.4, 19.6 which assert that the statements (17.8.3)–(17.8.5) hold for certain classical groups of low rank. This prepares the ground for the study of character sheaves on exceptional groups in the following two chapters.

Proposition 19.2. *Let $\mathscr{L} \in \mathscr{S}(T)$ be such that*

(a) *$(G, \mathscr{L})$ satisfies* (17.8.4).

(b) *$W_{\mathscr{L}}$ has all irreducible components of type A, except possibly for one component which is of type D_m $(4 \leqslant m \leqslant 8)$, $B_m (2 \leqslant m \leqslant 5)$, or C_m $(2 \leqslant m \leqslant 5)$.*

(c) $|W'_{\mathscr{L}}/W_{\mathscr{L}}| \leqslant 3$.

Then (17.8.3) *holds for* $(G, \mathscr{L})$.

Proof. We fix a family $\mathscr{F}$ in $\hat{W}_{\mathscr{L}}$ and let $\mathscr{F}'$ be the corresponding family in $\hat{W}'_{\mathscr{L}}$ (see 17.4). The family $\mathscr{F}$ consists of either a single representation E or of three representations E, M, N where E is a special representation [6, (4.1.4)].

Case 1. Assume first that $W'_{\mathscr{L}} = W_{\mathscr{L}}$. From results in [6, 4.5] we see that if $\mathscr{F} = \{E\}$, then there exists an involution $x \in W_{\mathscr{L}}$ such that $\alpha_x = E$, $\bar{l}(x) \equiv a(x) \pmod 2$, while if $\mathscr{F} = \{E, M, N\}$ then each of the four elements $E+M$, $E-M$, $E+N$, $E-N$ is of the form α_x for some involution $x \in W_{\mathscr{L}}$ such that $\bar{l}(x) \equiv a(x) \pmod 2$.

If $\mathscr{F} = \{E\}$, we see from 16.6 that $R_E^{\mathscr{L}}$ is a combination with integral $\geqslant 0$ coefficients of character sheaves, and from 14.13 that $R_E^{\mathscr{L}}$ is a single character sheaf A. From 17.18(a) we see that $\varepsilon_A = 1$.

If $\mathscr{F} = \{E, M, N\}$, we see from 16.6 that $R_E^{\mathscr{L}} + R_M^{\mathscr{L}}$, $R_E^{\mathscr{L}} - R_M^{\mathscr{L}}$, $R_E^{\mathscr{L}} + R_N^{\mathscr{L}}$, $R_E^{\mathscr{L}} - R_N^{\mathscr{L}}$ are combinations with integral $\geqslant 0$ coefficients of character sheaves. From 14.13 we see that the inner products (:) of these four elements are described by the matrix

$$\begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}.$$

It follows that there exist four distinct character sheaves A_1, A_2, A_3, $A_4 \in \hat{G}_{\mathscr{L}}$ such that

$$\begin{aligned} R_E^{\mathscr{L}} + R_M^{\mathscr{L}} &= A_1 + A_2 \\ R_E^{\mathscr{L}} - R_M^{\mathscr{L}} &= A_3 + A_4 \\ R_E^{\mathscr{L}} + R_N^{\mathscr{L}} &= A_1 + A_3 \\ R_E^{\mathscr{L}} - R_N^{\mathscr{L}} &= A_2 + A_4. \end{aligned}$$

Thus, we have

$$\begin{aligned} R_E^{\mathscr{L}} &= \tfrac{1}{2}(A_1 + A_2 + A_3 + A_4) \\ R_M^{\mathscr{L}} &= \tfrac{1}{2}(A_1 + A_2 - A_3 - A_4) \\ R_N^{\mathscr{L}} &= \tfrac{1}{2}(A_1 - A_2 + A_3 - A_4). \end{aligned}$$

Moreover, from 17.18(a) we see that $\varepsilon_{A_i} = 1$ $(1 \leqslant i \leqslant 4)$. Hence the pattern of (17.8.3) is established.

Case 2. Next, we assume that $W'_{\mathscr{L}}/W_{\mathscr{L}}$ has order 2 or 3 and $\Omega_{\mathscr{F}} = \{e\}$. Then E and M, N (if defined) do not extend to $W'_{\mathscr{L}}$-modules; $\mathscr{F}'$ consists of $E' = \operatorname{ind} E$ (if $\mathscr{F} = \{E\}$) or of $E' = \operatorname{ind} E$, $M' = \operatorname{ind} M$, $N' = \operatorname{ind} N$ (if $\mathscr{F} = \{E, M, N\}$). Here, $\operatorname{ind} = \operatorname{ind}_{W_{\mathscr{L}}}^{W'_{\mathscr{L}}}$. The arguments in Case 1 remain valid if we replace $R_E^{\mathscr{L}}$, $R_M^{\mathscr{L}}$, $R_N^{\mathscr{L}}$ by $R_{E'}^{\mathscr{L}}$, $R_{M'}^{\mathscr{L}}$, $R_{N'}^{\mathscr{L}}$.

Case 3. Assume that $W'_{\mathscr{L}}/W_{\mathscr{L}}$ has order 2 and that $\Omega_{\mathscr{F}} = \Omega_{\mathscr{L}}$. Then E and M, N (if defined) extend to $W'_{\mathscr{L}}$-modules; we shall denote by $\tilde{E}$ and $\tilde{M}$, $\tilde{N}$ (if M, N are defined) the preferred extensions (see 17.2), and by $\tilde{E}'$ and $\tilde{M}'$, $\tilde{N}'$ (if M, N are defined) the non-preferred extensions.

When $\mathscr{F} = \{E\}$, the following result can be extracted from [6, (7.6.6)]: there exist $x, x' \in W'_{\mathscr{L}}$ such that $\tilde{E} + \tilde{E}' = \alpha_x$, $l(x) \equiv a(x) \pmod 2$, $\tilde{E} - \tilde{E}' = (-1)^{l(x') - a(x')}\alpha_{x'}$. Now using 16.6 we see that $R_{\tilde{E}}^{\mathscr{L}} + R_{\tilde{E}'}^{\mathscr{L}}$ and $(-1)^{l(x') - a(x')}(R_{\tilde{E}}^{\mathscr{L}} - R_{\tilde{E}'}^{\mathscr{L}})$ are linear combinations with integral $\geqslant 0$ coefficients of character sheaves. From 14.13, the inner products (:) of these two elements are described by the matrix $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. Hence there exist four distinct character sheaves A_i $(1 \leqslant i \leqslant 4)$, such that $R_{\tilde{E}}^{\mathscr{L}} + R_{\tilde{E}'}^{\mathscr{L}} = A_1 + A_2$, $(-1)^{l(x') - a(x')}(R_{\tilde{E}}^{\mathscr{L}} - R_{\tilde{E}'}^{\mathscr{L}}) = A_3 + A_4$, and from 17.18(a), we see that $\varepsilon_{A_1} = \varepsilon_{A_2} = 1$, $\varepsilon_{A_3} = \varepsilon_{A_4} = (-1)^{l(x') - a(x')}$. We have

$$\begin{aligned} R_{\tilde{E}}^{\mathscr{L}} &= \tfrac{1}{2}(A_1 + A_2 + \varepsilon_{A_3} A_3 + \varepsilon_{A_4} A_4) \\ R_{\tilde{E}'}^{\mathscr{L}} &= \tfrac{1}{2}(A_1 + A_2 - \varepsilon_{A_3} A_3 - \varepsilon_{A_4} A_4) \end{aligned}$$

so that the pattern of (17.8.3) is established.

When $\mathscr{F} = \{E, M, N\}$, the following result can be extracted from [6, (7.6.7)]: There exists $x_i \in W'_{\mathscr{L}}$ $(1 \leqslant i \leqslant 8)$ such that

$$\begin{aligned}
\tilde{E} + \tilde{E}' + \tilde{M} + \tilde{M}' &= \alpha_{x_1}\\
\tilde{E} + \tilde{E}' - (\tilde{M} + \tilde{M}') &= \alpha_{x_2}\\
\tilde{E} + \tilde{E}' + (\tilde{N} + \tilde{N}') &= \alpha_{x_3}\\
\tilde{E} + \tilde{E}' - (\tilde{N} + \tilde{N}') &= \alpha_{x_4}\\
(-1)^{l(x_5) - a(x_5)}(\tilde{E} - \tilde{E}' + (\tilde{M} - \tilde{M}')) &= \alpha_{x_5}\\
(-1)^{l(x_6) - a(x_6)}(\tilde{E} - \tilde{E}' - (\tilde{M} - \tilde{M}')) &= \alpha_{x_6}\\
(-1)^{l(x_7) - a(x_7)}(\tilde{E} - \tilde{E}' + (\tilde{N} - \tilde{N}')) &= \alpha_{x_7}\\
(-1)^{l(x_8) - a(x_8)}(\tilde{E} - \tilde{E}' - (\tilde{N} - \tilde{N}')) &= \alpha_{x_8}
\end{aligned}$$

and $l(x_i) \equiv a(x_i) \pmod 2$ for $1 \leqslant i \leqslant 4$.

Now using 16.6, we see that the eight elements $R_?^{\mathscr{L}}$ where ? runs over the last eight expressions are linear combinations with integral $\geqslant 0$ coefficients of character sheaves. From 14.13, the inner products (:) of these eight $R_?^{\mathscr{L}}$ are described by the matrix

$$\left|\begin{array}{cccc|cccc|}
4 & 0 & 2 & 2 & & & & \\
0 & 4 & 2 & 2 & & \bigcirc & & \\
2 & 2 & 4 & 0 & & & & \\
2 & 2 & 0 & 4 & & & & \\
\hline
 & & & & 4 & 0 & 2 & 2 \\
 & \bigcirc & & & 0 & 4 & 2 & 2 \\
 & & & & 2 & 2 & 4 & 0 \\
 & & & & 2 & 2 & 0 & 4
\end{array}\right. .$$

It follows that there exist 16 distinct character sheaves A_i $(1 \leqslant i \leqslant 16)$ such that

$$\begin{aligned}
R_{\tilde{E}}^{\mathscr{L}} + R_{\tilde{E}'}^{\mathscr{L}} + R_{\tilde{M}}^{\mathscr{L}} + R_{\tilde{M}'}^{\mathscr{L}} &= A_1 + A_2 + A_3 + A_4\\
R_{\tilde{E}}^{\mathscr{L}} + R_{\tilde{E}'}^{\mathscr{L}} - R_{\tilde{M}}^{\mathscr{L}} - R_{\tilde{M}'}^{\mathscr{L}} &= A_5 + A_6 + A_7 + A_8\\
R_{\tilde{E}}^{\mathscr{L}} + R_{\tilde{E}'}^{\mathscr{L}} + R_{\tilde{N}}^{\mathscr{L}} + R_{\tilde{N}'}^{\mathscr{L}} &= A_1 + A_2 + A_5 + A_6\\
R_{\tilde{E}}^{\mathscr{L}} + R_{\tilde{E}'}^{\mathscr{L}} - R_{\tilde{N}}^{\mathscr{L}} - R_{\tilde{N}'}^{\mathscr{L}} &= A_3 + A_4 + A_7 + A_8\\
(-1)^{l(x_5) - a(x_5)}(R_{\tilde{E}}^{\mathscr{L}} - R_{\tilde{E}'}^{\mathscr{L}} + R_{\tilde{M}}^{\mathscr{L}} - R_{\tilde{M}'}^{\mathscr{L}}) &= A_9 + A_{10} + A_{11} + A_{12}
\end{aligned}$$

$$(-1)^{l(x_6)-a(x_6)}(R_{\tilde{E}}^{\mathscr{L}} - R_{\tilde{E}'}^{\mathscr{L}} - R_{\tilde{M}}^{\mathscr{L}} + R_{\tilde{M}'}^{\mathscr{L}}) = A_{13} + A_{14} + A_{15} + A_{16}$$
$$(-1)^{l(x_7)-a(x_7)}(R_{\tilde{E}}^{\mathscr{L}} - R_{\tilde{E}'}^{\mathscr{L}} + R_{\tilde{N}}^{\mathscr{L}} - R_{\tilde{N}'}^{\mathscr{L}}) = A_9 + A_{10} + A_{13} + A_{14}$$
$$(-1)^{l(x_8)-a(x_8)}(R_{\tilde{E}}^{\mathscr{L}} - R_{\tilde{E}'}^{\mathscr{L}} - R_{\tilde{N}}^{\mathscr{L}} + R_{\tilde{N}'}^{\mathscr{L}}) = A_{11} + A_{12} + A_{15} + A_{16}.$$

From 17.18(a), we see that $\varepsilon_{A_i} = 1$ $(1 \leqslant i \leqslant 8)$, and $\hat{\varepsilon}_{A_i} = (-1)^{l(x_j)-a(x_j)}$ for $9 \leqslant i \leqslant 16$, $5 \leqslant j \leqslant 8$.

From this we can express each of $R_{\tilde{E}}^{\mathscr{L}}$, $R_{\tilde{E}'}^{\mathscr{L}}$, $R_{\tilde{M}}^{\mathscr{L}}$, $R_{\tilde{M}'}^{\mathscr{L}}$, $R_{\tilde{N}}^{\mathscr{L}}$, $R_{\tilde{N}'}^{\mathscr{L}}$, as an explicit combination of $\varepsilon_{A_i} A_i$ $(1 \leqslant i \leqslant 16)$ with coefficients of form $\pm\frac{1}{4}$, and we see that the pattern of (17.8.3) is established.

Case 4. Assume that $W'_{\mathscr{L}}/W_{\mathscr{L}}$ has order 3 and that $\Omega_{\mathscr{F}} = \Omega_{\mathscr{L}}$. Then E and M, N (if defined) extend to $W'_{\mathscr{L}}$-modules; we shall denote by $\tilde{E}$ and $\tilde{M}$, $\tilde{N}$ (if M, N are defined) the preferred extensions (see 17.1), and by $\tilde{E}_\phi$ and $\tilde{M}_\phi$, $\tilde{N}_\phi$ (if M, N are defined) the extensions obtained from $\tilde{E}$, $\tilde{M}$, $\tilde{N}$ by tensoring with a non-trivial character ϕ of $\Omega_{\mathscr{L}}$ (regarded as a representation of $W'_{\mathscr{L}}$ with kernel $W_{\mathscr{L}}$). Let ϕ', ϕ'' be the two non-trivial characters of $\Omega_{\mathscr{L}}$. When $\mathscr{F} = \{E\}$, the following result can be extracted from [6, (7.6.6)]: for any $\omega \in \Omega_{\mathscr{L}}$, there exists $x_\omega \in \omega W_{\mathscr{L}}$ such that $\tilde{E} + \phi'(\omega)\tilde{E}_{\phi'} + \phi''(\omega)\tilde{E}_{\phi''} = \alpha_{x_\omega}$ and $l(x_\omega) \equiv a(x_\omega)$ (mod 2).

Using 16.6, we see that the three elements $R_{\tilde{E}}^{\mathscr{L}} + \phi'(\omega)R_{\tilde{E}_{\phi'}}^{\mathscr{L}} + \phi''(\omega)R_{\tilde{E}_{\phi''}}^{\mathscr{L}}$ $(\omega \in \Omega_{\mathscr{L}})$ are linear combinations with integral $\geqslant 0$ coefficients of character sheaves. From 14.13, the inner products (:) of these elements are described by the matrix

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Hence there exist nine distinct character sheaves $A_{\omega,i}$ $(\omega \in \Omega_{\mathscr{L}},\ 1 \leqslant i \leqslant 3)$ such that

$$R_{\tilde{E}}^{\mathscr{L}} + \phi'(\omega)R_{\tilde{E}_{\phi'}}^{\mathscr{L}} + \phi''(\omega)R_{\tilde{E}_{\phi''}}^{\mathscr{L}} = A_{\omega,1} + A_{\omega,2} + A_{\omega,3} \qquad (\omega \in \Omega_{\mathscr{L}}).$$

Moreover, from 17.18(a), we see that $\varepsilon_{A_{\omega,i}} = 1$ for all ω, i. We have

$$R_{\tilde{E}}^{\mathscr{L}} = \frac{1}{3}\sum_{\omega,i} A_{\omega,i}$$
$$R_{\tilde{E}_\phi}^{\mathscr{L}} = \frac{1}{3}\sum_{\omega,i} \phi(\omega^{-1})A_{\omega,i}, \qquad \text{for} \quad \phi = \phi' \text{ or } \phi'',$$

and the pattern of (17.8.3) is established. When $\mathscr{F} = \{E, M, N\}$, the following result can be extracted from [6, (7.6.7)]: for any $\omega \in \Omega_{\mathscr{L}}$, there exist x_ω, y_ω, z_ω, $u_\omega \in \omega W_{\mathscr{L}}$ such that

$$(\tilde{E}+\tilde{M})+\phi'(\omega)(\tilde{E}_{\phi'}+\tilde{M}_{\phi'})+\phi''(\omega)(\tilde{E}_{\phi''}+\tilde{M}_{\phi''})=\alpha_{x_\omega}$$
$$(\tilde{E}-\tilde{M})+\phi'(\omega)(\tilde{E}_{\phi'}-\tilde{M}_{\phi'})+\phi''(\omega)(\tilde{E}_{\phi''}-\tilde{M}_{\phi''})=\alpha_{y_\omega}$$
$$(\tilde{E}+\tilde{N})+\phi'(\omega)(\tilde{E}_{\phi'}+\tilde{N}_{\phi'})+\phi''(\omega)(\tilde{E}_{\phi''}+\tilde{N}_{\phi''})=\alpha_{z_\omega}$$
$$(\tilde{E}-\tilde{N})+\phi'(\omega)(\tilde{E}_{\phi'}-\tilde{N}_{\phi'})+\phi''(\omega)(\tilde{E}_{\phi''}-\tilde{N}_{\phi''})=\alpha_{u_\omega}$$

and $l(x_\omega)\equiv a(x_\omega) \pmod 2$, $l(y_\omega)\equiv a(y_\omega) \pmod 2$, $l(z_\omega)\equiv a(z_\omega) \pmod 2$, $l(u_\omega)\equiv a(u_\omega) \pmod 2$. Using 16.6 we see that the four elements $R_?^{\mathscr{L}}$ where ? runs over the last four expressions (for fixed ω) are linear combinations with integral $\geqslant 0$ coefficients of character sheaves. The inner products (:) of these four $R_?^{\mathscr{L}}$ are described by the matrix

$$\begin{pmatrix} 6 & 0 & 3 & 3 \\ 0 & 6 & 3 & 3 \\ 3 & 3 & 6 & 0 \\ 3 & 3 & 0 & 6 \end{pmatrix}.$$

Moreover two $R_?^{\mathscr{L}}$ corresponding to distinct ω have $(:)=0$. It follows that there exist 36 distinct character sheaves $A_{\omega,i}$ ($\omega \in \Omega_{\mathscr{L}}$, $1 \leqslant i \leqslant 12$) such that

$$R_{\bar{E}}^{\mathscr{L}}+R_{\bar{M}}^{\mathscr{L}}+\phi'(\omega)(R_{\bar{E}_{\phi'}}^{\mathscr{L}}+R_{\bar{M}_{\phi'}}^{\mathscr{L}})+\phi''(\omega)(R_{\bar{E}_{\phi''}}^{\mathscr{L}}+R_{\bar{M}_{\phi''}}^{\mathscr{L}})$$
$$=A_{\omega,1}+A_{\omega,2}+A_{\omega,3}+A_{\omega,4}+A_{\omega,5}+A_{\omega,6}$$
$$R_{\bar{E}}^{\mathscr{L}}-R_{\bar{M}}^{\mathscr{L}}+\phi'(\omega)(R_{\bar{E}_{\phi'}}^{\mathscr{L}}-R_{\bar{M}_{\phi'}}^{\mathscr{L}})+\phi''(\omega)(R_{\bar{E}_{\phi''}}^{\mathscr{L}}-R_{\bar{M}_{\phi''}}^{\mathscr{L}})$$
$$=A_{\omega,7}+A_{\omega,8}+A_{\omega,9}+A_{\omega,10}+A_{\omega,11}+A_{\omega,12}$$
$$R_{\bar{E}}^{\mathscr{L}}+R_{\bar{N}}^{\mathscr{L}}+\phi'(\omega)(R_{\bar{E}_{\phi'}}^{\mathscr{L}}+R_{\bar{N}_{\phi'}}^{\mathscr{L}})+\phi''(\omega)(R_{\bar{E}_{\phi''}}^{\mathscr{L}}+R_{\bar{N}_{\phi''}}^{\mathscr{L}})$$
$$=A_{\omega,1}+A_{\omega,2}+A_{\omega,3}+A_{\omega,10}+A_{\omega,11}+A_{\omega,12}$$
$$R_{\bar{E}}^{\mathscr{L}}-R_{\bar{N}}^{\mathscr{L}}+\phi'(\omega)(R_{\bar{E}_{\phi'}}^{\mathscr{L}}-R_{\bar{N}_{\phi'}}^{\mathscr{L}})+\phi''(\omega)(R_{\bar{E}_{\phi''}}^{\mathscr{L}}-R_{\bar{N}_{\phi''}}^{\mathscr{L}})$$
$$=A_{\omega,4}+A_{\omega,5}+A_{\omega,6}+A_{\omega,7}+A_{\omega,8}+A_{\omega,9}.$$

From 17.18(a), we see that $\varepsilon_{A_{\omega,i}}=1$ for all ω,i. We can now express each of $R_{\bar{E}}^{\mathscr{L}}$, $R_{\bar{M}}^{\mathscr{L}}$, $R_{\bar{N}}^{\mathscr{L}}$, $R_{\bar{E}_{\phi'}}^{\mathscr{L}}$,... as an explicit combination of the $A_{\omega,i}$ with coefficients of form $\frac{1}{6}\times$ sixth root of 1, and we see that the pattern of (17.8.3) is established. This completes the proof of the proposition.

PROPOSITION 19.3. *Assume that $G/\mathscr{Z}_G$ has all its irreducible factors of type A, except possibly for one factor which is of type D_m $(4 \leqslant m \leqslant 7)$ or C_m $(2 \leqslant m \leqslant 3)$. Assume also that $|\mathscr{Z}_G/\mathscr{Z}_G^0| \leqslant 3$. Then* (17.8.3)–(17.8.5) *hold for G.*

Proof. We may assume that $\dim G \geqslant 1$ and that the proposition is already proved for groups satisfying the same assumptions as G, but of dimension strictly smaller than that of G. If we can prove (17.8.4) for G then (17.8.3) will also hold for G, by 19.2. Thus, it is enough to prove that (17.8.4), (17.8.5) hold for G.

Let $G_1, G_2, ..., G_r$ be the set of almost simple closed normal subgroups of G. Applying the results in 17.16 to the finite covering map $G_1 \times G_2 \times \cdots \times G_r \to G$ given by multiplication in G, and applying 17.11 to the product $G_1 \times G_2 \times \cdots \times G_r$, we see that we are reduced to the case where G is almost simple. Since the case where G is almost simple of type A is covered by 18.5, we see that we are reduced to the case where G is almost simple of type D_m $(4 \leqslant m \leqslant 7)$ or C_m $(2 \leqslant m \leqslant 3)$, and the centre of G has at most two elements. To check the statement (17.8.4) for G it is enough, using the induction hypothesis, to check that any cuspidal character sheaf A on G is clean and satisfies $\varepsilon_A = \hat{\varepsilon}_A$. We now consider the classification of irreducible cuspidal perverse sheaves on G, following [4]. (This list contains as a sub-list the cuspidal character sheaves of G; at this stage we do not know that the two lists coincide.) We shall write for any G:

(19.3.1) $\operatorname{Irr}^0 G =$ set of irreducible cuspidal perverse sheaves on G.

(a) $G = SO_5(k)$, char $k \neq 2$. There are exactly two complexes A', A'' in $\operatorname{Irr}^0 G$. They have the same support: the closure of the class of su whre s is a semisimple element with $Z_G(s) \cong O_4(k)$ and u is a regular unipotent element in $Z_G^0(s)$. Then A' and A'' are clean by 7.11(d) and 18.5 for $SO_4(k)$. Assume that $A' \in \hat{G}_{\mathscr{L}}$. To verify the parity condition (15.13.1) for A' we use 18.3. The possibilities for $\mathscr{L}$ are restricted by 17.12: we must have $W'_{\mathscr{L}} = W_{\mathscr{L}} = W$ or $W'_{\mathscr{L}} = W_{\mathscr{L}}$ of type $A_1 \times A_1$. In both cases, N in 18.3 is even and the representations of the corresponding Hecke algebra have traces in $Q[u, u^{-1}]$. Thus, 18.3 is applicable. We see that if A' or A'' is in $\hat{G}$ then it is clean and satisfies the parity condition. (If neither A', A'' is in $\hat{G}$ then there is nothing to prove.) Thus G satisfies (17.8.4). Now, by 19.2, we see that (17.8.3) holds for G.

Let $\mathscr{E}_i$ $(i = 1, 2)$ be the two local systems in $\mathscr{S}(T)$ whose stabilizer in W is W itself. From 11.2(e) it follows that $\hat{G}_{\mathscr{E}_1} \neq \hat{G}_{\mathscr{E}_2}$. Let $\mathscr{L}$ be either $\mathscr{E}_1$ or $\mathscr{E}_2$. We have $W'_{\mathscr{L}} = W_{\mathscr{L}} = W$. Let $A \in \hat{G}_{\mathscr{L}}$ be the character sheaf corresponding under (17.8.3) to the family $\mathscr{F} \subset \hat{W}$ such tat $\mathscr{G}_{\mathscr{F}} = \mathbb{Z}/2\mathbb{Z}$, and to the pair $(g_2, \varepsilon) \in \mathscr{M}(\mathscr{G}_{\mathscr{F}})$; here g_2 is the element $\neq e$ of $\mathbb{Z}/2\mathbb{Z}$ and ε is the non-trivial character of $\mathscr{G}_{\mathscr{F}}$. From (17.8.3) we see that $(\varepsilon_A A\colon R_\rho^{\mathscr{L}}) = \frac{1}{2}$,

$(\varepsilon_A A: R_{\varepsilon_1}^{\mathscr{L}}) = -\frac{1}{2}$, $(\varepsilon_A A: R_{\varepsilon_2}^{\mathscr{L}}) = -\frac{1}{2}$, where ρ is the two-dimensional irreducible representation of W and ε_1, ε_2 are its one-dimensional representations other than 1 and sign; we also see that $(\varepsilon_A A: R_1^{\mathscr{L}}) = 0$, $(\varepsilon_A A: R_{\text{sign}}^{\mathscr{L}}) = 0$. It is easy to check that the character of the virtual representation $\rho - \varepsilon_1 - \varepsilon_2$ of W vanishes on elements with some eigenvalue 1 in the reflection representation of W. It follows that $(\varepsilon_A A: \chi(K_w^{\mathscr{L}})) = 0$ (see 6.5) whenever $w \in W$ has some eigenvalue 1. From 18.2, it follows that A is cuspidal. We thus find two cuspidal character sheaves of G (one for $\mathscr{L} = \mathscr{E}_1$, on for $\mathscr{L} = \mathscr{E}_2$). As the set of cuspidal character sheaves of G is contained in the set $\{A', A''\}$, these two sets must coincide and (17.8.5) is verified.

(a′) $G = Sp_4(k)$, char $k \neq 2$. There is a unique complex in $\text{Irr}^0 G$. It is $A = \pi^* A' = \pi^* A''$ (A', A'' as in (a)), where $\pi: Sp_4(k) \to SO_5(k)$ is the standard double covering. Using (a) and the arguments in 17.16, we see that A is a character sheaf of G, that it is clean and that it satisfies $\varepsilon_A = \hat{\varepsilon}_A$.

(a″) G is simple of type B_2, char $k = 2$. There is a unique complex A in $\text{Irr}^0 G$. Its support is the unipotent variety of G. If A is a character sheaf, then it is clean by 7.9, and, as in the proof in (a) it satisfies the parity condition. Thus (17.8.4) is satisfied by G. (If A is not a character sheaf, there is nothing to verify.) By 19.2, we see that (17.8.3) holds for G. Arguing as in (a) with $\mathscr{L} = \bar{\mathbf{Q}}_l$, we see that $\hat{G}_{\mathscr{L}}$ contains a cuspidal character sheaf which is necessarily A. Thus, G satisfies (17.8.5).

(b) $G = PSp_6(k)$, char $k \neq 2$, or a simple group of type C_3, char $k = 2$. The set $\text{Irr}^0 G$ is empty, hence there is nothing to check.

(c) $G = Sp_6(k)$, char $k \neq 2$. The set $\text{Irr}^0 G$ consists of two complexes A', A''. The centre of G acts (11.5) nontrivially on both A', A''. The support of A' is the closure of a unipotent conjugacy class and A'' is obtained by applying t_z^* to A' for $z = -1 \in G$ (see 17.17). If $A' \in \hat{G}$ then it is clean by 7.9; from (17.17.1) it then follows that $A'' \in \hat{G}$ and is also clean. If $A'' \in \hat{G}$, then by (17.17.1) we have $A' \in \hat{G}$, hence again both A', A'' are clean. In any case, G is clean. If $A' \in \hat{G}_{\mathscr{L}}$ we show that $\varepsilon_{A'} = \hat{\varepsilon}_{A'}$ as follows. The possibilities for $\mathscr{L}$ are restricted by 17.12: we must have $W'_{\mathscr{L}} = W_{\mathscr{L}} = W$ or $W_{\mathscr{L}} = A_1 \times A_1 \times A_1$, $\Omega_{\mathscr{L}}$ of order 2 acting non-trivially on $W_{\mathscr{L}}$, or $W_{\mathscr{L}} = A_3$, $\Omega_{\mathscr{L}}$ of order 2 acting-nontrivially on $W_{\mathscr{L}}$, or $W_{\mathscr{L}} = B_2$, $\Omega_{\mathscr{L}}$ of order 2. In the case where $W'_{\mathscr{L}} = W_{\mathscr{L}} = W$, we see from 11.10 that the centre of G acts trivially on any character sheaf in $\hat{G}_{\mathscr{L}}$; thus A' cannot be in $\hat{G}_{\mathscr{L}}$. In the remaining cases, the hypothesis of 18.3 are verified: in all cases, N in 18.3 is even and the representations of the corresponding Hecke algebras have traces in $\mathrm{Q}[u, u^{-1}]$. From 18.3 we see that $\varepsilon_{A'} = \hat{\varepsilon}_{A'}$. The same argument applies to A'' if $A'' \in \hat{G}_{\mathscr{L}}$. Thus, G satisfies (17.8.4). By 19.2, we see that G also satisfies (17.8.3). Now let $\mathscr{L} \in \mathscr{S}(T)$ be such that $W_{\mathscr{L}}$ is of type B_2 and $\Omega_{\mathscr{L}}$ is of order 2. Let $\mathscr{F}$ be the family $\{\rho, \varepsilon_1, \varepsilon_2\}$ of $W_{\mathscr{L}} = B_2$ (see (a)), and let $\mathscr{F}'$ be the corresponding family of $W'_{\mathscr{L}}$. Then $\mathscr{G}_{\mathscr{F}'} = \mathscr{G}_{\mathscr{F}} \times \Omega_{\mathscr{L}}$. Consider the character sheaves $\tilde{A}'$, $\tilde{A}''$ in $\hat{G}_{\mathscr{L}}$ corresponding under (17.8.3) to $((g_2, g_2), \varepsilon \boxtimes 1)$ or $((g_2, g_2), \varepsilon \boxtimes \varepsilon)$ in $\mathscr{M}(\mathscr{G}_{\mathscr{F}'})$. Here g_2 denotes the element $\neq e$ of $\mathscr{G}_{\mathscr{F}}$ or $\Omega_{\mathscr{L}}$, ε denotes the non-trivial character of $\mathscr{G}_{\mathscr{F}}$ or $\Omega_{\mathscr{L}}$; when we write $\varepsilon \boxtimes 1$, the factor ε refers to $\mathscr{G}_{\mathscr{F}}$ and the factor 1 refers to $\Omega_{\mathscr{L}}$. We now use the fact that the character of the virtual representation $(\rho \boxtimes \varepsilon - \rho \boxtimes 1 - \varepsilon_1 \boxtimes \varepsilon + \varepsilon_1 \boxtimes 1 - \varepsilon_2 \boxtimes \varepsilon + \varepsilon_2 \boxtimes 1)$ of $W'_{\mathscr{L}} = W_{\mathscr{L}} \times \Omega_{\mathscr{L}}$ vanishes on elements of $W'_{\mathscr{L}}$ which have some eigenvalue 1 in the reflection representation of W. As in (a), we see that $\tilde{A}'$, $\tilde{A}''$ are cuspidal. Hence they are A' and A'' and (17.8.5) is verified for G.

(d) $G = PSO_8(k)$, char $k \neq 2$. There are exactly four complexes A_i $(1 \leqslant i \leqslant 4)$ in $\mathrm{Irr}^0 G$. They have the same support: the closure of the class of su, where s is a semisimple element such that $Z_G^0(s)$ is isogenous to $SL_2(k) \times SL_2(k) \times SL_2(k) \times SL_2(k)$ and u is a regular unipotent element in $Z_G^0(s)$. The A_i are clean by 7.11(d) and 18.5 for $Z_G^0(s)$. To verify the parity condition (15.3.1) for A_i (assumed to be in $\hat{G}_{\mathscr{L}}$) we use 18.3. The possibilities for $\mathscr{L}$ are restricted by 17.12: we must have $W'_{\mathscr{L}} = W_{\mathscr{L}} = W$ or $W'_{\mathscr{L}} = W_{\mathscr{L}}$ of type $A_1 \times A_1 \times A_1 \times A_1$. In both cases, N in 18.3 is even and the representations of the corresponding Hecke algebras have traces in $Q[u, u^{-1}]$; thus, 18.3 is applicable. It follows that G satisfies (17.8.4). Using 19.2, we see that (17.8.3) holds for G.

Let $\mathscr{E}_i$ $(1 \leqslant i \leqslant 4)$ be the four local systems in $\mathscr{S}(T)$ whose stabilizer in W is W itself. From 11.2(e), it follows that the sets $\hat{G}_{\mathscr{E}_i}$ $(1 \leqslant i \leqslant 4)$ are disjoint. Let $\mathscr{L}$ be one of the $\mathscr{E}_i$. We have $W'_{\mathscr{L}} = W_{\mathscr{L}} = W$. Define $A \in \hat{G}_{\mathscr{L}}$ exactly as in (a). From (17.8.3), we see that $(\varepsilon_A A : R_{\rho_8}^{\mathscr{L}}) = \frac{1}{2}$, $(\varepsilon_A A : R_{\rho_6}^{\mathscr{L}}) = -\frac{1}{2}$, $(\varepsilon_A A : R_{\rho_2}^{\mathscr{L}}) = -\frac{1}{2}$, where $\{\rho_8, \rho_6, \rho_2\}$ is the unique family in $\hat{W}$ with three members, dim $\rho_i = i$. Moreover, we have $(\varepsilon_A A : R_{\rho}^{\mathscr{L}}) = 0$ for all other $\rho \in \hat{W}$. It is easy to see that the character of the virtual representation $\frac{1}{2}(\rho_8 - \rho_6 - \rho_2)$ of W is concentrated on elements without eigenvalue 1 in the reflection representation of W. As in (a), it follows that A is cuspidal and that each of A_1, A_2, A_3, A_4 is a character sheaf. Thus, (17.8.5) holds for G.

(d′) $G = SO_8(k)$, char $k \neq 2$. Let π: $SO_8(k) \to PSO_8(k)$ the standard double covering. There are exactly two complexes A', A'' in $\mathrm{Irr}^0 G$. We may arrange notation so that $A' = \pi^* A_1 = \pi^* A_2$, $A'' = \pi^* A_3 = \pi^* A_4$. Using (d) and the arguments in 17.16, we see that A', A'' are clean character sheaves satisfying the parity condition.

(d″) G is simple of type D_4, char $k = 2$. There is a unique complex A in $\mathrm{Irr}^0 G$. Its support is the unipotent variety of G. If A is a character sheaf, then it is clean by 7.9 and as in the proof of (d) it satisfies the parity condition. Thus (17.8.4) is satisfied for G. By 19.2, we see that (17.8.3) holds for G. Arguing as in (d) with $\mathcal{L}=\bar{\mathbf{Q}}_l$ we see that $\hat{G}_{\mathcal{L}}$ contains a cuspidal character sheaf which is necessarily A. Thus, G satisfies (17.8.5).

(e) G is simple of type D_5 or D_6, $\operatorname{char} k=2$, or $G=PSO_{10}(k)$, $SO_{10}(k)$, $PSO_{12}(k)$, or $SO_{12}(k)$, $\operatorname{char} k\neq 2$. The set $\mathrm{Irr}^0 G$ is empty, hence there is nothing to prove.

(f) $G=\frac{1}{2}\mathrm{Spin}_{12}(k)$, $\operatorname{char} k\neq 2$. (The half spin group $\frac{1}{2}\mathrm{Spin}_{4n}(k)$ is the quotient $(\neq SO_{4n}(k))$ of $\mathrm{Spin}_{4n}(k)$ by a central subgroup of order 2, for $n\geqslant 3$.) There are exactly four complexes A_1, A_2, A_3, A_4 in $\mathrm{Irr}^0 G$. They have non-trivial action of the centre of G. They have the same support: the closure of the class of su, where s is a semisimple element such that $Z_G^0(s)$ is isogenous to $SL_4(k)\times SL_4(k)$ and u is a regular unipotent element in $Z_G^0(s)$. Each A_i is clean by 7.11(d) and 18.5. Hence G is clean. Let $\mathcal{L}\in\mathcal{S}(T)$ be such that $W_{\mathcal{L}}$ is of type A_5 and $\Omega_{\mathcal{L}}$ is of order 2 acting non-trivially on $W_{\mathcal{L}}$. (Up to W-conjugacy, there are two such $\mathcal{L}$.) Let E be the unique 16-dimensional irreducible representation of $W_{\mathcal{L}}$, let $\tilde{E}$ be its preferred extension (17.2) to $W'_{\mathcal{L}}$ and let $\tilde{E}'$ be the other extension of E to a $W'_{\mathcal{L}}$-module. From 17.19 we see that there exists $x\in W'_{\mathcal{L}}$ such that $\tilde{E}-\tilde{E}'=(-1)^{l(x)-\bar{l}(x)+1}\alpha_x$, $\bar{l}(x)\not\equiv a(x)\pmod 2$. Such x must necessarily be in $W'_{\mathcal{L}}-W_{\mathcal{L}}$, since for any $y\in W_{\mathcal{L}}$, $\tilde{E}$ and $\tilde{E}'$ appear with the same coefficient in α_y. It is easy to check that for our $\mathcal{L}$, the non-trivial element of $\Omega_{\mathcal{L}}$ has odd length in W. Hence $l(v)\equiv\bar{l}(v)+1\pmod 2$ for all $v\in W'_{\mathcal{L}}-W_{\mathcal{L}}$. It follows that $\tilde{E}-\tilde{E}'=\alpha_x$, $l(x)\equiv a(x)\pmod 2$. From 17.18 we see that $R_{\tilde{E}}^{\mathcal{L}}-R_{\tilde{E}'}^{\mathcal{L}}$ is a $\mathbb{Z}$-linear combination of character sheaves A such that $\varepsilon_A=1$. From 14.13, it follows that $(R_{\tilde{E}}^{\mathcal{L}}-R_{\tilde{E}'}^{\mathcal{L}}:R_{\tilde{E}}^{\mathcal{L}}-R_{\tilde{E}'}^{\mathcal{L}})=2$, hence we have $R_{\tilde{E}}^{\mathcal{L}}-R_{\tilde{E}'}^{\mathcal{L}}=\pm A'\pm A''$ where $A'\neq A''$ are two character sheaves. We have $\varepsilon_{A'}=\varepsilon_{A''}=1$. Now let E_1 be any irreducible representation of $W_{\mathcal{L}}$ and let $\tilde{E}_1$, $\tilde{E}'_1$ be its extensions to $W'_{\mathcal{L}}$. It is known from [6, 5.16] that $\tilde{E}_1-\tilde{E}'_1=\pm\alpha_y$ for some $y\in W'_{\mathcal{L}}-W_{\mathcal{L}}$. From 17.18 we see that $R_{\tilde{E}_1}^{\mathcal{L}}-R_{\tilde{E}'_1}^{\mathcal{L}}$ is a $\mathbb{Z}$-linear combination of character sheaves A such that $\varepsilon_A=-1$, whenever $E_1\neq E$. Hence $(A'\colon R_{\tilde{E}_1}^{\mathcal{L}}-R_{\tilde{E}'_1}^{\mathcal{L}})=0$. On the other hand, for any E_1, the character of $\tilde{E}_1+\tilde{E}'_1$ is concentrated on $W_{\mathcal{L}}$ and it follows from 11.10 that $R_{\tilde{E}_1}^{\mathcal{L}}+R_{\tilde{E}'_1}^{\mathcal{L}}$ is a $\bar{\mathbf{Q}}_l$-linear combination of character sheaves A with trivial action of the centre of G. The same result shows that A' has non-trivial action of the centre of G. It follows that $(A'\colon R_{\tilde{E}_1}^{\mathcal{L}}+R_{\tilde{E}'_1}^{\mathcal{L}})=0$. We deduce that $(A'\colon R_{\tilde{E}}^{\mathcal{L}})=-(A'\colon R_{\tilde{E}'}^{\mathcal{L}})=\pm 1/2$, $(A'\colon R_{\tilde{E}_1}^{\mathcal{L}})=(A'\colon R_{\tilde{E}'_1}^{\mathcal{L}})=0$ for $E_1\neq E$.

The virtual representation $\tilde{E}-\tilde{E}'$ of $W'_{\mathcal{L}}$ has character concentrated on elements in $W'_{\mathcal{L}}-W_{\mathcal{L}}$ without eigenvalue 1 in the reflection representation V of W. (This is proved as follows. Let $V=V_0\oplus V_0^{\perp}$ be the $W_{\mathcal{L}}$-stable decomposition of V with $V_0^{\perp}$ one-dimensional, and let γ be the generator of $\Omega_{\mathcal{L}}$. Any element γw, $w\in W_{\mathcal{L}}$, acts as -1 on $V_0^{\perp}$ and as $-w_0 w$ on V_0 where w_0 is the longest element in $W_{\mathcal{L}}$. Its trace on $\tilde{E}-\tilde{E}'$ is, up to sign, the trace of $w_0 w$ on E. Hence, it is enough to show the following: if $\mathrm{Tr}(w_0 w, E) \neq 0$ then $-w_0 w$ has no eigenvalue 1 on V_0. We can choose an isomorphism $W_{\mathscr{L}} \approx \mathrm{Sp}_4(F_2)$; then E becomes the Steinberg representation of $\mathrm{Sp}_4(F_2)$. Its character is zero on elements of order divisible by 2. Hence we are reduced to the following obvious statement: if $\sigma \in W_{\mathscr{L}}$ has odd order then it has no eigenvalue -1 on V_0.)

It follows that $(A' : \chi(K_w^{\mathscr{L}})) = 0$ (see 6.5) whenever $w \in W'_{\mathscr{L}}$ has some eigenvalue 1 on V. From 18.2 it follows that A' is cuspidal. Similarly, A'' is cuspidal. Thus, $\hat{G}_{\mathscr{L}}$ contains at least two cuspidal character sheaves. Since there are two choices for $\mathscr{L}$, as above, we see that A_i $(1 \leqslant i \leqslant 4)$ are exactly the cuspidal character sheaves of G, so that (17.8.5) is verified. To verify the parity condition it is enough to show that $\varepsilon_{A'} = \hat{\varepsilon}_{A'}$ for A' as above. We have seen already that $\varepsilon_{A'} = 1$. From (18.3.4) we see that $\hat{\varepsilon}_{A'} = 1$ since $\dim G$ is even. Thus, (17.8.4) is verified for G.

(g) G is simple of type D_7, char $k = 2$ or $G = PSO_{14}(k)$ or $SO_{14}(k)$, char $k \neq 2$. The set $\mathrm{Irr}^0 G$ is empty, hence there is nothing to prove.

This completes the proof of the proposition.

PROPOSITION 19.4. *Assume that G satisfies one of the following*:

(a) $G/\mathscr{Z}_G^0 = SO_7(k)$

(b) $G = SO_9(k)$, char $k \neq 2$.

(c) $G = PSO_{16}(k)$, char $k \neq 2$.

Then (17.8.3)–(17.8.5) *hold for G.*

Proof. (a) Since $\mathrm{Irr}^0 G$ (19.3.1) is empty, the statement (17.8.4) for G follows form the analogous statement for Levi subgroups of proper parabolic subgroups, where 19.3 applies. Using 19.2, we see that (17.8.3) holds for G. The statement (17.8.5) is empty in our case.

(b) In this case $\mathrm{Irr}^0 G$ consists of a single complex A. It support is the closure of a unipotent class in G. Since (17.8.5) holds for the Levi subgroups of proper parabolic subgroups by 19.3 and (a), we see from 7.9 that A is clean if it is a character sheaf. It follows that G is clean (without assumption on A). We now assume that $A \in \hat{G}_{\mathscr{L}}$ and prove that $\varepsilon_A = \hat{\varepsilon}_A$. The possibilities for $\mathscr{L}$ are restricted by 17.12: $W'_{\mathscr{L}} = W_{\mathscr{L}}$ must be of type B_4, $B_3 \times A_1$, or $B_2 \times B_2$. In each case, N in 18.3 is even and the representations of the corresponding Hecke algebras have traces in $Q[u, u^{-1}]$. By 18.3, we see that $\varepsilon_A = \hat{\varepsilon}_A$. Hence (17.8.4) holds for G. We now prove that $A \in \hat{G}$. Let $\mathscr{L} \in \mathscr{S}(T)$ be such that $W'_{\mathscr{L}} = W_{\mathscr{L}}$ is of type $B_2 \times B_2$. The proof of 19.2 (Case 1) applies without change as far as the families with one or three members in $W_{\mathscr{L}}$ are concerned and establishes (17.8.3) for them. We now consider the remaining family $\mathscr{F}$; it consists of nine representations $E \boxtimes E'$ where E, E' run over the representations ρ, ε_1, ε_2 (see the proof of

19.3 (a)) of the Weyl group B_2. As in the proof of 19.2 (Case 1) we see that the following are combinations with integral $\geqslant 0$ coefficients of character sheaves:

$$\begin{array}{ll} R^{\mathscr{L}}_{\rho \boxtimes \rho} + R^{\mathscr{L}}_{\rho \boxtimes \varepsilon_i} + R^{\mathscr{L}}_{\varepsilon_j \boxtimes \rho} + R^{\mathscr{L}}_{\varepsilon_j \boxtimes \varepsilon_i}, & i, j \in \{1, 2\} \\ R^{\mathscr{L}}_{\rho \boxtimes \rho} - R^{\mathscr{L}}_{\rho \boxtimes \varepsilon_i} - R^{\mathscr{L}}_{\varepsilon_j \boxtimes \rho} + R^{\mathscr{L}}_{\varepsilon_j \boxtimes \varepsilon_i}, & i, j \in \{1, 2\} \\ R^{\mathscr{L}}_{\rho \boxtimes \rho} + R^{\mathscr{L}}_{\rho \boxtimes \varepsilon_i} - R^{\mathscr{L}}_{\varepsilon_j \boxtimes \rho} - R^{\mathscr{L}}_{\varepsilon_j \boxtimes \varepsilon_i}, & i. j \in \{1, 2\} \\ R^{\mathscr{L}}_{\rho \boxtimes \rho} - R^{\mathscr{L}}_{\rho \boxtimes \varepsilon_i} + R^{\mathscr{L}}_{\varepsilon_j \boxtimes \rho} - R^{\mathscr{L}}_{\varepsilon_j \boxtimes \varepsilon_i}, & i, j \in \{1, 2\}. \end{array}$$

By 14.13, the inner product (:) of any two of these 16 expressions is known (it is 4, 2, 1, or 0). This forces the decomposition pattern of these expressions: there are 16 character sheaves $A_1, A_2, ..., A_{16}$ such that $(A_i : R^{\mathscr{L}}_{E \boxtimes E'}) = \pm\frac{1}{4}$ with the pattern of signs described by (17.8.3); from 17.18(a) we see that $\varepsilon_{A_i} = 1$. (In our case, $\mathscr{G}_{\mathscr{F}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.) We now consider the character sheaf A_{i_0}, $i_0 \in [1, 16]$, which under (17.8.3) corresponds to the pair $((g_2, g_2), \varepsilon \boxtimes \varepsilon) \in \mathscr{M}(\mathscr{G}_{\mathscr{F}})$. (Here, g_2 is the element $\neq e$ of $\mathbb{Z}/2\mathbb{Z}$ and ε is the non-trivial character of $\mathbb{Z}/2\mathbb{Z}$.) We have

$$\begin{array}{ll} (A_{i_0} : R^{\mathscr{L}}_{E \boxtimes E'}) = \frac{1}{4}, & \text{if} \quad E \boxtimes E' = \rho \boxtimes \rho, \varepsilon_i \boxtimes \varepsilon_j, i, j \in \{1, 2\} \\ \qquad = -\frac{1}{4}, & \text{otherwise.} \end{array}$$

It is easy to check that the character of the virtual representation $\rho \boxtimes \rho + \sum_{i,j} \varepsilon_i \boxtimes \varepsilon_j - \sum_i \rho \boxtimes \varepsilon_i - \sum_i \varepsilon_i \boxtimes \rho$ of $W_{\mathscr{L}}$ vanishes on elements which have some eigenvalue 1 in the reflection representation of W. It follows that $(A_{i_0} : \chi(K^{\mathscr{L}}_w)) = 0$ whenever $w \in W_{\mathscr{L}}$ has some eigenvalue 1. From 18.2, it follows that A_{i_0} is cuspidal. Hence $A = A_{i_0}$ and (17.8.5) is verified for G. We have also verified (17.8.3) for one particular $\mathscr{L}$. For the other $\mathscr{L}$, the proof of 19.2 is applicable; thus (17.8.3) holds for all $\mathscr{L}$.

(c) In this case, $\mathrm{Irr}^0 G$ consists of a single complex A. The proof of (b) applies with minor changes; the various $\mathscr{L}$ such that $\hat{G}_{\mathscr{L}}$ can possibly contain a cuspidal character sheaf have $W'_{\mathscr{L}} = W_{\mathscr{L}}$ of type D_8, $D_6 \times A_1 \times A_1$, $D_5 \times A_3$, $D_4 \times A_4$, and for the $\mathscr{L}$ such that $W_{\mathscr{L}}$ is of type $D_4 \times D_4$, we see exactly as in (b) that $\hat{G}_{\mathscr{L}}$ contains a cuspidal character sheaf (which must be A).

19.5. Let $G = \mathrm{Spin}_{10}(k)$, char $k \neq 2$. From [4] it follows that $\mathrm{Irr}^0 G$ consists of eight complexes A_i $(1 \leqslant i \leqslant 8)$. We may arrange notation, so that A_1, A_2 have the same support, the closure of a unipotent class, and A_i $(3 \leqslant i \leqslant 8)$ are of form $t_z^* A_1$ or $t_z^* A_2$ (17.17) where z runs through the non-trivial elements of the centre of G. Moreover, the centre of G acts on each A_i by characters of order 4. We now state the following result.

PROPOSITION 19.6. *With the notations in* 19.5, A_i $(1 \leqslant i \leqslant 8)$ *are clean, cuspidal character sheaves of* G.

Proof. If L is a Levi subgroup of a proper parabolic subgroup P of G, then either $L/\mathscr{Z}_L$ is a product of groups of type A, hence, 18.5 applies to it, or $\mathscr{Z}_L/\mathscr{Z}_L^0$ has order at most 2, hence, 19.3 applies to it. In particular, (17.8.4) and (17.8.5) hold for L. Now using 7.9, we see that if A_1 (or A_2) is a character sheaf, then it is clean. Since any other A_i is related to A_1 or A_2 by t_z^* (see 19.5) it follows that those A_i which are character sheaves are clean. Hence G is clean.

Let $\mathscr{L} \in \mathscr{S}(T)$ be such that $W_{\mathscr{L}}$ is of type A_2 and $\Omega_{\mathscr{L}}$ is cyclic of order 4, acting non-trivially on $W_{\mathscr{L}}$. (Note that $\mathscr{L}$ is uniquely determined up to W-conjugacy. It corresponds to a semisimple class in the dual group $PSO_{10}(k)$: the class containing the image of a semisimple element in $SO_{10}(k)$ with two eigenvalues 1, two eigenvalues -1, three eigenvalues $i = \sqrt{-1}$, and three eigenvalues $-i$.)

Let E be the two-dimensional irreducible representation of $W_{\mathscr{L}}$, 1 the unit representation, and σ the sign representation. For each character θ: $\Omega_{\mathscr{L}} \to \bar{\mathbf{Q}}_l^*$ we define the $W'_{\mathscr{L}}$-modules $\tilde{E}_\theta = E \otimes \theta$, $\tilde{1}_\theta = \tilde{1} \otimes \theta$, $\tilde{\sigma}_\theta = \tilde{\sigma} \otimes \theta$ extending E, 1, and σ, as in 17.3. Fix a generator ω of $\Omega_{\mathscr{L}}$. Let s_1, s_2 be the simple reflections of $W_{\mathscr{L}}$, so that $\omega s_1 \omega^{-1} = s_2$, $\omega s_2 \omega^{-1} = s_1$. We have

$$\alpha_{\omega s_2 s_1} = \alpha_{\omega s_1 s_2} = (-1)^{l(\omega)+1} \sum_\theta \theta(\omega) \tilde{E}_\theta$$

$$\alpha_\omega = (-1)^{l(\omega)} \sum_\theta \theta(\omega) \tilde{1}_\theta$$

$$\alpha_{\omega s_1 s_2 s_1} = (-1)^{l(\omega)} \sum_\theta \theta(\omega) \tilde{\sigma}_\theta$$

$$\alpha_{\omega s_1} = \alpha_{\omega s_2} = 0.$$

The same formulas remain valid when ω is replaced by ω^{-1}. On the other hand, for $i = 0$ or 2, we have

$$\alpha_{\omega^i s_1} = \alpha_{\omega^i s_2} = \sum_\theta \theta(\omega^i) \tilde{E}_\theta$$

$$\alpha_{\omega^i e} = \sum_\theta \theta(\omega^i) \tilde{1}_\theta$$

$$\alpha_{\omega^i s_1 s_2 s_1} = \sum_\theta \theta(\omega^i) \tilde{\sigma}_\theta.$$

From 17.18, we see that $\sum_\theta \theta(\omega) R^{\mathscr{L}}_{\tilde{E}_\theta}$ is a $\mathbb{Z}$-linear combination of character sheaves A such that $\varepsilon_A = (-1)^{l(\omega)+1}$ and such that A is mapped by 11.9 to the coset $\omega W_{\mathscr{L}}$. From the same result we see that $\sum_\theta \theta(\omega) R^{\mathscr{L}}_{\tilde{1}_\theta}$, $\sum \theta(\omega) R^{\mathscr{L}}_{\tilde{\sigma}_\theta}$ are $\mathbb{Z}$-linear combinations of character sheaves B such that $\varepsilon_B = (-1)^{l(\omega)}$; so these B are not among the A above. From 17.18(b) we see that for any $\rho \in \hat{W}_{\mathscr{L}}$, $\sum_\theta \theta(\omega^i) R^{\mathscr{L}}_{\tilde{\rho}_\theta}$ $(\omega^i \neq \omega)$ is a $\mathbb{Z}$-linear combination of character sheaves C which are mapped by 11.9 to the coset $\omega^i W_{\mathscr{L}} \neq \omega W_{\mathscr{L}}$; so these C are not among the A above. It follows that if $A \in \hat{G}_{\mathscr{L}}$ is such that $(A\colon \sum_\theta \theta(\omega) R^{\mathscr{L}}_{\tilde{E}_\theta}) \neq 0$ then we have:

$$\Big(A\colon \sum_\theta \theta(\omega^i) R^{\mathscr{L}}_{\tilde{E}_\theta}\Big) = 0, \qquad \text{if} \quad \omega^i \neq \omega$$

$$\Big(A\colon \sum_\theta \theta(\omega^i) R^{\mathscr{L}}_{\tilde{1}_\theta}\Big) = 0, \qquad \text{for all} \quad i$$

$$\Big(A\colon \sum_\theta \theta(\omega^i) R^{\mathscr{L}}_{\tilde{\sigma}_\theta}\Big) = 0, \qquad \text{for all} \quad i.$$

Hence we have

$$(A\colon R^{\mathscr{L}}_{\tilde{1}_\theta}) = 0, \qquad \text{for all } \theta$$

$$(A\colon R^{\mathscr{L}}_{\tilde{\sigma}_\theta}) = 0, \qquad \text{for all } \theta$$

$$(A\colon R^{\mathscr{L}}_{\tilde{E}_\theta}) = r(\omega)\, \theta(\omega)^{-1}, \qquad \text{for all } \theta.$$

where $r(\omega) \neq 0$ is independent of θ.

From this we can deduce (as in the proof of 19.3 (f)) that A is cuspidal, using 18.2. It is enough to check that the character of $\sum_\theta \theta(\omega)^{-1} \tilde{E}_\theta$ vanishes at all elements of $W'_{\mathscr{L}}$ which have some eigenvalue 1 in the reflection representation of W. The value of this character at $\omega^i x$ $(x \in W_{\mathscr{L}})$ is zero if $\omega^i \neq \omega$ and is $\pm \mathrm{Tr}(s_1 s_2 s_1 x, E)$ if $\omega^i = \omega$. The last trace is nonzero precisely when $x = s_1, s_2$ or $s_1 s_2 s_1$. Thus we must prove: if $x \in W_{\mathscr{L}}$ is a reflection, then ωx has no eigenvalue 1 on the reflection representation V of W. We identify the roots with vectors in V. We denote the simple roots by α_i $(1 \leqslant i \leqslant 5)$ so that $\alpha_1, \alpha_4, \alpha_5$ correspond to ends of the Dynkin diagram, α_3 to a branch point, and α_3 is joined to $\alpha_2, \alpha_4, \alpha_5$. Let s_i be the simple reflection corresponding to α_i. We may assume that $W_{\mathscr{L}}$ is generated by s_1, s_2. There are exactly two elements of order 4 in W which map α_1 to α_2 and α_2 to α_1. One of them maps $\alpha_4 \to \alpha_5 \to -\alpha_4 \to -\alpha_5 \to \alpha_4$ and the other maps $\alpha_4 \to -\alpha_5 \to -\alpha_4 \to \alpha_5 \to \alpha_4$; both map α_3 to $-\alpha_3$ + a combination of $\alpha_1, \alpha_2, \alpha_4, \alpha_5$. It follows that ω must be one of these two elements. From this, we see that the characteristic polynomial of $s_1\omega$ or $s_2\omega$ on V is $(q^3+1)(q^2+1)$ and the characteristic polynomial of $s_1 s_2 s_1 \omega$ on V is

$(q+1)^3(q^2+1)$. Neither of these polynomials has $q=1$ as a root. This proves that A is a cuspidal character sheaf. It is one of the A_i $(1 \leqslant i \leqslant 8)$ in 19.5.

From 19.5 we see that $t_z^* A$ ($z \in$ centre of G) are distinct. They are components of $\sum_\theta \theta(\omega) R_{\tilde{E}_\theta}^{\mathscr{L}}$, with the same multiplicity as A, since $\sum_\theta \theta(\omega) R_{\tilde{E}_\theta}^{\mathscr{L}}$ is invariant under all t_z^* (see (17.17.3)). Since $(\sum_\theta \theta(\omega) R_{\tilde{E}_\theta}^{\mathscr{L}} : \sum_\theta \theta(\omega) R_{\tilde{E}_\theta}^{\mathscr{L}}) = 4$ (by 4.13), it follows that $\sum_\theta \theta(\omega) R_{\tilde{E}_\theta}^{\mathscr{L}} = \pm \sum_z t_z^* A$ (z runs over the centre of G). An analogous result with the same proof holds for $\sum_\theta \theta(\omega^{-1}) R_{\tilde{E}_\theta}^{\mathscr{L}}$; it is up to sign a sum of four distinct cuspidal character sheaves. Moreover, these must be different from the $t_z^* A$ above since by 17.18(b), the first four are mapped by 11.9 to $\omega W_{\mathscr{L}}$ and the last four are mapped to $\omega^{-1} W_{\mathscr{L}}$. Thus, there exist at least eight different cuspidal character sheaves on G. It follows that each of the A_i $(1 \leqslant i \leqslant 8)$ in 19.5 is a cuspidal character sheaf. From this we deduce, as we have seen at the beginning of the proof, that each A_i is clean. The proposition is proved.

20. Groups of Type E_6, E_7, G_2

20.1. The main results in this chapter are 20.3, 20.5, 20.6 which assert that the statements (17.8.3)–(17.8.5) hold for the groups E_6, E_7, G_2, at least under certain restrictions on char k.

Proposition 20.2. *Let $\mathscr{L} \in \mathscr{S}(T)$ be such that $(G, \mathscr{L})$ satisfies* (17.8.4) *and $W'_{\mathscr{L}} = W_{\mathscr{L}}$. Assume that one of the following conditions is satisfied*:

(a) *$W_{\mathscr{L}}$ is of type $E_6 \times A_n$.*

(b) *$W_{\mathscr{L}}$ is of type $E_7 \times A_n$.*

(c) *$W_{\mathscr{L}}$ is of type G_2.*

Then (17.8.3) *holds for* $(G, \mathscr{L})$.

Proof. We fix a family $\mathscr{F}$ in $\hat{W}_{\mathscr{L}}$. When $\mathscr{F}$ consists of one or three representations, the argument in the proof of 19.2 (Case 1) applies without change and shows that the $R_E^{\mathscr{L}}$ $(E \in \mathscr{F})$ decompose according to the pattern of (17.8.3). Assume now that $\mathscr{F}$ is a family consisting of two representations. Then we are in case (b) and we have $\mathscr{F} = \{E, E'\}$ where $E = 512_a \boxtimes \rho$, $E' = 512'_a \boxtimes \rho$ where 512_a, $512'_a$ are as in 17.19 and ρ is an irreducible representation of the A_n-factor of $W_{\mathscr{L}}$. From [6, 5.22], we see that there exist $x, x' \in W_{\mathscr{L}}$ such that

$$\alpha_x = E - E', \qquad \bar{l}(x) \equiv a(x) + 1 \pmod 2$$

$$\alpha_{x'} = E + E', \qquad \bar{l}(x') \equiv a(x') \pmod 2.$$

From 16.6 it follows that $R_E^{\mathscr{L}} - R_{E'}^{\mathscr{L}}$ and $R_E^{\mathscr{L}} + R_{E'}^{\mathscr{L}}$ are combinations with integral $\geqslant 0$ coefficients of character sheaves. By 14.13, the inner products $(\ :\ )$ of these two elements are described by the matrix $\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)$. It follows that there exist four distinct character sheaves A_i $(1 \leqslant i \leqslant 4)$ in $\hat{G}_{\mathscr{L}}$ such that $R_E^{\mathscr{L}} - R_{E'}^{\mathscr{L}} = A_1 + A_2$, $R_E^{\mathscr{L}} + R_{E'}^{\mathscr{L}} = A_3 + A_4$. Thus, we have $R_E^{\mathscr{L}} = \frac{1}{2}(A_1 + A_2 + A_3 + A_4)$, $R_{E'}^{\mathscr{L}} = \frac{1}{2}(-A_1 - A_2 + A_3 + A_4)$. Moreover, by 17.18(a) we have $\varepsilon_{A_1} = \varepsilon_{A_2} = -1$, $\varepsilon_{A_3} = \varepsilon_{A_4} = 1$. Hence the pattern of (17.8.3) is established.

Next, we assume that in case (a) with $n = 0$, $\mathscr{F}$ is the family consisting of 80_s, 60_s, 90_s, 10_s, 20_s (notations of [6, 4.11]). By [6, 7.3] each of the virtual representations $80_s + \varepsilon \cdot 60_s + 10_s$, $80_s + \varepsilon \cdot 60_s + 90_s$, $2 \cdot 80_s - 10_s$, $2 \cdot 80_s - 90_s$, $80_s - 20_s$ $(\varepsilon = \pm 1)$ is of the form α_x for some $x \in W_{\mathscr{L}}$ such that $\tilde{l}(x) \equiv a(x) \pmod 2$. From 16.6, we see that the expressions $R_{80_s}^{\mathscr{L}} + \varepsilon R_{60_s}^{\mathscr{L}} + R_{10_s}^{\mathscr{L}}$, $R_{80_s}^{\mathscr{L}} + \varepsilon R_{60_s}^{\mathscr{L}} + R_{90_s}^{\mathscr{L}}$, $2 \cdot R_{80_s}^{\mathscr{L}} - R_{10_s}^{\mathscr{L}}$, $2R_{80_s}^{\mathscr{L}} - R_{90_s}^{\mathscr{L}}$, $R_{80_s}^{\mathscr{L}} - R_{20_s}^{\mathscr{L}}$ $(\varepsilon = \pm 1)$ are combinations with integral $\geqslant 0$ coefficients of character sheaves. Using [6, 7.7(iii)] we deduce that the pattern of decomposition of each R_E $(E \in \mathscr{F})$ is as in (17.8.3). (For each $A \in \hat{G}_{\mathscr{L}}$, we have $\varepsilon_A = 1$ by 17.18(a).) The same argument applies whenever $\mathscr{F}$ is a family with five representations. Essentially the same argument, using [6, 7.7(ii)] instead of [6, 7.7(iii)], applies in the case where $\mathscr{F}$ is a family consisting of four representations (which can only arise in case (c)). Since in our case, there are no families with more than five representations, the proposition is proved.

Proposition 20.3. *Assume that G is one of the following:*

(a) *an adjoint group of type E_6,*

(b) *a simply connected group of type E_6, with $\operatorname{char} k \neq 2$,*

(c) *an adjoint group of type E_7.*

Then (17.8.3)–(17.8.5) *hold for G.*

Proof. In many respects the proof is similar to that of 19.3.

(a) To any Levi subgroup of a proper parabolic subgroup of G, we may apply 19.3. Hence to check (17.8.4) for G it is enough to check that any cuspidal character sheaf of G is clean and satisfies the parity condition. The complexes in $\operatorname{Irr}^0 G$ are classified as follows [4]. If $\operatorname{char} k \neq 3$, $\operatorname{Irr}^0 G$ consists of six complexes with the same support: the closure of the conjugacy class of su where s is a semisimple element whose connected centralizer is isogenous to $SL_3(k) \times SL_3(k) \times SL_3(k)$ and u is a regular unipotent element in $Z_G^0(s)$; these complexes are clean by 7.11 (d) and 18.5.

If $\operatorname{char} k = 3$, $\operatorname{Irr}^0 G$ consists of two complexes with the same support: the unipotent variety of G; if one of these complexes is a character sheaf, then it is clean by 7.9, since (17.8.5) is known to hold for proper Levi subgroups. Hence G is clean.

If $\hat{G}_{\mathscr{L}}$ contains a cuspidal complex, we see from 17.12 that the possibilities for $\mathscr{L}$ are restricted: we must have $W'_{\mathscr{L}} = W_{\mathscr{L}}$ of type E_6, $A_2 \times A_2 \times A_2$ or $A_5 \times A_1$. In each of these cases, 18.3 shows that the parity condition is satisfied.

Thus, G satisfies (17.8.4). Now using 20.2(a) and 19.2, we deduce that G satisfies (17.8.3). To prove (17.8.5) it is enough to prove the following statement: if $W_{\mathscr{L}} = W$ then $\hat{G}_{\mathscr{L}}$ contains at least two distinct cuspidal character sheaves. (When char $k \neq 3$, there are precisely three $\mathscr{L}$ such that $W_{\mathscr{L}} = W$; when char $k = 3$, there is only one such $\mathscr{L}$.)

We consider $\mathscr{L} \in \mathscr{S}(T)$ such that $W_{\mathscr{L}} = W$. Let A_θ be the character sheaf in $\hat{G}_{\mathscr{L}}$, which under (17.8.3) corresponds to the family $\mathscr{F} = \{80_s, 60_s, 90_s, 10_s, 20_s\}$ (notation of [6, 4.11]) and to the element $(g_3, \theta) \in \mathscr{M}(\mathscr{G}_{\mathscr{F}})$ where g_3 is an element of order 3 of $\mathscr{G}_{\mathscr{F}} \cong \mathfrak{S}_3$ and θ is a non-trivial character of $Z_{\mathfrak{S}_3}(g_s) \cong \mathbb{Z}/3\mathbb{Z}$.

From (17.8.3) it follows that

$$(\varepsilon_{A_\theta} A_\theta : R_{80_s}^{\mathscr{L}}) = \tfrac{1}{3}, \qquad (\varepsilon_{A_\theta} A_\theta : R_{60_s}^{\mathscr{L}}) = 0, \qquad (\varepsilon_{A_\theta} A_\theta : R_{90_s}^{\mathscr{L}}) = -\tfrac{1}{3};$$
$$(\varepsilon_{A_\theta} A_\theta : R_{10_s}^{\mathscr{L}}) = -\tfrac{1}{3}, \qquad (\varepsilon_{A_\theta} A_\theta : R_{20_s}^{\mathscr{L}}) = \tfrac{1}{3},$$

and

$$(\varepsilon_{A_\theta} A_\theta : R_E^{\mathscr{L}}) = 0 \qquad \text{for all} \quad E \in \hat{W}_{\mathscr{L}}, E \notin \mathscr{F}.$$

In order to prove that A_θ is cuspidal it is enough, using 18.2 as in the proof of 19.3(a), to show that the character of the virtual representation $80_s - 90_s - 10_s + 20_s$ of W vanishes on all elements of W which have some eigenvalue 1 in the reflection representation W. This is easily verified, using for example the character table of W. Thus the A_θ for the two choices of θ are cuspidal and the proposition is proved in our case.

(b) If char $k = 3$, this is proved exactly as in (a). Hence, we may assume that char $k \neq 2, 3$. As in (a), to check (17.8.4) for G it is enough to check that any cuspidal character sheaf of G is clean and satisfies the parity condition. The complexes in $\mathrm{Irr}^0 G$ are classified as follows (see [4]). The set $\mathrm{Irr}^0 G$ consists of 14 complexes. Two of these have trivial action of $\mathscr{Z}_G$ and are of the form $\pi^* A$ where A is one of the cuspidal character sheaves of $G/\mathscr{Z}_G$ described in the proof of (a). (Here, $\pi\colon G \to G/\mathscr{Z}_G$ is the canonical map.) Hence these two complexes are clean character sheaves satisfying the parity condition. In addition, there are six complexes in $\mathrm{Irr}^0 G$ whose support has the following form: the closure of the conjugacy class of an element su where s is a semisimple element whose centralizer is isogenous to $SL_2(k) \times SL_6(k)$, and u is a regular unipotent element in $Z_G(s)$; these complexes are clean by 7.11(d) and 18.5.

Next, there are two complexes A', A'' in $\mathrm{Irr}^0 G$ with the same support: the closure of a unipotent class in G. Finally, $t_z^* A'$, $t_z^* A''$ (see 17.7), where z is a non-trivial element in $\mathscr{Z}_G$, are in $\mathrm{Irr}^0 G$. If one of the complexes A', A'' is a character sheaf, then it is clean, by 7.9. The same is then true for $t_z^* A'$, $t_z^* A''$. Thus, G is clean. Assume now that $A \in \hat{G}_{\mathscr{L}}$ is a cuspidal character sheaf. The possibilities for $\mathscr{L}$ are restricted by 17.12: we must have $W'_{\mathscr{L}} = W_{\mathscr{L}} = W$ or $W'_{\mathscr{L}} = W_{\mathscr{L}}$ of type $A_5 \times A_1$ or $W_{\mathscr{L}}$ of type $A_2 \times A_2 \times A_2$ with $\Omega_{\mathscr{L}}$ of order 3 acting by cyclic permutation of the factors, or $W_{\mathscr{L}}$ of type D_4 with $\Omega_{\mathscr{L}}$ of order 3 acting non-trivially on $W_{\mathscr{L}}$, or $W_{\mathscr{L}}$ of type $A_1 \times A_1 \times A_1 \times A_1$ with $\Omega_{\mathscr{L}}$ of order 3 acting non-trivially on $\Omega_{\mathscr{L}}$. Using 18.3, we see that A satisifes the parity condition. Thus, G satisfies (17.8.4). Using 19.2 and 20.1(a), we see that G satisfies (17.8.3). It is then enough to show that G has at least 14 distinct character sheaves. As we have seen earlier in the proof (as a consequence of (a)), G has at least two character sheaves with trivial action of $\mathscr{Z}_G$.

Let $\mathscr{L} \in \mathscr{S}(T)$ be such that $W_{\mathscr{L}}$ is of type D_4 and $\Omega_{\mathscr{L}}$ is of order 3, acting non-trivially on $W_{\mathscr{L}}$. Then $\mathscr{L}$ is uniquely determined up to W-conjugacy. It is enough to show that $\hat{G}_{\mathscr{L}}$ contains at least 12 character sheaves which are mapped by 11.9 to some nontrivial coset in $W'_{\mathscr{L}}/W_{\mathscr{L}}$. Let $\mathscr{F} \subset \hat{W}_{\mathscr{L}}$ be the family $\{\rho_8, \rho_6, \rho_2\}$ (notations as in the proof of 19.3(d)), and let $\mathscr{F}'$ be the corresponding family $\{(\tilde{\rho}_8)_\theta, (\tilde{\rho}_6)_\theta, (\tilde{\rho}_2)_\theta\}$ in $\hat{W}'_{\mathscr{L}}$. (Notations are as in the proof of 19.2 (Case 4); θ is any character of $\Omega_{\mathscr{L}}$.) We have $\mathscr{G}_{\mathscr{F}'} = \Omega_{\mathscr{F}} \times \mathbb{Z}/2\mathbb{Z}$. We denote by g any element of $\mathbb{Z}/2\mathbb{Z}$, by ω some generator of $\Omega_{\mathscr{F}}$, and by ε the non-trivial character of $\mathbb{Z}/2\mathbb{Z}$. Then $\theta \boxtimes \varepsilon$ is a character of $\mathscr{G}_{\mathscr{F}'}$ and we denote by $A_{\omega,g,\theta}$ the character sheaf in $\hat{G}_{\mathscr{L}}$ corresponding under (17.8.3) to $\mathscr{F}'$ and to $(\omega g, \theta \boxtimes \varepsilon) \in \mathscr{M}(\mathscr{G}_{\mathscr{F}'})$. When ω, g, θ vary, we get 12 different character sheaves. We shall show that $A_{\omega,g,\theta}$ is cuspidal. From (17.8.3), we see that

$$(\varepsilon_1 A_{\omega,g,\theta} : R^{\mathscr{L}}_{(\tilde{\rho}_8)_{\theta'}}) = \tfrac{1}{6}\theta'(\omega)^{-1}, \qquad (\varepsilon_1 A_{\omega,g,\theta} : R^{\mathscr{L}}_{(\tilde{\rho}_2)_{\theta'}}) = -\tfrac{1}{6}\theta'(\omega)^{-1},$$
$$(\varepsilon_1 A_{\omega,g,\theta} : R^{\mathscr{L}}_{(\tilde{\rho}_6)_{\theta'}}) = \tfrac{1}{6}\theta'(\omega)^{-1}\varepsilon(g), \qquad (\varepsilon_1 A_{\omega,g,\theta} : R^{\mathscr{L}}_{E_1}) = 0$$

for any $E_1 \in \hat{W}'_{\mathscr{L}} - \mathscr{F}'$. Here, $\varepsilon_1 \in \{\pm 1\}$. To show that $A_{\omega,g,\theta}$ is cuspidal, it is enough, using 18.2 as in the proof of 19.3(a), to show that the character of the virtual representation

$$\sum_{\theta'} \theta'(\omega)^{-1}((\tilde{\rho}_8)_{\theta'} - (\tilde{\rho}_2)_{\theta'} + \varepsilon(g)(\tilde{\rho}_6)_{\theta'})$$

of $W'_{\mathscr{L}}$ vanishes on all elements of $W'_{\mathscr{L}}$ which have some eigenvalue 1 in the reflection representation of W. This, in turn, follows from the following statement.

(20.3.1) The characters of both virtual representations $\tilde{\rho}_8 - \tilde{\rho}_2$ and $\tilde{\rho}_6$ of $W'_{\mathscr{L}}$ vanish on all elements of form $\omega x \in W'_{\mathscr{L}}$ ($\omega =$ generator of $\Omega_{\mathscr{F}}$, $x \in W_{\mathscr{L}}$) which have some eigenvalue 1 in the reflection representation of W.

This can be checked as follows. We may assume that $W_{\mathscr{L}}$ is generated by four simple reflections of W. The normalizer W' of $W_{\mathscr{L}}$ in W is isomorphic to a Weyl group of type F_4 and $W'_{\mathscr{L}}$ is a subgroup of index 2 of W'. The representations $\tilde{\rho}_8$, $\tilde{\rho}_2$, $\tilde{\rho}_6$ of $W'_{\mathscr{L}}$ extend to representations of W'. We can then make use of the character table of a Weyl group of type F_4 and see that (20.3.1) holds. This completes the proof of the proposition in case (b).

(c) If L is the Levi subgroup of a proper parabolic subgroup of G, we may apply either (a) or 19.3 to $L/\mathscr{Z}_L^0$. Hence (17.8.4) holds for $L/\mathscr{Z}_L^0$ and for L. Hence to check (17.8.4) for G it is enough to check that any cuspidal character sheaf of G is clean and satisfies the parity condition. The complexes in $\mathrm{Irr}^0 G$ are classified as follows. If char $k \neq 2$, then $\mathrm{Irr}^0 G$ consists of four complexes with the same support: the closure of the conjugacy class of su where s is a semisimple element whose connected centralizer is isogenous to $SL_4(k) \times SL_4(k) \times SL_2(k)$ and u is a regular unipotent element in $Z_G^0(s)$; these complexes are clean by 7.11(d) and 18.5. If char $k = 2$, then $\mathrm{Irr}^0 G$ consists of two complexes; they have the same support: the unipotent variety of G. If one of these is a character sheaf, then it is clean by 7.9, since (17.8.5) is known to hold for proper Levi subgroups. Hence G is clean.

Let $\mathscr{L} \in \mathscr{S}(T)$ be such that $W'_{\mathscr{L}} = W_{\mathscr{L}} = W$. If char $k \neq 2$, then there are two such $\mathscr{L}$, and if char $k = 2$, there is a unique such $\mathscr{L}$. If we show that $\hat{G}_{\mathscr{L}}$ contains at least two cuspidal character sheaves A_1, A_2 with $\varepsilon_{A_1} = \varepsilon_{A_2} = -1$ then it will follow that (17.8.4) and (17.8.5) are verified for G. (We have necessarily $\hat{\varepsilon}_{A_1} = \hat{\varepsilon}_{A_2} = -1$, since G has odd dimension; see (18.3.4).) By [6, 5.22] there exist $x, x' \in W_{\mathscr{L}}$ such that

$$\alpha_x = 512_a - 512'_a, \qquad l(x) \equiv a(x) + 1 \qquad (\mathrm{mod}\ 2)$$
$$\alpha_{x'} = 512_a - 512'_a, \qquad l(x) \equiv a(x) \qquad (\mathrm{mod}\ 2)$$

(notations of 17.19). If we set $E = 512_a$, $E' = 512'_a$, it follows from 17.18 that $R_E^{\mathscr{L}} - R_{E'}^{\mathscr{L}}$ is a $\mathbb{Z}$-linear combination of character sheaves A such that $\varepsilon_A = -1$ and that $R_E^{\mathscr{L}} + R_{E'}^{\mathscr{L}}$ is a $\mathbb{Z}$-linear combination of character sheaves A' such that $\varepsilon_{A'} = 1$. By 14.13 we have $(R_E^{\mathscr{L}} - R_{E'}^{\mathscr{L}} : R_E^{\mathscr{L}} - R_{E'}^{\mathscr{L}}) = 2$, hence, there exist two character sheaves $A_1 \neq A_2$ such that $R_E^{\mathscr{L}} - R_{E'}^{\mathscr{L}} = \pm A_1 \pm A_2$. Moreover, we have $\varepsilon_{A_1} = \varepsilon_{A_2} = -1$ hence, $(A_i : R_E^{\mathscr{L}} + R_{E'}^{\mathscr{L}}) = 0$, for $i = 1, 2$. Any $E_1 \in \hat{W}_{\mathscr{L}}$ othen than E, E' is a Q-linear combination of elements α_y for y in a two-sided cell other than that of x (see 16.4). From 17.19, for all such y, we have $l(y) \equiv a(y)$ (mod 2); now using 17.18, we see that $R_{E_1}^{\mathscr{L}}$ is a $\mathbb{Q}$-

linear combination of character sheaves A such that $\varepsilon_A = 1$. It follows that $(A_i : R_{E_1}^{\mathscr{L}}) = 0$, for $i = 1, 2$ and $E_1 \neq E, E'$. We also see that

$$(A_i : R_E^{\mathscr{L}}) = -(A_i : R_{E'}^{\mathscr{L}}) = \pm\tfrac{1}{2} \qquad (i = 1, 2),$$

We shall show that A_1, A_2 are cuspidal. Using 18.2 as in the proof of 19.3(a), we see that it is enough to show that the character of the virtual representation $512_a - 512'_a$ of W vanishes on all elements of W which have some eigenvalue 1 in the reflection representation V of W. This is proved as follows. We identify W with $\mathrm{Sp}_6(F_2) \times \mathbb{Z}/2\mathbb{Z}$ and 512_a, $512'_a$ with the two extensions of the Steinberg representation of $\mathrm{Sp}_6(F_2)$ to $\mathrm{Sp}_6(F_2) \times \mathbb{Z}/2\mathbb{Z}$. (The $\mathbb{Z}/2\mathbb{Z}$-factor is generated by the longest element w_0 of W.) The character of the Steinberg representation is zero on elements of order divisible by 2. Since w_0 acts as -1 on V, we are reduced to the following obvious statement: if $w \in W$ has odd order then it has no eigenvalue -1 on V. (Compare with the proof in 19.3(f).) Thus, A_1, A_2 are cuspidal. This shows that (17.8.4), (17.8.5) hold for G. We may now apply 20.2(b) and we see that (17.8.3) also holds for G. This completes the proof of the proposition.

COROLLARY 20.4. *Assume that G is an adjoint group of type E_6. Let $f\colon G \to G$ be the non-trivial outer automorphism of G such that $f(B) = B$, $f(T) = T$. Let $\mathscr{L} = \bar{\mathbf{Q}}_l \in \mathscr{S}(T)$. Then for any $A \in \hat{G}_{\mathscr{L}}$, we have $f^*A \approx A$.*

Proof. We have $W_{\mathscr{L}} = W$. It is clear that f^* takes $\hat{G}_{\mathscr{L}}$ to itself. For any $E \in \hat{W}$, we have $f^*R_E^{\mathscr{L}} = R_{\bar{E}}^{\mathscr{L}}$ where $\bar{E}$ is the W-module obtained from E by composition with the automorphism of W induced by f. As this automorphism of W is inner (conjugation by the longest element), we have $\bar{E} = E$ hence, $f^*R_E^{\mathscr{L}} = R_E^{\mathscr{L}}$. Hence, for any $E \in \hat{W}$ and any $A \in \hat{G}_{\mathscr{L}}$ we have $(f^*A : R_E^{\mathscr{L}}) = (A : R_E^{\mathscr{L}})$. From (17.8.3) for $(G, \mathscr{L})$, we see that $A \in \hat{G}_{\mathscr{L}}$ is completely determined by the multiplicities $(A : R_E^{\mathscr{L}})$, $(E \in \hat{W})$, except when A is cuspidal.

Hence, $f^*A \approx A$ if $A \in \hat{G}_{\mathscr{L}}$ is non-cuspidal. Assume now that $A \in \hat{G}_{\mathscr{L}}$ is cuspidal and $\operatorname{char} k = 3$. Then A is completely described by a (non-trivial) one-dimensional representation of the group $Z(u)/Z^0(u)$ of components of the centralizer of a regular unipotent element $u \in G$. (The support of A is the closure of the class of u.) We can choose u such that $f(u) = u$; it is then enough to show that f acts trivially on $Z(u)/Z^0(u)$. This follows from the known fact that $Z(u)/Z^0(u)$ is a cyclic group (of order 3) generated by the image of u.

Assume next that $\operatorname{char} k \neq 3$. Let $s \in T$ be a semisimple element such that the simple root corresponding to the branch point takes the value $\theta \in k^*$ ($\theta^3 = 1$, $\theta \neq 1$) on s and all other simple roots take the value 1 on s. Then $Z^0(s)$ is isogenous to $SL_3(k) \times SL_3(k) \times SL_3(k)$. Let $u \in Z^0(s)$ be a regular unipotent element. We have $f(s) = s$ and we may assume that $f(u) = u$. The group of components $Z(su)/Z^0(su)$ is isomorphic to $C_3 \times C_3'$ where C_3, C_3' are cyclic groups of order 3 and C_3 is generated by the image of su. The action of f on $Z(su)/Z^0(su)$ is as follows: f is the identity on C_3 and f acts as $g \to g^{-1}$ on C_3'. The six cuspidal character sheaves of G are supported by the closure of the class of su; they are completely described by a one-dimensional representation of $Z(su)/Z^0(su) = C_3 \times C_3'$ which is non-trivial on C_3. From the description of the action of f on $C_3 \times C_3'$ given above it follows that there are exactly two cuspidal character sheaves of G which are fixed by f^*. If a cuspidal character sheaf is in $\hat{G}_{\mathscr{L}'}$, $\mathscr{L}' \neq \mathscr{L}$, then it is not fixed by f^*, since f maps $\mathscr{L}'$ to $\mathscr{L}'^{-1}$ which is not in the W-orbit of $\mathscr{L}'$. This implies that the two cuspidal character sheaves in $\hat{G}_{\mathscr{L}}$ must be fixed by f^*. The corollary is proved.

PROPOSITION 20.5. *Assume that G is simply connected of type E_7,* char $k \neq 3$. *Then* (17.8.3)–(17.8.5) *hold for G.*

Proof. If char $k = 2$, this is proved exactly as in 20.3(c). We assume now that char $k \neq 2, 3$ and we denote by z the non-trivial element in $\mathscr{Z}_G$. By 19.3 and 20.3(a), the statements (17.8.3)–(17.8.5) hold for $L/\mathscr{Z}_L^0$ where L is any Levi subgroup of a proper parabolic subgroup of G; hence they also hold for L.

According to [4], the set $\mathrm{Irr}^0 G$ consists of eight complexes. Two of these have trivial action of $\mathscr{Z}_G$ and are of form $\pi^* A$, where A is one of the cuspidal character sheaves of $G/\mathscr{Z}_G$ described in the proof of 20.3(c). (Here, π: $G \to G/\mathscr{Z}_G$ is the canonical map.) Hence, these two complexes are clean character sheaves satisfying the parity condition; they belong to $\hat{G}_{\mathscr{L}_0}$ where $\mathscr{L}_0 = \bar{\mathbf{Q}}_l$. In addition, there is a unique complex $A \in \mathrm{Irr}^0 G$ whose support is the closure of a unipotent class. If it is a character sheaf, then it is clean, by 7.9. The same is true for $t_z^* A \in \mathrm{Irr}^0 G$. Note that $t_z^* A \neq A$ since they have different support.

Next, there are two complexes A, $A'' \in \mathrm{Irr}^0 G$ with the same support: the closure of the conjugacy class of su, where s is a fixed semisimple element whose centralizer is isogenous to $SL_6(k) \times SL_3(k)$, and u is a regular unipotent element in $Z_G(s)$. Then A', A'' are clean by 7.11(d) and 18.5; the same holds for $t_z^* A'$, $t_z^* A'' \in \mathrm{Irr}^0 G$.

Note that $t_z^* A'$, $t_z^* A''$ have the same support which is different from the support of A', A'' since zs is not conjugate to s. This completes the list of complexes in $\mathrm{Irr}^0 G$. The complexes A, $t_z^* A$, A', A'', $t_z^* A'$, $t_z^* A''$ all have non-trivial action of $\mathscr{Z}_G$ and none of then is fixed by t_z^*. We see that G is clean.

To prove that G satisfies (17.8.4), it is now enough to prove that for any cuspidal character sheaf A of G on which $\mathscr{Z}_L$ acts non-trivally, we have $\varepsilon_A = \hat{\varepsilon}_A$.

If $A \in \hat{G}_{\mathscr{L}}$, then $\Omega_{\mathscr{L}}$ must be of order 2 (since $\mathscr{Z}_G$ acts non-trivially on A) and the possibilities for $\mathscr{L}$ are further restricted by 17.12. We must have

(i) $W_{\mathscr{L}}$ of type $A_3 \times A_3 \times A_1$, with $\Omega_{\mathscr{L}}$ switching the two A_3-factors.

(ii) $W_{\mathscr{L}}$ of type A_7, with $\Omega_{\mathscr{L}}$ acting non-trivially on $W_{\mathscr{L}}$.

(iii) $W_{\mathscr{L}}$ of type E_6, with $\Omega_{\mathscr{L}}$ acting non-trivially on $W_{\mathscr{L}}$.

(iv) $W_{\mathscr{L}}$ of type $D_4 \times A_1 \times A_1$, with $\Omega_{\mathscr{L}}$ acting non-trivially both on the D_4-factors and on the $A_1 \times A_1$ factor.

(v) $W_{\mathscr{L}}$ of type $A_2 \times A_2 \times A_2$, with $\Omega_{\mathscr{L}}$ switching two of the A_2-factors.

In all cases, $\Omega_{\mathscr{L}}$ has four orbits on the set of simple reflections of $W_{\mathscr{L}}$. If we are in case (i), (iv), or (v), the representations of the Hecke algebra corresponding to $W'_{\mathscr{L}}$ have traces in $\mathbb{Q}[u, u^{-1}]$, and from 18.3, we see that $\varepsilon_A = \hat{\varepsilon}_A$. Assume now that we are in case (ii) or (iii). Since A is cuspidal, we have $dA = \hat{\varepsilon}_A A$ (see 15.5). Since $\mathscr{Z}_G$ acts non-trivially on A, we have $t_z^* A \neq A$, as we have seen earlier in the proof. Now using 17.21, it follows that $\varepsilon_A = -1$. (It is easy to check that the generator of $\Omega_{\mathscr{L}}$ has odd length in W.) Since G has odd dimension and A is cuspidal, we have $\hat{\varepsilon}_A = -1$ (see (18.3.4)). Hence, again we have $\hat{\varepsilon}_A = \varepsilon_A$. This shows that G satisfies (17.8.4).

We now show that $(G, \mathscr{L})$ satisfies (17.8.3) for any $\mathscr{L} \in \mathscr{S}(T)$. If $\mathscr{L}$ is not as in (iii) above, this follows from 20.2 and 19.2. Hence we may assume that $\mathscr{L}$ is as in (iii) above. We may also assume that $W_{\mathscr{L}}$ is generated by a subset I of S. Any family in $W'_{\mathscr{L}}$ consists of 2, 6, or 10 representations. For families with 2 or 6 representations, we may argue exactly as in the proof of 19.2 (case 3).

Let us now consider the family $\mathscr{F}'$ in $\hat{W}'_{\mathscr{L}}$ consisting of the representations $\widetilde{80}_s$, $\widetilde{60}_s$, $\widetilde{90}_s$, $\widetilde{10}_s$, $\widetilde{20}_s$ (preferred extensions to $W'_{\mathscr{L}}$ of the representations 80_s, 60_s, 90_s, 10_s, 20_s of $W_{\mathscr{L}}$; see [6, 4.11]) and $\overline{80}_s$, $\overline{60}_s$, $\overline{90}_s$, $\overline{10}_s$, $\overline{20}_s$ (the non-preferred extensions). We have $W_{\mathscr{L}} = W_I = W'_{\mathscr{L},I}$ (notation of 15.6). Let $\mathscr{F}$ be the family $\{80_s, 60_s, 90_s, 10_s, 20_s\}$ of $\hat{W}_{\mathscr{L}}$. Let L_I, $R_E^{\mathscr{L},I}$ be as in 15.6 and let i_I^S be as in 15.3. If $A, A' \in (\hat{L}_I)_{\mathscr{L}}$, we have

$$(20.5.1) \quad (i_I^S A : i_I^S A') = (A : A') + (A : f^* A')$$

where $f\colon L_I \to L_I$ is the map given by conjugation by a representative in $N(T)$ of the non-trivial element in $\Omega_{\mathscr{L}}$. The proof of (20.5.1) is almost identical to that of (17.12.6); the only difference is that x in that proof is not necessarily 1; it can be any element of $\Omega_{\mathscr{L}}$. We have $\mathscr{L} = \pi_0^* \mathscr{E}_0 | T$, where $\pi_0\colon L_I \to L_I/(L_I)_{\text{der}}$ is as in 17.9 and $\mathscr{E}_0 \in \mathscr{S}(L_I/L_I)_{\text{der}})$ is the unique local system such that $\mathscr{E}_0 \neq \bar{\mathbf{Q}}_l$, $\mathscr{E}_0^{\otimes 2} = \bar{\mathbf{Q}}_l$. Let $\pi_1\colon L_I \to (L_I)_{\text{ad}}$ be the canonical projection, $\bar{T}$ the image of T under π_1, and let $\mathscr{L}_1$ be the local system $\bar{\mathbf{Q}}_l$

on $\bar{T}$. From 17.9 and 17.10 it follows that $A_1 \to \tilde{\pi}_1 A \otimes \pi_0^* \mathscr{E}_0$ is a bijection $(\widehat{(L_I)_{\mathrm{ad}}})_{\mathscr{L}_0} \to (\hat{L}_I)_{\mathscr{L}}$. Now using 20.4, and the isomorphism $f^*(\pi_0^*\mathscr{E}_0) = \pi_0^*\mathscr{E}_0$, we see that $f^*A' = A'$ for all $A' \in (\hat{L}_I)_{\mathscr{L}}$. Hence (20.5.1) becomes

$$(20.5.2) \quad (i_I^S A : i_I^S A') = 2(A : A').$$

This implies that for each character sheaf $A \in (\hat{L}_I)_{\mathscr{L}}$, there exist two character sheaves $\tilde{A} \neq \tilde{A}'$ on G such that

$$(20.5.3) \quad i_I^S A = \tilde{A} + \tilde{A}'.$$

Moreover, A is uniquely determined by either $\tilde{A}$ or $\tilde{A}'$. If $E \in \mathscr{F}$, we have $\mathrm{ind}_{W_{\mathscr{L}}}^{W'_{\mathscr{L}}} = \tilde{E} + \bar{E}$; using 15.7(i) it follows that $i_I^S(R_E^{\mathscr{L},I}) = R_{\tilde{E}}^{\mathscr{L}} + R_{\bar{E}}^{\mathscr{L}}$. Since (17.8.3) is already known for $(L_I, \mathscr{L})$ (it follows from 20.3(a)), we know that $(\hat{L}_I)_{\mathscr{L},\mathscr{F}}$ consists of eight character sheaves $A_1, A_2, \ldots, A_8$ and we know explicitly the coefficients $c_{i,E}$ in

$$R_E^{\mathscr{L},I} = \sum_{i=1}^{8} c_{i,E} A_i \qquad (E \in \mathscr{F}).$$

Applying i_I^S and using (20.5.3) we find

$$(20.5.4) \quad R_{\tilde{E}}^{\mathscr{L}} + R_{\bar{E}}^{\mathscr{L}} = \sum_{i=1}^{8} c_{i,E}(\tilde{A}_i + \tilde{A}'_i) \qquad (E \in \mathscr{F}).$$

In particular, the 16 character sheaves $\tilde{A}_i$, $\tilde{A}'_i$ $(1 \leqslant i \leqslant 8)$ are in $(\hat{G})_{\mathscr{L},\mathscr{F}'}$ and the multiplicities $(\tilde{A}_i : R_{\tilde{E}}^{\mathscr{L}} + R_{\bar{E}}^{\mathscr{L}}) = (\tilde{A}'_i : R_{\tilde{E}}^{\mathscr{L}} + R_{\bar{E}}^{\mathscr{L}}) = c_{i,E}$ are known.

Note also that $\tilde{A}_i$, $\tilde{A}'_i$ have trivial action of $\mathscr{Z}_G$. (Indeed, L_I has a connected centre; this implies that each $i_I^S A_i$ has trivial action of $\mathscr{Z}_G$, hence our assertion.)

This gives only a part of the pattern (17.8.3) for $\mathscr{F}'$. To get the full pattern we must also decompose the differences $R_{\tilde{E}}^{\mathscr{L}} - R_{\bar{E}}^{\mathscr{L}}$ $(E \in \mathscr{F})$. We shall do that using the following statement.

(20.5.5) For any $A \in \hat{G}_{\mathscr{L},\mathscr{F}'}$ with non-trivial $\mathscr{Z}_G$-action, we have $t_z^* A \neq A$.

Assume for a moment that (20.5.5) is proved. From [6, 7.10], we see that the following seven virtual representations of $W'_{\mathscr{L}}$ are of form α_x for some $x \in W'_{\mathscr{L}} - W_{\mathscr{L}}$ such that $\bar{l}(x) \equiv a(x) \pmod 2$:

$$\begin{aligned}
(20.5.6)\qquad & (\widetilde{80}_s-\overline{80}_s)+\varepsilon(\widetilde{60}_s-\overline{60}_s)+(\widetilde{10}_s-\overline{10}_s),\\
& (\widetilde{80}_s-\overline{80}_s)+\varepsilon(\widetilde{60}_s-\overline{60}_s)+(\widetilde{90}_s-\overline{90}_s),\\
& 2(\widetilde{80}_s-\overline{80}_s)-(\widetilde{10}_s-\overline{10}_s),\\
& 2(\widetilde{80}_s-\overline{80}_s)-(\widetilde{90}_s-\overline{90}_s),\\
& (\widetilde{80}_s-\overline{80}_s)-(\widetilde{20}_s-\overline{20}_s).
\end{aligned}$$

Here, $\varepsilon=\pm 1$.

By 16.6 and 17.18(b) the elements $R_?^{\mathscr{L}}$, where ? is one of the expressions (20.5.6), are combinations with integral $\geqslant 0$ coefficients of character sheaves with non-trivial action of $\mathscr{Z}_G$. Moreover, from (20.5.5) and (17.17.3) we see that these $R_?^{\mathscr{L}}$ are combinations with integer, $\geqslant 0$ coefficients of expressions $(A+t_z^*A)$ where A are character sheaves in $\hat{G}_{\mathscr{L},\mathscr{F}'}$ with non-trivial action of $\mathscr{Z}_G$. The inner product (:) of any two such $R_?^{\mathscr{L}}$ is known from 14.13. We may apply [6, 7.7(ii), (iii)] to the real vector space V spanned by all $A+t_z^*A$ (where A is any character sheaf in $\hat{G}_{\mathscr{L},\mathscr{F}'}$ with non-trivial action of $\mathscr{Z}_G$), with orthonormal basis $(1/\sqrt{2})(A+t_z^*A)$, and to the orthonormal set $(1/\sqrt{2})(R_{\tilde{E}}^{\mathscr{L}}-R_{\bar{E}}^{\mathscr{L}})$ $(E\in\mathscr{F})$.

This gives each of $(1/\sqrt{2})(R_{\tilde{E}}^{\mathscr{L}}-R_{\bar{E}}^{\mathscr{L}})$ as an explicit $\mathbb{Q}$-linear combination of $\frac{1}{2}(A^i+t_z^*A^i)$ $(1\leqslant i\leqslant 8)$ where $A^1,\dots,A^8$, $t_z^*A^1,\dots,t_z^*A^8$ are distinct character sheaves with non-trivial action of $\mathscr{Z}_G$. (Hence, they are distinct from $\tilde{A}_1,\dots,\tilde{A}_8,\tilde{A}'_1,\dots,\tilde{A}_8$ above.) This gives the pattern of decomposition of each $R_{\tilde{E}}^{\mathscr{L}}-R_{\bar{E}}^{\mathscr{L}}$ $(E\in\mathscr{F})$. Since the pattern of decomposition of each $R_{\tilde{E}}^{\mathscr{L}}+R_{\bar{E}}^{\mathscr{L}}$ is already known, we find the pattern of decomposition of each $R_{\tilde{E}}^{\mathscr{L}}$ and $R_{\bar{E}}^{\mathscr{L}}$ in terms of the 32 character sheaves $A^1,\dots,A^8$, $t_z^*A^1,\dots,t_z^*A^8$, $\tilde{A}_1,\dots,\tilde{A}_8$, $\tilde{A}'_1,\dots,\tilde{A}'_8$, and we see that the pattern of (17.8.3) is verified.

Let us now verify (20.5.5). If A (as in (20.5.5)) is cuspidal, then we have $t_z^*A\neq A$. (As we have seen earlier in the proof, for any $A'\in\mathrm{Irr}^0G$ with non-trivial $\mathscr{Z}_G$-action, we have $t_z^*A\neq A$.) Assume now that A is noncuspidal. Then A is a direct summand of a complex induced by a cuspidal character sheaf A' of the Levi subgroup L_J of a parabolic subgroup P_J of G of type D_6 or $A_1\times A_1\times A_1$. (In the last case, P_J is defined by the following three vertices of the Coxeter graph of E_7. One is the end point v_1 at distance one from the branch point, one is the end point v_2 at distance three from the branch point and one, v_3, is at distance two from both v_1 and v_2; see [6, 15.1].)

The case where P_J is of type D_6 cannot actually occur. Indeed, in that case we would have $\hat{\varepsilon}_A=1$ (see the proof of 15.5), hence, $\varepsilon_A=1$ (since (17.8.4) holds for G). On the other hand, from (18.3.3) we see that $\varepsilon_A=-1$ for all $A\in G_{\mathscr{L},\mathscr{F}'}$. (Indeed, in (18.3.1) we have N even for all $E\in\mathscr{F}'$.) Hence P_J must be of type $A_1\times A_1\times A_1$ (described above). In this case, it follows that the support of A is the closure of one of the following two sets

$$X_i = \bigcup_{x \in G} x(z_i(\mathscr{Z}^0_{LJ})_{\text{reg}} U_{LJ})x^{-1}, \qquad i = 1, 2,$$

where $\mathscr{Z}^0_{LJ})_{\text{reg}}$, U_{LJ} are defined as in (18.5.4) and $z_1 = e$, $z_2 = z$. Just as in the proof of (18.5.5) we see that $X_1 \cap X_2 = \varnothing$, hence, $\bar{X}_1 \neq \bar{X}_2$. We have $\bar{X}_2 = z\bar{X}_1$. Hence, if A has support $\bar{X}_1$ (resp. $\bar{X}_2$) then $t_z^* A$ has support $\bar{X}_2$ (resp. $\bar{X}_1$), so that $t_z^* A \neq A$, and (20.5.5) is verified. This also completes the verification of (17.8.3) for G. It remains to prove that G satisfies (17.8.5). We have seen already that if $\mathscr{L}_0 = \bar{\mathbf{Q}}_l$, then $\hat{G}_{\mathscr{L}_0}$ contains at least two cuspidal character sheaves. It is then enough to show that, if $\mathscr{L}$ is as in (iii) above, then $\hat{G}_{\mathscr{L}}$ contains at least six distinct cuspidal character sheaves. Consider the family $\mathscr{F}'$ in $W'_{\mathscr{L}}$ consisting of 10 representations. We have $\mathscr{G}_{\mathscr{F}'} = \Omega_{\mathscr{L}} \times \mathfrak{S}_3$. We consider the character sheaves $A_{(i)} \in \hat{G}_{\mathscr{L},\mathscr{F}'}$ $(1 \leqslant i \leqslant 6)$ which under (17.8.3) correspond to the elements $(\omega g_3, \nu \boxtimes \theta)$, $(\omega, \nu \boxtimes 1)$ in $\mathscr{M}(\mathscr{G}_{\mathscr{F}'})$; here ν is any character of $\Omega_{\mathscr{L}} \simeq \mathbb{Z}/2\mathbb{Z}$, ω is the generator of $\Omega_{\mathscr{L}}$, g_3 is an element of order 3 of $Z_{\mathfrak{S}_3}(g_3) \simeq \mathbb{Z}/3\mathbb{Z}$, θ is a non-trivial character of $Z_{\mathfrak{S}_3}(g_3)$, and 1 is the unit representation of $\mathfrak{S}_3$. We show that $A_{(i)}$ $(1 \leqslant i \leqslant 6)$ are cuspidal. Using 18.2 as in the proof of 19.3(a), we see that it is enough to show that the characters of the virtual representations

$$(\widetilde{80}_s - \overline{80}_s) - (\widetilde{90}_s - \overline{90}_s) - (\widetilde{10}_s - \overline{10}_s) + (\widetilde{20}_s - \overline{20}_s)$$

$$(\widetilde{80}_s - \overline{80}_s) + 3(\widetilde{60}_s - \overline{60}_s) + 2(\widetilde{90}_s - \overline{90}_s) + 2(\widetilde{10}_s - \overline{10}_s) + (\widetilde{20}_s - \overline{20}_s)$$

of $W'_{\mathscr{L}}$ vanish on all elements of $W'_{\mathscr{L}}$ which have some eigenvalue 1 in the reflection representation of W. This, in turn, is equivalent to the following statement. The characters of the virtual representations

$$80_s - 90_s - 10_s + 20_s$$

$$80_s + 3 \cdot 60_s + 2 \cdot 90_s + 2 \cdot 10_s + 20_s$$

of $W_{\mathscr{L}}$ vanish on all elements of $W_{\mathscr{L}}$ which have some eigenvalue -1 on the reflection representation of $W_{\mathscr{L}}$. This can be checked using the character table of $W_{\mathscr{L}}$. This completes the proof of the proposition.

PROPOSITION 20.6. *Assume that G is simple of type G_2 and* char $k \neq 2.3$. *Then* (17.8.3)–(17.8.5) *hold for G.*

Proof. By 18.5, the statements (17.8.3)–(17.8.5) hold for $L/\mathscr{Z}^0_L$ where L is any Levi subgroup of a proper parabolic subgroup of G; hence, they also hold for L. The complexes in $\mathrm{Irr}^0 G$ are classified as follows [4]. The set $\mathrm{Irr}^0 G$ consists of four complexes. One of them is supported by the closure of the conjugacy class of su where s is a semisimple element with centralizer $\approx SO_4(k)$ and u is a regular unipotent element in $Z(s)$. This complex is clean by 7.11(d) and 18.5. Two other complexes in $\mathrm{Irr}^0 G$ are supported by the closure of the conjugacy class of $s'u'$ where s' is a semisimple element with centralizer $\approx SL_3(k)$ and u' is a regular unipotent element in $Z(s)$. These complexes are clean by 7.11(d) and 18.5. Finally, there is a complex in $\mathrm{Irr}^0 G$ which is supported by the closure of the subregular unipotent class in G. If this is a character sheaf, then it is clean by 7.9. It follows that G is clean.

To prove that G satisfies (17.8.4) it is now enough to show that for any cuspidal character sheaf $A \in \hat{G}_{\mathscr{L}}$ we have $\varepsilon_A = \hat{\varepsilon}_A$. The possibilities for $\mathscr{L}$ are restricted by 17.12; we must have $W'_{\mathscr{L}} = W_{\mathscr{L}}$ of type G_2, A_2 or $A_1 \times A_1$. In each case, we see from 18.3 that $\varepsilon_A = \hat{\varepsilon}_A$. Thus, G satisfies (17.8.4). Now using 20.2(c) and 19.2, we see that (17.8.3) holds for G. To prove that (17.8.5) holds for G, it is enough to show that if $\mathscr{L} = \bar{\mathbf{Q}}_l$, then $\hat{G}_{\mathscr{L}}$ contains at least four cuspidal character sheaves. Let $A_1, A_2, A_3, A_4 \in \hat{G}_{\mathscr{L}}$ be the character sheaves corresponding under (17.8.3) to the family $\mathscr{F} = \{V, V', \varepsilon_1, \varepsilon_2\}$ of W (notations of [6, 4.8]), and to the elements $(1, \varepsilon)$, (g_2, ε), (g_3, θ), (g_3, θ^2) in $\mathscr{M}(\mathscr{G}_{\mathscr{F}}) = \mathscr{M}(\mathfrak{S}_3)$. Here g_2 is an element of order 2 of $\mathfrak{S}_3$, ε is the sign character of $\mathfrak{S}_3$ or its restriction to $Z_{\mathfrak{S}_3}(g_2)$, g_3 is an element of order 3 of $\mathfrak{S}_3$, and θ, θ^2 are the nontrivial characters of $Z_{\mathfrak{S}_3}(g_3)$. We shall show that A_1, A_2, A_3, A_4 are cuspidal. Using 18.2, as in the proof of 19.3(a), we see that it is enough to show that the characters of the virtual representations

$$V - 3V' + 2\varepsilon_1 + 2\varepsilon_2, \qquad V - V', \quad V - \varepsilon_1 - \varepsilon_2$$

of W vanish on all elements of W which have some eigenvalue 1 in the reflection representation of W. This is easily checked using the character table of W. This completes the proof of the proposition.

21. Groups of Type E_8 and F_4

21.1. In this chapter we shall prove that the statements (17.8.3)–(17.8.5) hold for the groups of type E_8 and F_4, assuming that we are in good characteristic.

Proposition 21.2. *Assume that G is simple of type E_8 and* char $k \neq 2, 3, 5$. *Then G satisfies* (17.8.4).

Proof. By 20.3, 19.3, the statements (17.8.3)–(17.8.5) hold for $L/\mathscr{Z}^0_L$ where L is a Levi subgroup of any proper parabolic subgroup of G; hence, these statements also hold for L.

According to [4], the set $\mathrm{Irr}^0 G$ consists of 13 complexes.

(a) There is a unique complex in $\mathrm{Irr}^0 G$ supported by the closure of a unipotent class; if it is a character sheaf, then it is clean, by 7.9.

(b) There is a unique complex in $\mathrm{Irr}^0 G$ supported by the closure of the class of su, where s is a semisimple element with $Z_G(s)$ isogenous to the product of $SL_2(k)$ with a simply connected group of type E_7 and u is a certain unipotent element in $Z_G(s)$; this complex is clean by 7.11(d), 18.5 (for $SL_2(k)$), and 20.5.

(c) There are two complexes in $\mathrm{Irr}^0 G$ with the same support: the closure of the class of su, where s is a semisimple element with $Z_G(s)$ isogenous to the product of $SL_3(k)$ with a simply connected group of type E_6 and u is a certain unipotent element in $Z_G(s)$; these complexes are clean by 7.11(d), 18.5 (for $SL_3(k)$), and 20.3(b).

(d) There are two complexes in $\mathrm{Irr}^0 G$ with the same support: the closure of the class of su, where s is a semisimple element with $Z_G(s)$ isogenous to $\mathrm{Spin}_{10}(k) \times SL_4(k)$ and u is a certain unipotent element in $Z_G(s)$; these complexes are clean by 7.11(d), 18.5 (for $SL_4(k)$), and 19.6.

(e) There are four complexes in $\mathrm{Irr}^0 G$ with the same support: the closure of the class of su, where s is a semisimple element with $Z_G(s)$ isogenous to $SL_5(k) \times SL_5(k)$ and u is a regular unipotent element in $Z_G(s)$; these complexes are clean by 7.11(d) and 18.5 (for $SL_5(k)$).

(f) There are two complexes in $\mathrm{Irr}^0 G$ with the same support: the closure of the class of su, where s is a semisimple element with $Z_G(s)$ isogenous to $SL_2(k) \times SL_3(k) \times SL_6(k)$ and u is a regular unipotent element in $Z_G(s)$; these complexes are clean by 7.11(d) and 18.5.

(g) Finally, there is a complex in $\mathrm{Irr}^0 G$ whose support is the closure of the class of su, where s is a semisimple element such that $Z_G(s)$ is isogenous to $\mathrm{Spin}_{16}(k)$ and u is a certain unipotent element in $Z_G(s)$. This complex is clean by 7.11(d) and 19.4(c). (Note that from [4] it follows that any complex in $\mathrm{Irr}^0\, \mathrm{Spin}_{16}(k)$ supported by the closure of a unipotent class, comes from a complex in $\mathrm{Irr}^0\, PSO_{16}(k)$, so that 19.4(c) is applicable.).

This completes the classification of complexes in $\mathrm{Irr}^0 G$ and shows that G is clean. Assume now that $A \in \hat{G}_{\mathscr{L}}$ is a cuspidal character sheaf. To complete the proof it is enough to show that $\varepsilon_A = \hat{\varepsilon}_A$. The possibilities for $\mathscr{L}$ are restricted by 17.12; we must have $W'_{\mathscr{L}} = W_{\mathscr{L}}$ of type E_8, $E_7 \times A_1$, $E_6 \times A_2$, $D_5 \times A_3$, $A_4 \times A_4$, $A_5 \times A_2 \times A_1$, A_8, $A_7 \times A_1$, D_8. In all cases but the first two, we may apply 18.3; the integer N in 18.3 is even in these cases. Hence, we may assume that $W_{\mathscr{L}}$ is of type E_8 or $E_7 \times A_1$. Assume first that $W_{\mathscr{L}}$ is of type E_8 (hence, $W_{\mathscr{L}} = W$). From (18.3.4) we see that $\hat{\varepsilon}_A = 1$. Assume that $\varepsilon_A = -1$. We shall reach a contradiction as follows.

Let A_i ($i = 1, 2$) be the two cuspidal character sheaves in $(\hat{L}_I)_{\mathscr{L}}$, where $I \subset S$ is such that $L_I/\mathscr{Z}^0_L$ is an adjoint group of type E_7 (see the proof of 20.3(c)). If $n \in G$ normalizes L_I, then conjugation by n leaves each A_i stable; indeed A_i come from cuspidal character sheaves of $L_I/\mathscr{Z}^0_{L_I}$ and conjugation by n induces an inner automorphism of $L_I/\mathscr{Z}^0_{L_I}$. It follows that in our case, (17.12.5) simplifies to

$$(i_I^S A_i : i_I^S A_j) = 2(A_i : A_j), \qquad i, j \in \{1, 2\}.$$

Hence, we can write $i_I^S A_1 = \tilde{A}_1 + \tilde{A}'_1$, $i_I^S A_2 = \tilde{A}_2 + \tilde{A}'_2$ where $\tilde{A}_1, \tilde{A}'_1, \tilde{A}_2, \tilde{A}'_2$ are four distinct character sheaves of G. By (17.8.3) for L_I, we have $R^{\mathscr{L},I}_{512_a} - R^{\mathscr{L},I}_{512'_a} = A_1 + A_2$. Applying i_I^S to both sides and using 15.7(i), we find

$$(21.7.1) \quad R^{\mathscr{L}}_{4096'_z} - R^{\mathscr{L}}_{4096'_x} + R^{\mathscr{L}}_{4096_x} - R^{\mathscr{L}}_{4096_z} = \tilde{A}_1 + \tilde{A}'_1 + \tilde{A}_2 + \tilde{A}'_2.$$

From 17.18 and 17.19 we see that both $R^{\mathscr{L}}_{4096'_z} - R^{\mathscr{L}}_{4096'_x}$ and $R^{\mathscr{L}}_{4096_x} - R^{\mathscr{L}}_{4096_z}$ are $\mathbb{Z}$-linear combinations of character sheaves; since both of these elements have self inner product (:) equal to 2, we see from (21.7.1) that each of them is the sum of two of the character sheaves $\tilde{A}_1, \tilde{A}'_1, \tilde{A}_2, \tilde{A}'_2$. From the definition of $\tilde{A}_1, \tilde{A}'_1, \tilde{A}_2, \tilde{A}'_2$, we see that neither of them is cuspidal. We deduce that our given cuspidal character sheaf A satisfies $(A : R^{\mathscr{L}}_{4096'_z} - R^{\mathscr{L}}_{4096'_x}) = (A : R^{\mathscr{L}}_{4096_x} - R^{\mathscr{L}}_{4096_z}) = 0$. From 17.8 and 17.9 we see that $R^{\mathscr{L}}_{4096'_z} + R^{\mathscr{L}}_{4096'_x}$ and $R^{\mathscr{L}}_{4096_x} + R^{\mathscr{L}}_{4096_z}$ are $\mathbb{Z}$-linear combinations of character sheaves A' such that $\varepsilon_{A'} = 1$. Since $\varepsilon_A = -1$, we must have $(A : R^{\mathscr{L}}_{4096'_z} + R^{\mathscr{L}}_{4096'_x}) = (A : R^{\mathscr{L}}_{4096_x} + R^{\mathscr{L}}_{4096_z}) = 0$. It follows that $(A : R^{\mathscr{L}}_{4096'_z}) = (A : R^{\mathscr{L}}_{4096'_x}) = (A : R^{\mathscr{L}}_{4096_x}) = (A : R^{\mathscr{L}}_{4096_z}) = 0$. For all $E \in \hat{W}$ such that $\dim E \neq 4096$, the character of the corresponding representation $E(u)$ of the Hecke algebra has values in $Q[u, u^{-1}]$. We may therefore use 18.3 and we see that $\varepsilon_A = \hat{\varepsilon}_A$, a contradiction.

The case where $W_{\mathscr{L}}$ is of type $E_7 \times A_1$ is treated in an entirely similar way. (In this case, the trouble is created by the four irreducible representations of degree 512 instead of those of degree 4096 for $W_{\mathscr{L}}$ of type E_8.)

This completes the poof.

Proposition 21.3. *Assume that G is simple of type F_4 and* char $k \neq 2, 3$. *Then G satisfies* (17.8.4).

Proof. By 19.3 and 19.4(a), the statements (17.8.3)–(17.8.5) hold for L where L is a Levi subgroup of any proper parabolic subgroup of G.

According to [4] the set $\mathrm{Irr}^0 G$ consists of seven complexes.

(a) There is a unique complex in $\mathrm{Irr}^0 G$ supported by the closure of a unipotent class; if it is a character sheaf, then it is clean, by 7.9.

(b) There is a unique complex in $\mathrm{Irr}^0 G$ supported by the closure of the class of su, where s is a semisimple element with $Z_G(s)$ isogenous to $\mathrm{Sp}_6(k) \times SL_2(k)$ and u is a certain unipotent element in $Z_G(s)$; this complex is clean by 7.11(d), 18.5 (for $SL_2(k)$), and 19.3.

(c) There are two complexes in $\mathrm{Irr}^0 G$ with the same support: the closure of the class of su, where s is a semisimple element with $Z_G(s)$ isogenous to $SL_3(k)\times SL_3(k)$ and u is a regular unipotent element in $Z_G(s)$; these complexes are clean by 7.11(d) and 18.5 (for $SL_3(k)$).

(d) There are two complexes in $\mathrm{Irr}^0 G$ with the same support: the closure of the class of su, where s is a semisimple element with $Z_G(s)$ isogenous to $SL_4(k)\times SL_2(k)$ and u is a regular unipotent element in $Z_G(s)$; these complexes are clean by 7.11(d) and 18.5.

(e) There is a complex in $\mathrm{Irr}^0 G$ supported by the closure of the conjugacy class of su, where s is a semisimple element such that $Z_G(s)\approx \mathrm{Spin}_9(k)$ and u is a certain unipotent element of $Z_G(s)$; this complex is clean by 7.11(d) and 19.4(b). (Note that from [4] it follows that any complex in $\mathrm{Irr}^0\ \mathrm{Spin}_9(k)$ supported by the closure of a unipotent class comes from a complex in $\mathrm{Irr}^0\ SO_9(k)$, so that 19.4(b) is applicable.)

This completes the classification of complexes in $\mathrm{Irr}^0 G$ and shows that G is clean. To complete the proof it is enough to show that $\varepsilon_A=\hat{\varepsilon}_A$ for any cuspidal character sheaf $A\in\hat{G}_{\mathscr{L}}$. The possibilities for $\mathscr{L}$ are restricted by 17.12; we must have $W'_{\mathscr{L}}=W_{\mathscr{L}}$ of type F_4, C_4, $B_3\times A_1$, $A_2\times A_2$, $A_3\times A_1$. In each case, 18.3 shows that $\hat{\varepsilon}_A=\varepsilon_A$. This completes the proof.

COROLLARY 21.4. *Assume that G is as in* 21.2 *and* 21.3.

(a) *If $\mathscr{L}\in\mathscr{S}(T)$ is a local system $\neq\bar{\mathbf{Q}}_l$ then* (17.8.3) *holds for* $(G,\mathscr{L})$.

(b) *If $\mathscr{L}\in\mathscr{S}(T)$ is the local system $\bar{\mathbf{Q}}_l$ and $\mathscr{F}\subset W_{\mathscr{L}}$ is a family with at most five representations then* (17.8.3) *holds for* $(G,\mathscr{L})$ *as far as $\mathscr{F}$ is concerned.*

Proof. (a) follows from 20.2(a), (b), and 19.2, since (17.8.4) holds for G, by 21.2, 21.3. We now prove (b). If $\mathscr{F}$ has one or three representations, the proof of 19.2 (case 1) applies without change. If $\mathscr{F}$ is a family with two or five representations, the argument in the proof of 20.2 applies without change. This completes the proof.

21.5. Assume that $\mathscr{L}_0=\bar{\mathbf{Q}}_l\in\mathscr{S}(T)$. According to [4, 9.2(d)], the endomorphism algebra of $K_e^{\mathscr{L}_0}=\bar{K}_e^{\mathscr{L}_0}$ is canonically isomorphic to the group algebra $\bar{\mathbf{Q}}_l[W]$. Hence, we have a canonical direct sum decomposition $K_e^{\mathscr{L}_0}=\bigoplus_{E\in\hat{W}}(E\otimes A_E)$, where A_E are character sheaves of G and $E\otimes A_E$ is an isotypical component for the action of W. Recall that in 17.8 we have defined m_E to be the image of $E\in\hat{W}$ under the imbedding (17.8.1) (we now have $\mathscr{L}=\mathscr{L}_0$). Thus, $m_E\in\coprod_{\mathscr{F}}\mathscr{M}(\mathscr{G}_{\mathscr{F}})$, where $\mathscr{F}$ runs over the families of W. With these notations, we can state the following result.

PROPOSITION 21.6. *Assume that G is clean. For E, $E' \in \hat{W}$ we have*

$$(A_{E'} : R_E^{\mathscr{L}_0}) = \{m_{E'}, m_E\}$$

where $\{\ ,\ \}$ *is defined as in* (17.8.2).

Proof. We may assume that k is an algebraic closure of the finite field F_q. We choose an F_q-rational structure on G as in 13.1. We assume that q is large enough so that each maximal torus in G defined over F_q contains regular elements defined over F_q. We shall show that the desired formula can be reduced to an analogous formula concerning irreducible representations of $G(F_q)$, which in turn, is a special case of the main theorem 4.23 in [6]. For each A_E ($E \in \hat{W}$) we have a canonical choice for the isomorphism $\phi_{A_E}: F^*A_E \xrightarrow{\sim} A_E$ (see (13.8.1)) with the following property: If $s_w \in G(F_q)$ is a regular semisimple element corresponding to the conjugacy class of $w \in W$, then $\chi_{A_E, \phi_{A_E}}(s_w) = (-1)^{\dim G} \mathrm{Tr}(w, E)$. It follows that the number ξ_{A_E} attached to A_E in 13.10(b) is equal to 1. (Indeed it is enough to test the eigenvalues of $\phi_{A_E}: V_{A_E,i,w} \to V_{A_E,i,w}$ in 13.10(b) for $w = e$.)

Let us write the identity in 14.14 for $\theta = 1$ and $g = s_w$, as above. In the right-hand side of that identity, we have a sum over all $A \in \hat{G}_{\mathscr{L}_0}$; however, the only A which can contribute to the sum are those for which $\chi_{A,\phi_A}(s_w) \neq 0$. For such A, the support of A contains some regular semisimple elements, hence A must be of form A_E ($E \in \hat{W}$). Moreover, in that formula, we have $\nu(A) = 1$ (see 14.7), since G is adjoint; we also have $\xi_{A_E} = 1$, as noticed above. Hence, in our case, the identity of 14.14 reads

$$\mathrm{Tr}(s_w, \mathscr{P}_E^1) = \sum_{E' \in \hat{W}} (A_{E'} : R_E^{\mathscr{L}_0}) \,\mathrm{Tr}(w, E') \qquad (E \in \hat{W}). \tag{21.6.1}$$

Here $\mathscr{P}_E^1$ is an irreducible principal series representation of $G(F_q)$. Its character at the regular semisimple element $s_w \in G(F_q)$ is equal by [3, 7.9] to the multiplicity of $\mathscr{P}_E^1$ in the virtual $G(F_q)$-representation $R_{T_{(w)}}^1$ of [3], where $T_{(w)}$ is the maximal torus of G containing s_w. This multiplicity is computed by [6, 4.23] (the relationship of $\mathscr{P}_E^1$ with the parametrization in [6, 4.23] is explained in [6, 10.2]). It follows that

$$\mathrm{Tr}(s_w, \mathscr{P}_E^1) = \sum_{E' \in \hat{W}} \{m_{E'}, m_E\} \,\mathrm{Tr}(w, E'). \tag{21.6.2}$$

Comparing now (21.6.1) and (21.6.2) we get the identity

$$\sum_{E' \in \hat{W}} ((A_{E'} : R_E^{\mathscr{L}_0}) - \{m_{E'}, m_E\}) \,\mathrm{Tr}(w, E') = 0$$

for all $w \in W$. Since the functions $w \to \mathrm{Tr}(w, E')$ on W are linearly independent ($E' \in \hat{W}$), it follows that $(A_{E'} : R_E^{\mathscr{L}}) = \{m_{E'}, m_E\}$, as desired.

21.7. Assume that G is as in 21.3 and that $\mathscr{L}_0 = \bar{\mathbf{Q}}_l \in \mathscr{S}(T)$. We wish to classify the non-cuspidal character sheaves in $\hat{G}_{\mathscr{L}_0}$. They are of two types:

(a) the 25 character sheaves A_E ($E \in \hat{W}$) (see 21.5);

(b) the components of $\mathrm{ind}_{P_I}^G(A^1)$ where P_I is of type B_2 and A^1 is the unique cuspidal character sheaf in $(\hat{L}_I)_{\mathscr{L}_0}$ (see 19.3(a)).

Next, assume that G is as in 21.2 and that $\mathscr{L}_0 = \bar{\mathbf{Q}}_l \in \mathscr{S}(T)$.
The non-cuspidal character sheaves in $\hat{G}_{\mathscr{L}_0}$ are of four types:

(c) the 112 character sheaves A_E ($E \in \hat{W}$) (see 21.5);

(d) the components of $\mathrm{ind}_{P_I}^G(A^1)$ where P_I is of type D_4 and A^1 is the unique cuspidal character sheaf in $(\hat{L}_I)_{\mathscr{L}_0}$ (see 19.3(d));

(e) the components of $\mathrm{ind}_{P_I}^G(A^i)$ ($i = 1, 2$) where P^I is of type E_6 and A^i are the two cuspidal character sheaves in $(\hat{L}_I)_{\mathscr{L}_0}$ (see the proof of 20.3(a));

(f) the components of $\mathrm{ind}_P^G(A^i)$ ($i = 1, 2$) where P_I is of type E_7 and A^i are the two cuspidal character sheaves in $(\hat{L}_I)_{\mathscr{L}_0}$ (see the proof of 20.3(c)).

In each of the cases (b), (d), (e), (f), the endomorphism algebra End $\mathrm{ind}_{P_I}^G(A^i)$ is isomorphic to a twisted group algebra of $N(L_I)/L_I$ ($N(L_I) =$ normalizer of L_I in G); see [4, 3.5, 3.6, (4.1.1)]. Note that in each case A^i is stabilized by the full $N(L_I)/L_I$. (This is obvious in cases (b), (d), and has been verified during the proof of 21.2 in case (f); in case (e) it follows from the arguments in the proof of (20.5.2).)

The twisting is described by a 2-cocycle of $N(L_I)/L_I$. The twisting is in fact trivial. To show this it is enough to show that the algebra End $\mathrm{ind}_{P_I}^G(A^i)$ has some one-dimensional representation or, equivalently, that there exists a character sheaf of G which appears with multiplicity 1 in $\mathrm{ind}_{P_I}^G(A^i)$; this is verified in Lemma 21.8 below. Assuming that this verification has been done, we see that:

–There are exactly 5 character sheaves of type (b). (The group $N(L_I)/L_I$ is a Coxeter group of type B_2, hence it has five irreducible representations.)

–There are exactly 25 character sheaves of type (d). (The group $N(L_I)/L_I$ is a Coxeter group of type F_4, hence, it has 25 irreducible representations.)

–There are exactly $6+6$ character sheaves of type (e). (The group $N(L_I)/L_I$ is a Coxeter group of type G_2.)

–There are exactly $2+2$ character sheaves of type (f). (The group $N(L_I)/L_I$ is of order 2.)

It follows that if G is as in 21.2, then $\hat{G}_{\mathscr{L}_0}$ contains exactly $153 = 112+25+12+4$ non-cuspidal character sheaves. If G is as in 21.3, then $\hat{G}_{\mathscr{L}_0}$ contains exactly $30=25+5$ non-cuspidal character sheaves.

LEMMA 21.8. *In each of the Cases* (b), (d), (e), (f) *in* 21.7, *there exists a character sheaf A of G such that $(A : \operatorname{ind}_{P_I}^G A^i)=1$. Here we have $i=1$ in Cases* (b), (d) *and $i=1,2$ in Cases* (e), (f).

Proof. In the case (f), this already has been verified, during the proof of 21.2.

In the case (b), let $\mathscr{F}$ be the family $\{4_2, 2_1, 2_3\}$ (notation of [6, 4.10]). From 21.6, we see that A_{4_2}, A_{2_1}, A_{2_3} are in $\hat{G}_{\mathscr{L}_0,\mathscr{F}}$. By 21.4, the statement (17.8.3) holds for $\mathscr{F}$. In particular, there is a fourth element A in $\hat{G}_{\mathscr{L}_0,\mathscr{F}}$ and

$$A + A_{4_2} + A_{2_1} + A_{2_3} = 2R_{4_2}^{\mathscr{L}_0}.$$

For any $E \in \hat{W}$ we have $(A_E : \operatorname{ind}_{P_I}^G A^1)=0$ since all irreducible components of $\operatorname{ind}_P^G A^1$ have support $\neq G$. Hence

$$\begin{aligned}
(A : \operatorname{ind}_{P_I}^G A^1) &= (A + A_{4_2} + A_{2_1} + A_{2_3} : \operatorname{ind}_{P_I}^G A^1)\\
&= 2(R_{4_2}^{\mathscr{L}_0} : \operatorname{ind}_{P_I}^G A^1)\\
&= 2(\operatorname{res}_{P_I}^G (R_{4_2}^{\mathscr{L}_0}) : A^1)\\
&= 2(R_{\operatorname{res} 4_2}^{\mathscr{L}_0} : A^1) && \text{(by 15.7(b))}\\
&= 1 && \text{(by (17.8.3) for } L_I)
\end{aligned}$$

so that A has the required property.

The case (d) is entirely similar: we replace in the previous argument $\{4_2, 2_1, 2_3\}$ by $\{112_z, 84_x, 28_x\}$ (notation of [6, 4.13]).

In case (e), let $\mathscr{F}$ be the family $\{1400_z, 1344_x, 1008_z, 448_z, 56_z\}$ (notation of [6, 4.13]). By 21.4, the statement (17.8.3) holds for $\mathscr{F}$. We have $\mathscr{G}_{\mathscr{F}} = \mathfrak{S}_3$. Let g_3 be an element of order 3 of $\mathscr{G}_{\mathscr{F}}$ and let θ, θ^2 be the non-trivial characters of its centralizer ($\approx \mathbb{Z}/3\mathbb{Z}$). Let r be the two-dimensional irreducible representation of $\mathscr{G}_{\mathscr{F}}$. Let A_θ (resp. $A_{\theta^2}, \tilde{A}$) be the character sheaf in $\hat{G}_{\mathscr{L}_0,\mathscr{F}}$ corresponding under (17.8.3) to $(g_3,\theta) \in \mathscr{M}(\mathscr{G}_{\mathscr{F}})$ (resp, (g_3, θ^2), $(1, r)$). From 21.6, we see that $A_{1008_z} \in \hat{G}_{\mathfrak{L}_0,\mathscr{F}}$ and that $(A_{1008_z} : R_E^{\mathscr{L}_0}) = (A : R_E^{\mathscr{L}_0})$ for all $E \in \mathscr{F}$; the last inner product is determined by (17.8.3) for $\mathscr{F}$. From (17.8.3) we also see that if $A' \in \hat{G}_{\mathscr{L}_0,\mathscr{F}}$ satisfies

$(A' : R_E^{\mathscr{L}_0}) = (\tilde{A} : R_E^{\mathscr{L}_0})$ for all $E \in \mathscr{F}$, then $A' = \tilde{A}$. It follows that $\tilde{A} = A_{1008_z}$ and hence its support is the whole of G. Since all components of $\operatorname{ind}_{P_I}^G(A^i)$ have support $\neq G$, it follows that $(\tilde{A} : \operatorname{ind}_{P_I}^G A^i) = 0$ and hence

$$(A_\theta + A_{\theta^2} : \operatorname{ind}_{P_I}^G A^i) = (A_\theta + A_{\theta^2} + \tilde{A} : \operatorname{ind}_{P_I}^G A^i).$$

From (17.8.3) for $\mathscr{F}$ it follows that $A_\theta + A_{\theta^2} + \tilde{A} = R_{1400_z}^{\mathscr{L}_0} - R_{1008_z}^{\mathscr{L}_0} + R_{56_z}^{\mathscr{L}_0}$. Hence,

$$\begin{aligned}(A_\theta + A_{\theta^2} : \operatorname{ind}_{P_I}^G(A^i)) &= (R_{1400_z}^{\mathscr{L}_0} - R_{1008_z}^{\mathscr{L}_0} + R_{56_z}^{\mathscr{L}_0} : \operatorname{ind}_{P_I}^G A^i)\\ &= (R_{\operatorname{res} 1400_z}^{\mathscr{L}_0} - R_{\operatorname{res} 1008_z}^{\mathscr{L}_0} + R_{\operatorname{res} 56_z}^{\mathscr{L}_0} : A^i)\\ &= 1 \qquad \text{(by (17.8.3) for } L_I).\end{aligned}$$

Hence, precisely one of A_θ, A_{θ^2} appears in $\operatorname{ind}_{P_I}^G(A^i)$ with multiplicity 1. This holds for both $i=1$ and 2. Since $\operatorname{ind}_{P_I}^G(A^1)$, $\operatorname{ind}_{P_I}^G(A^2)$ have no common components, it follows that one of A_θ, A_{θ^2} appears with multiplicity 1 in $\operatorname{ind}_{P_I}^G(A^1)$ and the other appears with multiplicity 1 in $\operatorname{ind}_{P_I}^G(A^2)$. The lemma is proved.

21.9. Assume that G is as in 21.2 or 21.3 and $\mathscr{L}_0 = \bar{\mathbf{Q}}_l \in \mathscr{S}(T)$. If $\mathscr{F} \subset \hat{W}$ is a family with a single representation E, we see from 21.6 that $A_E \in \hat{G}_{\mathscr{L},\mathscr{F}}$, and from 21.4(b), we see that $\hat{G}_{\mathscr{L}_0,\mathscr{F}} = \{A_E\}$. If $\mathscr{F} \subset \hat{W}$ is a family consisting of three representations $\{E, E', E''\}$, we see from 21.6 that $A_E, A_{E'}, A_{E''} \in \hat{G}_{\mathscr{L}_0,\mathscr{F}}$ and from 21.4(b) that $\hat{G}_{\mathscr{L}_0,\mathscr{F}} = \{A_E, A_{E'}, A_{E''}, \tilde{A}\}$, where $\tilde{A}$ is a fourth character sheaf. Just as in the proof of 21.8 we can compute $(\tilde{A} : \operatorname{ind}_{P_I}^G(A^1))$ where P_I, A^1 are as in 21.7(b) or (d); we find that it is non-zero, hence $\tilde{A}$ is a component of $\operatorname{ind}_{P_I}^G(A^1)$. We shall denote by $\mathscr{F}_0$ the unique family in $\hat{W}$ which consists of strictly more than five representations. If G is as in 21.3, the previous argument shows that $\bigcup_{\mathscr{F} \neq \mathscr{F}_0} \hat{G}_{\mathscr{L}_0\mathscr{F}}$ consists of 14 character sheaves of type 21.7(a) and of 2 character sheaves of type 21.7(b). It follows that

(21.9.1) $\hat{G}_{\mathscr{L}_0,\mathscr{F}_0}$ consists of 11 character sheaves of type 21.7(a) (the A_E such that $E \in \mathscr{F}_0$), of 3 character sheaves A_b^i $(1 \leqslant i \leqslant 3)$ of type 21.7(b), and of an unknown number of cuspidal character sheaves.

Assume now that G is as in 21.2. If $\mathscr{F} \subset \hat{W}$ is a family consisting of two representatives $\{E, E'\}$, then we see from 21.6 that $A_E, A_{E'} \in \hat{G}_{\mathscr{L}_0,\mathscr{F}}$; from the proof of 21.2 we see that $\hat{G}_{\mathscr{L}_0,\mathscr{F}}$ contains two character sheaves of type 21.7 (f), and from 21.4 (b) we see that $\hat{G}_{\mathscr{L}_0,\mathscr{F}}$ contains no further character sheaves.

If $\mathscr{F} \subset \hat{W}$ is a family consisting of five representations, we see from 21.6 that $A_E \in \hat{G}_{\mathscr{L}_0,\mathscr{F}}$ for all $E \in \mathscr{F}$. Exactly as in the proof of 21.8, we see that there is one character sheaf in $\hat{G}_{\mathscr{L}_0,\mathscr{F}}$ which has multiplicity 1 in $\operatorname{ind}_{P_I}^G(A^1)$

and another character sheaf in $\hat{G}_{\mathscr{L}_0,\mathscr{F}}$ which has multiplicity 1 in $\mathrm{ind}_{P_I}^G(A^2)$. (Here P_I, A^i are as in 21.7(*e*).) This accounts for seven out of the eight character sheaves in $\hat{G}_{\mathscr{L}_0,\mathscr{F}}$ (see 21.4(b)). Let A be the eight character sheaf in $\hat{G}_{\mathscr{L}_0,\mathscr{F}}$. By 21.4(b). A appears with coefficient 1/2 in $R_E^{\mathscr{L}_0}$ where E is the special representation in $\mathscr{F}$. All $A' \in \hat{G}_{\mathscr{L}_0,\mathscr{F}}$ other than A are of type other than 21.7(d), hence if we write $P_{\bar{I}}$, $L_{\bar{I}}$, $\tilde{A}^1$ instead of P_I, L_I, A^1 in 21.7(d), we have

$$\begin{aligned}(A : \mathrm{ind}_{P_{\bar{I}}}^G(\tilde{A}^1)) &= \tfrac{1}{2}(R_E^{\mathscr{L}_0} : \mathrm{ind}_{P_{\bar{I}}}^G(\tilde{A}^1))\\ &= \tfrac{1}{2}(\mathrm{res}\,{}_{P_I}^G R_E^{\mathscr{L}_0} : \tilde{A}^1)\\ &= \tfrac{1}{2}(R_{\mathrm{res}\,E}^{\mathscr{L}_0} : \tilde{A}^1).\end{aligned}$$

The last inner product may be computed using (17.8.3) for $L_{\bar{I}}$ and turns out to be non-zero. It follows that A is of type 21.7(d).

We see now that $\bigcup_{\mathscr{F} \neq \mathscr{F}_0} \hat{G}_{\mathscr{L}_0,\mathscr{F}}$ consists of 95 character sheaves of type 21.7(c), of 20 character sheaves of type 21.7(d), of 8 character sheaves of type 21.7(e), and of 4 character sheaves of type 21.7(f). It follows that

(21.9.2) $\hat{G}_{\mathscr{L}_0,\mathscr{F}_0}$ consists of 17 character sheaves of type 21.7(e) (the A_E such that $E \in \mathscr{F}_0$), of 5 character sheaves A_d^i $(1 \leqslant i \leqslant 5)$ of type 21.7(d), of 4 character sheaves A_e^i $(1 \leqslant i \leqslant 4)$ of type 21.7(e), and of an unknown number of cuspidal character sheaves.

21.10. We shall need a variant of Lemma 14.3. Assume that G is semisimple and let $(\Sigma, \mathscr{E})$ be a cuspidal pair for G (as in (7.1.2)). Let A be a perverse sheaf on G which is a direct summand of the complex K' induced by an irreducible cuspidal perverse sheaf of a Levi subgroup of a proper parabolic subgroup. We shall prove the following result.

(21.10.1) *If Σ is a regular conjugacy class of G and* char k *is not a bad prime for G, then the local system $\mathscr{H}^i A \mid \Sigma$ does not contain $\mathscr{E}$ as a direct summand.*

This is proved as follows. We may assume that $A = K'$. Using 7.11(a), and (14.2.1), (14.2.2), we see that we are reduced to the case where Σ is a unipotent class. From the results of [4], it is known that (in good characteristic), the regular unipotent class of G cannot carry a cuspidal pair unless G is isogenous to a product of groups $SL_n(k)$. Hence we may further assume that G is as in 18.5. In this case, by 18.5, G is clean and any admissible complex on G is a character sheaf. Hence Lemma 14.3 is applicable and the result follows.

We now state the following result.

LEMMA 21.11. *Assume that G is semisimple, simply connected, and that* char k *is not a bad prime for G. Let A_0 be an irreducible cuspidal perverse sheaf on G on which $\mathscr{Z}_G$ acts trivially and such that* $\operatorname{supp} A_0=\bar{\Sigma}$, *where Σ is a regular conjugacy class of G. Then A_0 is a character sheaf of G. More precisely, if w is a Coxeter element of minimal length of W and if $\mathscr{L}_0=\bar{\mathbf{Q}}_l\in\mathscr{S}(T)$, then $(A_0:\chi(K_w^{\mathscr{L}_0}))=1$.*

Proof. Recall from 2.4 that $K_w^{\mathscr{L}_0}=(\pi_w)_!\bar{\mathbf{Q}}_l$. According to Steinberg [18], we can choose $g\in\Sigma$ such that $g\in BwB$. We have a commutative diagram

$$\begin{tikzcd} G/\mathscr{Z}_G \arrow[r, "\beta"] \arrow[d, "\alpha"'] & G/Z_G(g) \arrow[d, "\alpha'"] \\ \pi_w^{-1}\Sigma \arrow[r, "\pi_w"] & \Sigma \end{tikzcd}$$

where β is the canonical map, $\alpha(x)=(xgx^{-1},xBx^{-1})$, $\alpha'(x)=xgx^{-1}$. Clearly, α' is an isomorphism. According to [16, 8.2] (which is just a reformulation of Steinberg's results in [18]), the map α is also an isomorphism. It follows that $K_w^{\mathscr{L}_0}|\Sigma=(\alpha')_!\beta_!\bar{\mathbf{Q}}_l$. We now factorise β as follows

$$G/\mathscr{Z}_G\xrightarrow{\beta_1}G/(\mathscr{Z}_G\cdot Z_G^0(g))\xrightarrow{\beta_2}G/Z_G(g).$$

According to 3.12, the group $Z_G^0(g)$ is unipotent; its dimension is $r=$ rank G. Since all fibres of β_1 are isomorphic to $Z_G^0(g)$, we have $(\beta_1)_!\bar{\mathbf{Q}}_l=\bar{\mathbf{Q}}_l[-2r]$. (We disregard the Tate twist.) On the other hand, β_2 is a principal covering with (finite) group $Z_G(g)/(\mathscr{Z}_G\cdot Z_G^0(g))$, hence, $\beta_!\bar{\mathbf{Q}}_l[2r]=(\beta_2)_!(\beta_1)_!\bar{\mathbf{Q}}_l[2r]=(\beta_2)_!\bar{\mathbf{Q}}_l=$ direct sum of all G-equivariant local systems on Σ corresponding to irreducible representations of $Z_G(g)/Z_G^0(g)$ which are trivial on $\mathscr{Z}_G$. All these local systems are one-dimensional and appear with multiplicity 1, since $Z_G(g)/Z_G^0(g)$ is abelian. In particular we see that, if $\mathscr{E}$ is the local system on Σ such that $A_0|\Sigma=\mathscr{E}[\dim\Sigma]$, then $\mathscr{E}$ appears with multiplicity 1 in the local system $\mathscr{H}^{2r}(K_w^{\mathscr{L}_0})|\Sigma$ and $\mathscr{E}$ does not appear in the local system $\mathscr{H}^i(K_w^{\mathscr{L}_0})|\Sigma$ for $i\neq 2r$. Hence,

$$\sum_i(-1)^i\ (\text{multiplicity of } \mathscr{E} \text{ in } \mathscr{H}^i(K_w^{\mathscr{L}_0})|\Sigma) \tag{21.11.1}$$

is equal to 1.

On the other hand, the expression (21.11.1) is clearly equal to

$$\sum_A (A:\chi(K_w^{\mathscr{L}_0}))\cdot m_A$$

(sum over all $A \in \hat{G}_{\mathscr{L}_0}$) where

$$m_A = \sum_i (-1)^i \text{ (multiplicity of } \mathscr{E} \text{ in } \mathscr{H}^i(A) \mid \Sigma).$$

It follows that

$$(21.11.2) \quad \sum_A (A : \chi(K_w^{\mathscr{L}_0})) m_A = 1.$$

From (21.10.1), we see that $m_A = 0$ whenever A is not cuspidal. It is also clear that $m_A = 0$ if A is cuspidal with support $\neq \bar{\Sigma}$, (Since Σ is a regular class, we have $\Sigma \subset \operatorname{supp} A \Rightarrow \bar{\Sigma} = \operatorname{supp} A$, for A cuspidal.) If A is cuspidal with support $\bar{\Sigma}$, then from the definition of m_A we see that $m_A = (-1)^{\dim \Sigma} = 1$ if $A \mid \Sigma \approx \mathscr{E}[\dim \Sigma]$ and $m_A = 0$, otherwise. It follows then from (21.11.2) that there is a unique cuspidal $A \in \hat{G}_{\mathscr{L}_0}$ such that $A \mid \Sigma \approx \mathscr{E}[\dim \Sigma]$ (hence $A = A_0$), and that $(A_0 : \chi(K_w^{\mathscr{L}_0})) = 1$. The lemma is proved.

Proposition 21.12. *Let G be as in 21.2 and let $\mathscr{L}_0 = \bar{\mathbf{Q}}_l \in \mathscr{S}(T)$. Then* (17.8.3) *holds for* $(G, \mathscr{L}_0)$ *and* (17.8.5) *holds for G.*

Proof. By 21.4(b), we know that (17.8.3) holds for $(G, \mathscr{L}_0)$ as far as $\mathscr{F}$ is concerned for any family $\mathscr{F} \neq \mathscr{F}_0$ where $\mathscr{F}_0$ is the unique family in $\hat{W}$ such that $\mathscr{G}_{\mathscr{F}_0} = \mathfrak{S}_5$.

In [6, p. 227] we have described 28 virtual representations $X_{1,1}, \ldots, X_{7,4}$ of W of the form α_y for some $y \in W$ such that $l(y) \equiv \alpha(y) \pmod 2$. By 16.6, the corresponding 28 elements $R_{X_{1,1}}^{\mathscr{L}_0}, \ldots, R_{X_{7,4}}^{\mathscr{L}_0}$ are combinations with integral $\geqslant 0$ coefficients of character sheaves in $\hat{G}_{\mathscr{L}_0, \mathscr{F}_0}$. Let us consider a Coxeter element w of minimal length in W. From (14.10.3) it follows that

$$(21.12.1) \quad \chi(K_w^{\mathscr{L}_0}) = \sum_{E \in \hat{W}} \operatorname{Tr}(w, E) R_E^{\mathscr{L}_0}.$$

As noted in [6, p. 310], any $E \in \mathscr{F}_0$ is a $\mathbb{Q}$-linear combination of the 28 virtual representations $X_{1,1}, \ldots, X_{7,4}$ and of $\sum_{E \in \mathscr{F}_0} \operatorname{Tr}(w, E) E$. It follows that in order to establish (17.8.3) for $\mathscr{F}_0$, it is enough to establish the pattern of decomposition of $R_{X_{1,1}}^{\mathscr{L}_0}, \ldots, R_{X_{7,4}}^{\mathscr{L}_0}$ and of $\sum_{E \in \mathscr{F}_0} \operatorname{Tr}(w, E) R_E^{\mathscr{L}_0}$. Some of the elements $X_{i,j}$ are of the form $J(\beta)$ where $J\colon \mathscr{R}(W_I) \to \mathscr{R}(W)$ is defined as in 17.13 with $I \subsetneq S$ and where β is a virtual representation of W_I of form α_y (relative to W_I) (see [6, pp. 173, 228]). From (17.13.5) it follows that the corresponding $R_{X_{i,j}}^{\mathscr{L}_0}$ are of form $j_I^S(R_\beta^{\mathscr{L}_0, I})$, hence, all irreducible components of $R_{X_{i,j}}^{\mathscr{L}_0}$ are also components of $i_I^S(R_\beta^{\mathscr{L}_0, I})$. Using this, we can tell what type of components $R_{X_{i,j}}^{\mathscr{L}_0}$ can have. In particular, we see that

(21.12.2) $R_{X_{i,1}}^{\mathscr{L}_0}$ $(1 \leqslant i \leqslant 7)$ are combinations of A_E $(E \in \mathscr{F}_0)$ only; the elements $R_{X_{i,2}}^{\mathscr{L}_0}$ $(1 \leqslant i \leqslant 5)$, $R_{X_{1,3}}^{\mathscr{L}_0}$, $R_{X_{2,4}}^{\mathscr{L}_0}$ are combinations of A_E $(E \in \mathscr{F}_0)$ and of A_d^j $(1 \leqslant j \leqslant 5)$ (see (21.9.2)); the element $R_{X_{3,3}}^{\mathscr{L}_0}$ is a combination of A_E $(E \in \mathscr{F}_0)$ and of A_e^j $(1 \leqslant j \leqslant 4)$ (see (21.9.2)); the element $R_{X_{2,3}}^{\mathscr{L}_0}$ is a combination of A_E $(E \in \mathscr{F}_0)$, of A_d^j $(1 \leqslant j \leqslant 5)$, and of A_e^j $(1 \leqslant j \leqslant 4)$.

Using 14.13 and (21.12.1) we see that

$$(\chi(K_w^{\mathscr{L}_0}) : \chi(K_w^{\mathscr{L}_0})) = \sum_{E, E' \in \hat{W}} \operatorname{Tr}(w, E) \operatorname{Tr}(w, E') = 30$$

since the order of the centralizer of w in W is 30. This can be expressed in an equivalent form:

$$\text{(21.12.3)} \qquad \sum_{A \in \hat{G}_{\mathscr{L}_0}} (A : \chi(K_w^{\mathscr{L}_0}))^2 = 30.$$

Since $(A : R_E^{\mathscr{L}_0})$ is already known for any $E \in \hat{W}$ and any $A \notin \hat{G}_{\mathscr{L}_0, \mathscr{F}_0}$, we can also compute (by (21.12.1)) $(A : \chi(K_w^{\mathscr{L}_0}))$ for such A. We find that it is ± 1 for exactly 20 character sheaves $A \in \hat{G}_{\mathscr{L}_0} - \hat{G}_{\mathscr{L}_0, \mathscr{F}_0}$, and it is zero for the remaining character sheaves in $\hat{G}_{\mathscr{L}_0} - \hat{G}_{\mathscr{L}_0, \mathscr{F}_0}$. Now using (21.12.3) we deduce that

$$\text{(21.12.4)} \qquad \sum_{A \in \hat{G}_{\mathscr{L}_0, \mathscr{F}_0}} (A : \chi(K_w^{\mathscr{L}_0}))^2 = 10.$$

From 21.11, we see that the four complexes (say K_{10}, K_{11}, K_{12}, K_{13}) in 21.2(e) and the two complexes (say K_6, K_7) in 21.2(f) are in $\hat{G}_{\mathscr{L}_0}$, and they have inner product 1 with $\chi(K_w^{\mathscr{L}_0})$; by 21.9, they are necessarily in $\hat{G}_{\mathscr{L}_0, \mathscr{F}_0}$. Hence from (21.12.4) we deduce

$$\text{(21.12.5)} \qquad \sum (A : \chi(K_w^{\mathscr{L}_0}))^2 = 4,$$

sum over all $A \in \hat{G}_{\mathscr{L}_0, \mathscr{F}_0}$, $A \neq K_6$, K_7, K_{10}, K_{11}, K_{12}, K_{13}. If $A = A_E$ where $E \in \mathscr{F}_0$, then $(A : \chi(K_w^{\mathscr{L}_0}))$ is computable from (21.12.1) and 21.6; we find that this equals 1 if $E = 70_y$, the unique 70–dimensional representation in $\mathscr{F}_0$, and is zero for all other E in $\mathscr{F}_0$. It follows that

$$\text{(21.12.6)} \qquad \sum (A : \chi(K_w^{\mathscr{L}_0}))^2 = 3,$$

sum over all $A \in \hat{G}_{\mathscr{L}_0, \mathscr{F}_0}$, $A \neq K_6$, K_7, K_{10}, K_{11}, K_{12}, K_{13}; A not of form A_E. Using 14.13 and (21.12.1) we can compute $(\chi(K_w^{\mathscr{L}_0}) : R_{X_{1,2}}^{\mathscr{L}_0}) = 2$. The only character sheaves which can appear both in $\chi(K_w^{\mathscr{L}_0})$ and $R_{X_{1,2}}^{\mathscr{L}_0}$ are A_E $(E = 70_y)$ and A_d^i $(1 \leqslant i \leqslant 5)$. We have seen that A_{70_y} appears with multiplicity 1 in $\chi(K_w^{\mathscr{L}_0})$ and, by 21.6, it also appears with multiplicity 1 in $R_{X_{1,2}}^{\mathscr{L}_0}$. Hence,

$$(21.12.7) \quad \sum_{1\leqslant i\leqslant 5} (A_d^i : \chi(K_w^{\mathscr{L}_0}))(A_d^i : R_{X_{1,2}}^{\mathscr{L}_0}) = 1.$$

It follows that $(A_d^i : \chi(K_w^{\mathscr{L}_0})) \neq 0$ for some $i \in [1, 5]$. Similarly, from

$$(21.12.8) \quad (\chi(K_w^{\mathscr{L}_0}) : R_{X_{3,3}}^{\mathscr{L}_0}) = -2$$

it follows that $(A_e^j : \chi(K_w^{\mathscr{L}_0})) \neq 0$ for some $j \in [1, 4]$. From the definition of A_e^j we see that the Verdier dual of A_e^j is $A_e^{j'}$ for some $j' \neq j$, $j' \in [1, 4]$. On the other hand it is easy to see that the components of $\chi(K_w^{\mathscr{L}_0})$ are permuted by Verdier duality (since $\mathscr{L}_0$ is self-dual). It follows that there are at least two indices $j, j' \in [1, 4]$ such that $(A_e^j : \chi(K_w^{\mathscr{L}_0})) \neq 0$, $(A_e^{j'} : \chi(K_w^{\mathscr{L}_0})) \neq 0$.

Now using (21.12.6) we see that there is a unique index i in $[1, 5]$ (say $i = 3$) such that $(A_d^i : \chi(K_w^{\mathscr{L}_0})) \neq 0$, and exactly two indices j, j' in $[1, 4]$ (say 1 and 2) such that $(A_e^j : \chi(K_w^{\mathscr{L}_0})) \neq 0$, $(A_e^{j'} : \chi(K_w^{\mathscr{L}_0})) \neq 0$; moreover, these three inner products are ± 1. Using (21.12.7) and (21.12.8) it follows that

$$(A_d^3 : \chi(K_w^{\mathscr{L}_0})) = 1, \quad (A_e^1 : \chi(K_w^{\mathscr{L}_0})) = (A_e^2 : \chi(K_w^{\mathscr{L}_0})) = -1.$$

We also see from (21.12.6) that there are no cuspidal character sheaves other than K_6, K_7, K_{10}, K_{11}, K_{12}, K_{13} which appear in $\chi(K_w^{\mathscr{L}_0})$. From 16.6 and (21.12.1) we see that

$$\sum_{E\in\mathscr{F}_0} \operatorname{Tr}(w, E)\, R_E^{\mathscr{L}_0} = \sum_{A\in\hat{G}_{\mathscr{L}_0,\mathscr{F}_0}} (A : \chi(K_w^{\mathscr{L}_0}))A.$$

It follows that

$$(21.12.9) \quad \sum_{E\in\mathscr{F}_0} \operatorname{Tr}(w, E) R_E^{\mathscr{L}_0} = A_{70_y} + A_d^3 - A_e^1 - A_e^2 + K_6 + K_7 + K_{10} + K_{11} + K_{12} + K_{13}.$$

Let us now consider the euclidean space H with orthonormal basis given by the 39 objects A_E $(E\in\mathscr{F}_0)$, A_d^i $(1\leqslant i\leqslant 5)$, $A_e^i(1\leqslant i\leqslant 4)$, and K_i $(1\leqslant i\leqslant 13)$. (The last 13 objects correspond to the 13 irreducible cuspidal perverse sheaves on G described in 21.2(a)–(g); 6 of them are already known to be character sheaves by (21.12.9).) We can regard $R_{X_{1,1}}^{\mathscr{L}_0}, \ldots, R_{X_{7,4}}^{\mathscr{L}_0}$ as 28 vectors in H. They have integral and $\geqslant 0$ coordinates; some of the coordinates are known a priori to be zero, by (21.12.2). The mutual inner products of these vectors are known by 14.13. The inner products of these vectors with the vector given by the right-hand side of (21.12.9) are also known by 14.13. The coefficients of A_E $(E \in \mathscr{F}_0)$ in these 28 vectors are explicitly known by 21.6.

We also know that both A_e^3, A_e^4 appear with coefficient >0 in at least one of our 28 vectors. These properties determine uniquely the pattern of coefficients of our 28 vectors; in particular, they force each K_i $(7 \leqslant i \leqslant 13)$ to appear with coefficient >0 in at least one of the 28 vectors. (Hence, (17.8.5) holds for G.) The pattern is the one described in the table in [6, pp. 304, 305] which should be interpreted as follows. The rows $X_{1,1},\ldots, X_{7,4}$ in that table correspond to our vectors; the first 17 columns correspond to the A_E $(E \in \mathscr{F}_0)$, the next 5 columns correspond to the A_d^i, the next 4 columns correspond to the A_e^i, and the last 13 columns correspond to the K_i. The columns of that table are put into 1–1 correspondence with the elements of $\mathscr{M}(\mathfrak{S}_5)$ in [6, pp. 369, 370]; this also defines a 1–1 correspondence $\hat{G}_{\mathscr{L}_0, \mathscr{F}_0} \leftrightarrow \mathscr{M}(\mathfrak{S}_5)$. This completes the proof.

PROPOSITION 21.13. *Let G be as in 21.3 and let $\mathscr{L}_0 = \bar{\mathbf{Q}}_l \in \mathscr{S}(T)$. Then* (17.8.3) *holds for* $(G, \mathscr{L}_0)$ *and* (17.8.5) *holds for G.*

Proof. The proof is entirely similar to that of 21.12. We must check (17.8.3) for $(G, \mathscr{L}_0)$ only as far as $\mathscr{F}_0$ is concerned, where $\mathscr{F}_0$ is the unique family in $\hat{W}$ with $\mathscr{G}_{\mathscr{F}_0} = \mathfrak{S}_4$ (see 21.4(b)).

In [6, p. 227] we have described 19 virtual representations $Y_{1,1},\ldots, Y_{1,5}$ of W of the form α_y for some $y \in W$ such that $l(y) \equiv a(y)$ (mod 2). By 16.6, the corresponding 19 elements $R_{Y_{1,1}}^{\mathscr{L}_0},\ldots, R_{Y_{1,5}}^{\mathscr{L}_0}$ are combinations with integral $\geqslant 0$ coefficients of character sheaves in $G_{\mathscr{L}_0, \mathscr{F}_0}$. Let us consider a Coxeter element w of minimal length in W. As in 21.12, we see that $\chi(K_w^{\mathscr{L}_0}) = \sum_{E \in \hat{W}} \mathrm{Tr}(w, E) R_E^{\mathscr{L}_0}$ is a combination of 12 character sheaves, with coefficients 1, and that

$$(21.13.1) \qquad \sum_{E \in \mathscr{F}_0} \mathrm{Tr}(w, E) R_E^{\mathscr{L}_0} = A_{6_2} - A_b^1 + K_1 + K_2 + K_3 + K_4$$

where 6_2 is the exterior square of the reflection representation of W, A_b^1 is as in (21.9.1), K_1, K_2 are the two complexes in 21.3(e), and K_3, K_4 are the two complexes in 21.3(d).

One checks that any $E \in \mathscr{F}_0$ is a $\mathbb{Q}$-linear combination of the 19 virtual representations $Y_{1,1},\ldots, Y_{1,5}$ and of $\sum_{E \in \mathscr{F}_0} \mathrm{Tr}(w, E) E$. It follows that in order to establish (17.8.3) for $\mathscr{F}_0$ it is enough to establish the pattern of decomposition of $R_{Y_{1,1}}^{\mathscr{L}_0},\ldots, R_{Y_{1,5}}^{\mathscr{L}_0}$ and of $\sum_{E \in \mathscr{F}_0} \mathrm{Tr}(w, E) R_E^{\mathscr{L}_0}$.

We now consider the euclidean space H with orthonormal basis given by the 21 objects A_E $(E \in \mathscr{F}_0)$, A_b^i $(1 \leqslant i \leqslant 3)$, and K_i $(1 \leqslant i \leqslant 7)$. (The last 7 objects correspond to the seven irreducible cuspidal perverse sheaves on G described in 21.3(a)–(e); the first four of them are already known to be character sheaves; see (21.13.1).) We can regard $R^{\mathscr{L}_0}_{Y_{1,1}},\ldots, R^{\mathscr{L}_0}_{Y_{1,5}}$ as 19 vectors in H. These vectors have the following properties:

–They have integral $\geqslant 0$ coordinates.

–$R^{\mathscr{L}_0}_{Y_{i,1}}$ $(1 \leqslant i \leqslant 5)$ are combinations of A_E $(E \in \mathscr{F}_0)$ only.

–$R^{\mathscr{L}_0}_{Y_{3,2}}$, $R^{\mathscr{L}_0}_{Y_{4,2}}$, $R^{\mathscr{L}_0}_{Y_{5,2}}$, $R^{\mathscr{L}_0}_{Y_{5,3}}$ are combinations of A_E $(E \in \mathscr{F}_0)$ and A^i_b $(1 \leqslant i \leqslant 3)$.

–The mutual inner products of these 19 vectors are known by 14.13.

–The inner products of these 19 vectors with the vector given by the right-hand side of (21.13.1) are known by 14.13.

–The coefficients of A_E $(E \in \mathscr{F}_0)$ in these 19 vectors are explicitly known by 21.6.

These properties determine uniquely the pattern of coefficients of our 19 vectors; in particular, they force each K_i $(5 \leqslant i \leqslant 7)$ to appear with coefficient >0 in at least one of the 19 vectors. (Hence, (17.8.5) holds for G.) The pattern is the one described in the table in [6, p. 306] which should be interpreted as follows. The rows $Y_{1,1},\ldots, Y_{1,5}$ in that table correspond to our 19 vectors; the first 11 columns correspond to the A_E $(E \in \mathscr{F}_0)$, the next 3 columns correspond to the A^i_b and the last 7 columns correspond to the K_i. The columns of that table are put into 1–1 correspondence with the elements of $\mathscr{M}(\mathfrak{S}_4)$ in [6, pp. 371, 372]; this also defines a 1–1 correspondence $\hat{G}_{\mathscr{L}_0,\mathscr{F}_0} \leftrightarrow \mathscr{M}(\mathfrak{S}_4)$. This completes the proof.

Selected References from Parts I–III

3. P. Deligne and G. Lusztig, Representations of reductive groups over finite fields, *Ann. of Math.* **103** (1976), 103–161.
4. G. Lusztig, Intersection cohomology complexes on a reductive group, *Invent. Math.* **75** (1984), 205–272.
5. G. Lusztig, Character sheaves, I, *Adv. in Math.* **56** (1985), 193–237.
6. G. Lusztig, Characters of reductive groups over a finite field, *Ann. of Math. Studies* 107, Princeton Univ. Press, Princeton, N.J., 1984.
13. G. Lusztig, Character sheaves, II, *Adv. in Math.*, in press.

Additional References

16. G. Lusztig, Coxeter orbits and eigenspaces of Frobenius, *Invent. Math.* **38** (1976), 101–159.
17. G. Lusztig, Character sheaves, III, *Adv. in Math.*, in press.
18. R. Steinberg, Regular elements in semisimple algebraic groups, *Publ. Math. IHES* **25** (1965), 49–80.

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Character Sheaves, V

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This paper is part of a series [5, 13, 17, 24] devoted to the study of a class $\hat{G}$ of irreducible perverse sheaves (called character sheaves) on a connected reductive algebraic group G over an algebraically closed field k. (The numbering of sections, subsections, and references will continue that of the earlier parts.)

Most results in this paper hold under a very mild restriction on the characteristic of k, see (23.0.1). For simplicity, in this introduction, we assume that the characteristic of k is good for G; this implies in particular that (23.0.1) holds. One of our main results is Theorem 23.1 which gives a classification of the character sheaves of G on which the group of components of the centre acts faithfully; moreover, it gives a multiplicity formula rather analogous to the main theorem (4.23) in [6]; it also states that G is clean (in the sense of (13.9.2)), it satisfies the parity condition (15.13) and that the class of character sheaves on G coincides with the class of admissible complexes defined in [4]. In the case of groups of type A and exceptional groups, this has been essentially done in part IV [24]; the case of classical groups is dealt with in this paper (Sections 22 and 23). One of the applications of our results is the computation of the local intersection cohomology sheaves $\mathscr{H}^a IC(\bar{C}, \mathscr{E})$ of the closure $\bar{C}$ of any unipotent class C in G with coefficients in any G-equivariant irreducible local system $\mathscr{E}$ on C. For $G = GL_n(k)$, this was done in [22]; for the other simple G, it has been done in [27, 28, 20] for those $(C, \mathscr{E})$ which are assumed to be in the image of Springer's correspondence [7].

In this paper, we complete this computation by removing the last assumption on $(C, \mathscr{E})$. (See Theorem 24.8.) The computation uses in an essential way the theory of character sheaves.

In Section 25, it is shown that in the case where G is defined over F_q, the characteristic functions χ_{A,Φ_A} (see (25.2.1)) of the character sheaves A which are themselves defined over F_q, form an orthonormal basis of the space of class functions on $G(F_q)$. It may be conjectured that this is the

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same (up to multiplication by roots of 1) as the basis of "almost-characters" of $G(F_q)$; see [6, 13.6].

Section 22 contains classical groups in characteristic 2. Section 23 shows the classification of character sheaves and the multiplicity formula. Local intersection cohomology with twisted coefficients of the closure of a unipotent class are contained in Section 24 and class functions on a reductive group over a finite field are given in Section 25.

22. Classical Groups in Characteristic 2

22.1. In this section we assume that k has characteristic 2 and we shall verify properties (17.8.3)–(17.8.5) for G, simple of type B, C, or D over k. Let us recall the content of these properties. Property (17.8.3) gives a parametrization of $\hat{G}$ and some multiplicity formulas. Property (17.8.4) for G states that G is clean and any character sheaf A on G satisfies the parity condition $\varepsilon_A = \hat{\varepsilon}_A$, se (15.13.1). Property (17.8.5) for G states that any irreducible cuspidal perverse sheaf on G is a character sheaf, see (7.1.1).

Lemma 22.2. *Assume that G is simple of type B_n or C_n $(n \geqslant 2)$. Then*

(a) *If $n = m^2 + m$ $(m \in \mathbb{N})$, then* $\mathrm{Irr}^0 G$ *consists of a single complex; it is supported by the closure of a unipotent class. If n is not of the form $m^2 + m$ then* $\mathrm{Irr}^0 G$ *is empty.*

(b) *G satisfies (17.8.4) and $\hat{\varepsilon}_A = 1$ for all $A \in \hat{G}$.*

Proof. We may assume that (in the case $n > 2$) the lemma is already proved for G replaced by a simple group of type $B_{n'}$ or $C_{n'}$ $(2 \leqslant n' < n)$ over k; we may also assume that G is adjoint. Statement (a) follows from [4]. We now prove (b). Let L be the Levi subgroup of a proper parabolic subgroup of G. Then $L/\mathscr{Z}_L^0$ is a product of $PGL_{n_i}(k)$'s and possibly a simple group of type $B_{n'}$ or $C_{n'}$ $(2 \leqslant n' < n)$. Using the induction hypothesis and 18.5, 17.11, we see that (17.8.4) holds for $L/\mathscr{Z}_L^0$, hence it also holds for L, by 17.10. It is then enough to check that any cuspidal character sheaf A of G is clean and satisfies $\varepsilon_A = \hat{\varepsilon}_A = 1$. The equality $\hat{\varepsilon}_A = 1$ follows from (a). The cleanness of A follows from (a) and from 7.9. To prove the parity condition for A, we use 18.3. If $A \in \hat{G}_{\mathscr{L}}$, then from (17.2.4) we see that $\mathscr{L}$ must satisfy $W_{\mathscr{L}} = W'_{\mathscr{L}} = W$. The number N in 18.3 is now even, by (a). Using 18.3 and the rationality of the representations of the Hecke algebra of W, we see that $\varepsilon_A = \hat{\varepsilon}_A$. The lemma is proved.

Lemma 22.3. *Assume that G is simple of type D_n $(n \geqslant 4)$. Then:*

(a) *If $n = 4m^2 (m \in \mathbb{N})$, then* $\mathrm{Irr}^0 G$ *consists of a single complex; it is* *supported by the closure of a unipotent class. If n is not of the form $4m^2$ then* $\mathrm{Irr}^0 G$ *is empty.*

(b) *G satisfies* (17.8.4) *and $\hat{\varepsilon}_A = 1$ for all $A \in \hat{G}$.*

Proof. It is essentially the same as that of 2.22 and will be omitted.

22.4. In this subsection, we assume that (W, S) is a Weyl group of type B_n or C_n and $n = d^2 + d$. We shall describe the irreducible representations of W in terms of symbols, as in [6, 4.5]. Let $\mathscr{F}_0 \subset \hat{W}$ be the family consisting of all representations in $\hat{W}$ whose symbol contains exactly the entries $0, 1, ..., 2d$ ($d+1$ of them in the first row and the remaining d in the second row of the symbol). As in [6, 4.5], we attach to $\mathscr{F}_0$ an F_2-vector space V of dimension $2d$ with a symplectic form $(\,,\,)$: $V \times V \to F_2$. By definition, V is the set of subsets of even cardinality of $\{0, 1, ..., 2d\}$; the group structure on V is given by $M + M' = (M \cup M') - (M \cap M')$ $M, M' \in V$. The symplectic form is $(M, M') = |M \cap M'| \bmod 2$. We identify $\mathscr{F}_0$ with a subset of V as follows: to the irreducible representation E in $\mathscr{F}_0$ whose symbol has second row $M \subset \{0, 1, 2, ..., 2d\}$, we attach the element $v_E \in V$ defined by $v_E = \{i \in M \mid i \text{ even}\} \cup \{1 \leqslant i \leqslant 2d-1 \mid i \text{ odd}, i \notin M\}$. Let $\tilde{V}$ be the image of the imbedding $\mathscr{F}_0 \subset V$. For $y \in \tilde{V}$, we denote by E_y the corresponding object in $\mathscr{F}_0$. Let e_i be the 2-element subset $\{i-1, i\}$ of $\{0, 1, 2, ..., 2d\}$, $1 \leqslant i \leqslant 2d$. Then $e_1, ..., e_{2d}$ is a basis of V. Let η_0: $V \to F_2$ be the linear form defined by $\eta_0(e_i) = 1$ for $1 \leqslant i \leqslant 2d$.

We shall need the following fact.

(22.4.1) *The character of the virtual representation $\sum_{y \in \tilde{V}} (-1)^{\eta_0(y)} E_y$ of W vanishes on all elements of W which have no eigenvalue* 1 *in the reflection representation of W.*

To prove this result we shall use some results in [6] on the representation theory of the finite group $Sp_{2n}(F_q)$ (q is a large power of 2). It is known that in our case ($n = d^2 + d$), the group $Sp_{2n}(F_q)$ has a unique unipotent cuspidal representation ρ. From [6, 9.5] and its proof we see that to each $\eta \in \mathrm{Hom}(V, F_2)$ there corresponds a unipotent representation ρ_η of $Sp_{2n}(F_q)$ such that:

(a) the character of ρ_η on a regular semisimple element of type w in $Sp_{2n}(F_q)$ is equal to $\sum_{y \in \tilde{V}} (-1)^{\eta(y)} \mathrm{Tr}(w, E_y)$

(b) If $\eta', \eta'' \in \mathrm{Hom}(V, F_2)$ satisfy for some i the identities $\eta'(y) + \eta''(y) = (y, e_i)$ for all $y \in V$ and $\eta'(e_i) = \eta''(e_i) = 0$, then $\rho_{\eta'} + \rho_{\eta''}$ is a summand of a representation induced from the Levi subgroup of a proper parabolic subgroup (over F_q). Moreover, from [6, (8.5.6)] it follows that ρ must be equal to ρ_η for some $\eta \in \mathrm{Hom}(V, F_2)$. Since ρ is cuspidal, we see from (b) that η must be equal to η_0, so that $\rho = \rho_{\eta_0}$. If w has some eigenvalue 1 in the reflection representation of W, then the regular semisimple elements of type w in $Sp_{2n}(F_q)$ are contained in an isotropic torus, hence the character of ρ_{η_0} vanishes at them, since ρ_{η_0} is cuspidal. This, together with (a) above yields (22.4.1).

22.5. In this subsection, we assume that (W, S) is a Weyl group of type D_n, and $n = d^2 \geqslant 4$. Let W' be the semidirect product of W and $\mathbb{Z}/2\mathbb{Z}$, with $\mathbb{Z}/2\mathbb{Z}$ acting on W nontrivially, preserving S. We shall describe the irreducible representations of W in terms of symbols as in [6, 4.6]. Let $\mathscr{F}_0 \subset \hat{W}$ be the family consisting of all representations in $\hat{W}$ whose symbol contains exactly the entries $0, 1, \ldots, 2d-1$ (d of them in one row, and the remaining d in the other row of the symbol). Let $\mathscr{F}'_0$ be the set of irreducible representations of W' whose restrictions to W are in $\mathscr{F}_0$. Let V be the F_2-vector space of all subsets of even cardinality of $\{0, 1, \ldots, 2d-1\}$; the group structure and the symplectic form on V are defined just as in 22.4; in the present case, however, the symplectic form has a one-dimensional radical Rad V; it is spanned by $\{0, 1, \ldots, 2d-1\}$. As in [6, 4.6], we identify $\mathscr{F}_0$ with a subset $\tilde{V}^+$ of $V^+ = V/\mathrm{Rad}\, V$; as in [6, 4.18] we identify $\mathscr{F}'_0$ with a subset $\tilde{V}$ of V. For $y \in \tilde{V}^+$ (resp. $y' \in \tilde{V}$) we write E_y (resp. $E'_{y'}$) for the corresponding object of $\mathscr{F}_0$ (resp. $\mathscr{F}'_0$). One defines a basis $e_1, e_2, \ldots, e_{2d-1}$ of V as in 22.4; we shall denote in the same way the images of e_i under $V \to V/\mathrm{Rad}\, V$. Let $\eta'_0 \in \mathrm{Hom}(V, F_2)$ be the linear form defined by $\eta'_0(e_i) = 1$ for all i. If d is even, then η'_0 is zero on Rad V and hence defines a linear form $\eta_0 \in \mathrm{Hom}(V^+, F_2)$. (Note that Rad V is spanned by $e_1 + e_3 + e_5 + \cdots + e_{2d-1}$.)

We shall need the following two facts.

(22.5.1) If d is even, the character of the virtual representation $\sum_{y \in \tilde{V}^+} (-1)^{\eta_0(y)} E_y$ of W vanishes on all elements of W which have some eigenvalue 1 in the reflection representation of W.

(22.5.2) If d is odd, the character of the virtual representation $\sum_{y' \in \tilde{V}} (-1)^{\eta'_0(y')} E_{y'}$ of W' vanishes on all elements of W' which have some eigenvalue 1 in the reflection representation of W, extended to W'.

These two statements are proved using the representation theory of split (resp. twisted) even orthogonal groups over F_q (q = large power of 2) in the same way as (22.4.1) was proved using the representation theory of $Sp_{2n}(F_q)$. We omit further details.

PROPOSITION 22.6. *Assume that* G *is simple of type* B_n *or* C_n $(n \geqslant 2)$. *Then* (17.8.3)–(17.8.5) *hold for* G.

Proof. We shall only consider the case C_n; the case B_n is identical. We may assume that $n \geqslant 3$ (see 19.3) and that the proposition is true when n is replaced by n', $2 \leqslant n' < n$. From 22.2(b) we see that (17.8.4) holds for G. The map $G \to G_{\mathrm{ad}}$ is bijective and the character sheaves do not feel the difference between G and G_{ad}. Hence we may assume that G is adjoint. Let L be a Levi subgroup of a proper parabolic subgroup of G. Then $L/\mathscr{Z}_L^0$ is a product of $PGL_{n_i}(k)$'s and possibly a simple group of type $C_{n'}$, $2 \leqslant n' < n$. Using the induction hypothesis and 18.5, 17.11, we see that (17.8.3) holds for $L/\mathscr{Z}_L^0$ hence it also holds for L, by 17.10. Let $\mathscr{L} \in \mathscr{S}(T)$ be such that $\mathscr{L} \neq \bar{\mathbb{Q}}_l$. Then there exists $w \in W$ and $I \subsetneqq S$ such that $\mathscr{L}' = w^*\mathscr{L}$ satisfies $W'_{\mathscr{L}'} = W'_{\mathscr{L}',I}$. From 17.12, we see that (17.8.3) for $\hat{G}_{\mathscr{L}'}$ is a consequence of the analogous statement for L_I, which is already known. Using 17.15, we see that (17.8.3) holds for $\hat{G}_{\mathscr{L}}$.

We now begin the proof of (17.8.3) for $\hat{G}_{\mathscr{L}}$ in the case where $\mathscr{L} = \bar{\mathbb{Q}}_l$. This decomposes into statements for each $\hat{G}_{\mathscr{L},\mathscr{F}}$ where $\mathscr{F}$ is any family in $\hat{W}$. Let $\mathscr{F}$ be a family in $\hat{W}$ such that there exists $I \subsetneqq S$ and a family $\mathscr{F}_1$ in $\hat{W}_I$ such that $\mathscr{F}$ is "smoothly induced" by $\mathscr{F}_1$(see 17.13). Using 17.13 and the fact that (17.8.3) is already known for L_I, we see that (17.8.3) holds for $\hat{G}_{\mathscr{L},\mathscr{F}}$. Using 17.14, we see also that (17.8.3) holds for any family $\mathscr{F}$ in $\hat{W}$ such that the family $\mathscr{F} \otimes \mathrm{sign}$ is smoothly induced from a family of a proper parabolic subgroup of W. By 22.2(a), we may therefore assume in the rest of the proof that $n = d^2 + d$. We shall take $\mathscr{F} = \mathscr{F}_0$, (we shall use the notations of 22.4). The statement (17.8.3) for $\hat{G}_{\mathscr{L},\mathscr{F}_0}$, (which we are trying to prove) can be reformulated as follows, see [6, 4.5].

(22.6.1) There exists a bijection $\hat{G}_{\mathscr{L},\mathscr{F}_0} \leftrightarrow \mathrm{Hom}(V, F_2)$ $(A_\eta \leftrightarrow \eta)$, such that $(A_\eta : R_{E_y}^{\mathscr{L}}) = 2^{-d}(-1)^{\eta(y)}$ for all $y \in \tilde{V}$ and all $\eta \in \mathrm{Hom}(V, F_2)$.

The proof of (22.6.1) will follow closely the proof of the main theorem in [6] for classical groups, given in [6, 9.1]–[6, 9.5].

As in [6, 9.1], with the basis $e_1, \ldots, e_{2d}$ of V (see 2.4) one can associate a collection $\mathscr{T}(V)$ of lagrangian subspaces of V; the union of subspaces in $\mathscr{T}(V)$ is precisely $\tilde{V}$. According to [6, (9.5.2)] for any $C \in \mathscr{T}(V)$ and any linear form $\xi: C \to F_2$, there exists $x \in W$ such that $\sum_{y \in C}(-1)^{\xi(y)} E_y = \alpha_x$. Here α_x is as in (16.2.7). Applying now 16.6(a), we see that $\rho_{c,\xi} =^{\mathrm{def}} \sum_{y \in C}(-1)^{\xi(y)} R_{E_y}^{\mathscr{L}}$ is a linear combination of character sheaves in $\hat{G}_{\mathscr{L},\mathscr{F}_0}$ with integral, $\geqslant 0$ coefficients, for any $C \in \mathscr{T}(V)$, $\xi \in \mathrm{Hom}(C, F_2)$. If (C', ξ') is another pair like (C, ξ), we have

(22.6.2) $(\rho_{C,\xi} : \rho_{C',\xi'}) =$ number of linear forms $\eta: V \to F_2$ such that $\eta \mid C = \xi$, $\eta \mid C' = \xi'$.

This follows immediately from 14.13.

Now let $D=\{\eta',\eta''\}$ be a two element subset of $\mathrm{Hom}(V,F_2)$ such that $\eta'+\eta''$ is equal to the inner product with the standard basis element e_i of V and $\eta'(e_i)=\eta''(e_i)=0$. Using [6, (4.5.4)], we see that there exists a subset $I\subset S$, $|I|=|S|-1$ and a family $\mathscr{F}_1$ in $\hat{W}_I$ such that the truncated induction $J\colon \mathscr{R}(W_I)\to\mathscr{R}(W)$ (see 17.13) takes each representation in $\mathscr{F}_1$ to the sum of two distinct representations $E_y+E_{y+e_i}$ in $\mathscr{F}_0$. Here $y\in\tilde{V}$, $y+e_i\in\tilde{V}$, $(y,e_i)=0$. Moreover, J defines a bijection between $\mathscr{F}_1$ and the set $\{y\in\tilde{V}\mid (y,e_i)=0\}$ modulo the equivalence relation $y\sim y+e_i$. Let $E'_{\bar{y}}$ be the representation in $\mathscr{F}_1$ corresponding to the class $\bar{y}$ of y. The statement (17.8.3) for L_I, $\mathscr{L}=\bar{\mathbb{Q}}_l$, $\mathscr{F}_1$ is already known; it implies that for each $\zeta\in\mathrm{Hom}(V,F_2)$, $\zeta(e_i)=0$, there exists $A_\zeta\in\hat{L}_I$ such that $(A_\zeta\colon R^{\mathscr{L}}_{E'_{\bar{y}}})=2^{-(d-1)}(-1)^{\zeta(y)}$, for all $y\in\tilde{V}$, $(y,e_i)=0$. Here, (:) is with respect to L_I. We now take $\zeta=\eta'$ and we consider the object $i_I^S A_{\bar{\eta}'}\in\mathscr{K}_0(G)$ (see 15.3). It is a linear combination with integral, $\geqslant 0$ coefficients of character sheaves of G; we denote by ρ_D the result of omiting from this sum the character sheaves which are not in $\hat{G}_{\mathscr{L},\mathscr{F}_0}$.

Then for any $y\in\tilde{V}$, we have

$$\begin{aligned}(\rho_D\colon R^{\mathscr{L}}_{E_y})&=(i_I^S A_{\bar{\eta}'},R^{\mathscr{L}}_{E_y}) &&\text{by definition of } \hat{G}_{\mathscr{L},\mathscr{F}_0}\\ &=(A_{\bar{\eta}'}\colon R^{\mathscr{L}}_{\mathrm{res}_I^S E_y}) &&\text{by 15.7(b).}\end{aligned}$$

By definition of $E'_{\bar{y}}$, we have

$$\mathrm{res}_I^S E_y=\begin{cases}E'_{\bar{y}}+\Phi & \text{if } (y,e_i)=0\\ \Phi & \text{if } (y,e_i)\neq 0,\end{cases}$$

where Φ is a sum of irreducible representations of W_I in families $\neq\mathscr{F}_1$. Hence

$$\begin{aligned}(A_{\bar{\eta}'}\colon R^{\mathscr{L}}_{\mathrm{res}_I^S E_y})&=\begin{cases}(A_{\bar{\eta}'}\colon R^{\mathscr{L}}_{E'_{\bar{y}}}) & \text{if } (y,e_i)=0\\ 0 & \text{if } (y,e_i)\neq 0\end{cases}\\ &=\begin{cases}2^{-(d-1)}(-1)^{\eta'(y)} & \text{if } (y,e_i)=0\\ 0 & \text{if } (y,e_i)\neq 0\end{cases}.\end{aligned}$$

Thus,

$$(\rho_D\colon R^{\mathscr{L}}_{E_y})=\begin{cases}2^{-(d-1)}(-1)^{\eta'(y)} & \text{if } (y,e_i)=0\\ 0 & \text{if } (y,e_i)\neq 0\end{cases}.$$

It follows that for any $C\in\mathscr{T}(V)$ and any $\xi\in\mathrm{Hom}(C,F_2)$, we have

$$(\rho_D\colon\rho_{C,\xi})=\sum_{\substack{y\in C\\ (y,e_i)=0}}(-1)^{\xi(y)+\eta'(y)}2^{-(d-1)},$$

hence

$$(\rho_D : \rho_{C,\xi}) = \text{number of } \eta \in D \text{ such that } \eta \mid C = \xi. \tag{22.6.3}$$

Next, we note that any $A \in \hat{G}_{\mathscr{L},\mathscr{F}_0}$ appears with >0 coefficient in some $\rho_{C,\xi}$ as above. Indeed, given A, we can find $y \in \tilde{V}$ such that $(A : R_{E_y}^{\mathscr{L}}) \neq 0$. Now let $C \in \mathscr{T}(V)$ be such that $y \in C$. Then, clearly,

$$R_{E_y}^{\mathscr{L}} = 2^{-m} \sum_{\xi} (-1)^{\xi(y)} \rho_{C,\xi} \tag{22.6.4}$$

sum over all $\xi \in \operatorname{Hom}(C, F_2)$. Hence, for some ξ we have $(A : \rho_{C,\xi}) \neq 0$, as desired. Applying now [6, 9.2] to the elements $\rho_{C,\xi}$, ρ_D of $\mathscr{K}_0(G)$ (which satisfy (22.6.2), (22.6.3)), we deduce that there exists a bijection $\eta \leftrightarrow A_\eta$ between $\operatorname{Hom}(V, F_2)$ and $\hat{G}_{\mathscr{L},\mathscr{F}_0}$ such that

$$\rho_{C,\xi} = \sum_{\substack{\eta \\ \eta \mid C = \xi}} A_\eta$$

for all $C \in \mathscr{T}(V)$ and all $\xi \in \operatorname{Hom}(C, F_2)$ and

$$\rho_D = A_{\eta'} + A_{\eta''}$$

for all $D = \{\eta', \eta''\}$ as above. Using now (22.6.4), we see that for any $y \in \tilde{V}$, and any $\eta \in \operatorname{Hom}(V, F_2)$, we have

$$\begin{aligned} (A_\eta : R_{E_y}^{\mathscr{L}}) &= 2^{-d} \sum_{\substack{\xi \\ \xi \in C}} (-1)^{\xi(y)} (A_\eta : \rho_{C,\xi}) \\ &= 2^{-d} (-1)^{\eta(y)}, \end{aligned}$$

where C is any subspace in $\mathscr{T}(V)$ containing y. This completes the proof of (17.8.3) for $\hat{G}_{\mathscr{L},\mathscr{F}_0}$. To complete the proof, it is enough to show that if η_0 is as in 22.4 then A_{η_0} is cusipal. (This, together with 22.2(a) will show that (17.8.5) holds for G.) Using 18.2, as in the proof of 19.3(a), we see that we are reduced to the statement (22.4.1). This completes the proof.

PROPOSITION 22.7. *Assume that G is simple of type D_n $(n \geqslant 4)$. Then* (17.8.3)–(17.8.5) *holds for G.*

Proof. The proof is very similar to that of 22.6; we shall only sketch it. We can assume that G is adjoint. We again use induction on n. From 22.3(b), we see that (17.8.4) holds for G. To verify (17.8.3), we are reduced to the case $\hat{G}_{\mathscr{L},\mathscr{F}_0}$, $n = d^2$, where $\mathscr{L} = \bar{Q}_l$ and $\mathscr{F}_0$ is as in 22.5. As in 22.6, we see that (with the notations in 22.5) there exists a bijection $\eta \leftrightarrow A_\eta$ between $\operatorname{Hom}(V^+. F_2)$ and $\hat{G}_{\mathscr{L},\mathscr{F}_0}$ such that $(A_\eta : R_{E_y}^{\mathscr{L}}) = 2^{-(d-1)}(-1)^{\eta(y)}$ for all $y \in \tilde{V}$ and all $\eta \in \operatorname{Hom}(V^+, F_2)$. This establishes (17.8.3) for $\hat{G}_{\mathscr{L},\mathscr{F}_0}$. To establish (17.8.5), it is then enough (by 22.3(a)) to show that, in case where d is even, A_{η_0} is cuspidal; this follows from (22.5.1). This completes the proof.

23. Classification of Character Sheaves and the Multiplicity Formula

23.0. We recall that G is a connected reductive algebraic group over k, an algebraically closed field of characteristic $p \geqslant 0$. Let $\mathscr{L} \in \mathscr{S}(T)$ (see 2.2), and let χ: $\mathscr{Z}_G/\mathscr{Z}_G^0 \to \bar{Q}_l^*$ be a character. For each family $\mathscr{F}' \subset \hat{W}'_{\mathscr{L}}$, let $\mathscr{M}(\mathscr{G}_{\mathscr{F}'})^\chi$ be the subset of $\mathscr{M}(\mathscr{G}_{\mathscr{F}'})$ consisting of all pairs (x, σ) such that the $\Omega_{\mathscr{F}}$-component (see (17.6.1)) of x is mapped by (11.8.1) to χ. For each $E \in \mathscr{F}'$, let $m_E \in \mathscr{M}(\mathscr{G}_{\mathscr{F}'})$ be as in 17.8. For any $z \in \mathscr{Z}_{\mathscr{G}}/\mathscr{Z}_G^0$, let t_z^* be as in 17.17 and let σ_z be the image of z under the map $\mathscr{Z}_G/\mathscr{Z}_G^0 \to \mathrm{Hom}(\Omega_{\mathscr{L}}, \bar{Q}_l^*)$ dual to (11.8.1); we denote also by σ_z the restriction of σ_z to $\Omega_{\mathscr{F}}$ and also the corresponding character $\mathscr{G}_{\mathscr{F}'} = \Omega_{\mathscr{F}} \mathscr{G}_{\mathscr{F}} \to \bar{Q}_l^*$, trivial on $\mathscr{G}_{\mathscr{F}}$.

We shall need the following notation. We have a partition $\hat{G} = \coprod_\chi \hat{G}^\chi$ where $\hat{G}^\chi$ consists of all character sheaves of G on which $\mathscr{Z}_G/\mathscr{Z}_G^0$ acts (see 11.5) according to χ (a character $\mathscr{Z}_G/\mathscr{Z}_G^0 \to \bar{Q}_l^*$). This induces a partition $\hat{G}_{\mathscr{L}} = \coprod_\chi \hat{G}_{\mathscr{L}}^\chi$ for each $\mathscr{L} \in \mathscr{S}(T)$. In the case where the parity condition is satisfied, it also induces a partition $\hat{G}_{\mathscr{L},\mathscr{F}} = \coprod \hat{G}_{\mathscr{L},\mathscr{F}}^\chi$ for each $\mathscr{L} \in \mathscr{S}(T)$ and each family $\mathscr{F} \subset \hat{W}'_{\mathscr{L}}$. Similarly, the set $\mathrm{Irr}^0 G$ (see (19.3.1)) has a natural partition $\mathrm{Irr}^0 G = \coprod_\chi (\mathrm{Irr}^0 G)^\chi$, where χ runs over the characters of $\mathscr{Z}_G/\mathscr{Z}_G^0$.

In the following theorem, we shall make the following assumptions on p.

$$\begin{aligned} &\text{If } p = 5, \text{ then } G \text{ has no factors of type } E_8.\\ &\text{If } p = 3, \text{ then } G \text{ has no factors of type } E_7, E_8, F_4, G_2. \qquad (23.0.1)\\ &\text{If } p = 2, \text{ then } G \text{ has no factors of type } E_6, E_7, E_8, F_4, G_2.\end{aligned}$$

(factor $=^{\mathrm{def}}$ almost simple, closed, normal subgroup).

Theorem 23.1. *Assume that p satisfies* (23.0.1). *Then the following holds for G.*

(a) *G is clean and any character sheaf A on G satisfies $\varepsilon_A = \hat{\varepsilon}_A$.*

(b) *Any irreducible cuspidal perverse sheaf A on G is a character sheaf.*

(c) *Consider $\mathscr{L} \in \mathscr{S}(T)$ and let χ: $\mathscr{Z}_G/\mathscr{Z}_G^0 \to \bar{Q}_l^*$ be a faithful character. Then there exists a bijection*

$$\hat{G}_{\mathscr{L}}^\chi \leftrightarrow \coprod_{\mathscr{F}'} \mathscr{M}(\mathscr{G}_{\mathscr{F}'})^\chi \qquad (A \leftrightarrow \bar{m}_A),$$

where $\mathscr{F}'$ runs over all families in $\hat{W}'_{\mathscr{L}}$, such that

$$(A: R_E^{\mathscr{L}}) = \hat{\varepsilon}_A\{\bar{m}_A, m_E\}$$

for all $A \in \hat{G}_{\mathscr{L}}$ and all $E \in \hat{W}'_{\mathscr{L}}$, and such that $z \in \mathscr{Z}_G/\mathscr{Z}_G^0$, $A \in \hat{G}_{\mathscr{L}}$, $\bar{m}_A = (x, \sigma) \in \mathscr{M}(\mathscr{G}_{\mathscr{F}'})^\chi \Rightarrow \bar{m}_{t_z^* A} = (x, \sigma \otimes \sigma_z)$.

Property (a) is just (17.8.4), property (b) is just (17.8.5). Property (c) is a variant of (17.8.3). It can be formulated for arbitrary χ (not necessarily faithful) in the same way, but we shall prove it only for faithful χ; we shall refer to it as "property $(17.8.3)_\chi$ for G, $\mathscr{L}$". When this property is satisfied for all $\mathscr{L}$ we shall say that "$(17.8.3)_\chi$ holds for G." It is clear that if G satisfies $(17.8.3)_\chi$ for all χ, then G satisfies (17.8.3).

The proof of the theorem will be given in 23.21. Most of this section will be concerned with the case where G is of type B, C, or D and $p \neq 2$. The strategy of proof in these cases is rather similar to that in the previous section. Using an inductive hypothesis one first shows, using 7.9 and the classification of $\mathrm{Irr}^0 G$ in [4] that G is clean. We then show that if G is not a spin-group then it satisfies the parity condition; the proof of (17.8.3) and after it that of (17.8.5) are then carried out as in the previous section. There are additional difficulties for spin-groups since for them the parity condition will not be known until a late stage in the proof. In all sections concerned with classical groups (23.2–23.7, 23.12–23.20) it will be assumed that $p \neq 2$.

23.2. We now describe the sets $(\mathrm{Irr}^0 G)^\chi$ in the case where G is semisimple of type B, C, or D and χ is a faithful character of $\mathscr{Z}_G$. The results in this section can be extracted from [4].

(a) $G = PSp_{2n}(k)$ $(n \geqslant 1)$. If n is odd then $\mathrm{Irr}^0 G$ is empty. We now assume that n is even. Then to each unordered pair (N_1, N_2) of triangular numbers $\geqslant 0$ such that $n = N_1 + N_2$ one can associate one complex $\bar{A}_{N_1,N_2} \in \mathrm{Irr}^0 G$, if $N_1 \neq N_2$, and two complexes $\bar{A}_{N_1,N_2}, \bar{A}'_{N_1,N_2} \in \mathrm{Irr}^0 G$, if $N_1 = N_2$, so that all complexes in $\mathrm{Irr}^0 G$ are obtained exactly once. The support of $\bar{A}_{N_1,N_2}$ is the conjugacy class of su, where s is a semisimple element in G with $Z^0(s)$ doubly covered by $Sp_{2N_1}(k) \times Sp_{2N_2}(k)$ and u is a certain unipotent element in $Z^0(s)$. Moreover, $\bar{A}'_{N_1,N_1}$ has the same support as $\bar{A}_{N_1,N_1}$. We have $s = e$ if and only if one of N_1, N_2 is zero.

(b) $G = Sp_{2n}(k)$, $\chi \neq 1$ $(n \geqslant 1)$. If n is even, then $(\mathrm{Irr}^0 G)^\chi$ is empty. Assume now that n is odd. There is a 1–1 correspondence $N_1, N_2 \to A_{N_1,N_2}$ between the set of ordered pairs N_1, N_2 of triangular numbers such that $n = N_1 + N_2$ and the set $(\mathrm{Irr}^0 G)^\chi$. The support of A_{N_1,N_2} is the closure of the conjugacy class of su, where s is a semisimple element in G with $Z(s)$ isomorphic to $Sp_{2N_1}(k) \times Sp_{2N_2}(k)$ and u is a certain unipotent element in

$Z(s)$. If z is the non-trivial element of $\mathscr{Z}_G$, then $t_z^* A_{N_1,N_2} = A_{N_2,N_1}$. Since n is odd, we have $N_1 \neq N_2$ and $\operatorname{supp} t_z^* A_{N_1,N_2} \neq \operatorname{supp} A_{N_1,N_2}$. We have $s = e$ if and only if $N_1 = 0$ and $s = z$ if and only if $N_2 = 0$.

(c) $G = PSO_m(k)$ $(m \geqslant 3)$. We assume that m is either odd or divisible by 8; otherwise, $\operatorname{Irr}^0 G$ is empty. To each unordered pair (N_1, N_2) of squares, which are not both odd, such that $m = N_1 + N_2$, one can associate:

(i) one complex $\bar{A}_{N_1,N_2} \in \operatorname{Irr}^0 G$, if N_1 or N_2 is zero,

(ii) two complexes $\bar{A}_{N_1,N_2}, \bar{A}'_{N_1,N_2} \in \operatorname{Irr}^0 G$, if $N_1 > 0$, $N_2 > 0$, $N_1 \neq N_2$,

(iii) four complexes $\bar{A}_{N_1,N_2}, \bar{A}'_{N_1,N_1}, \bar{A}''_{N_1,N_2}, \bar{A}'''_{N_1,N_2}$, if $N_1 = N_2$,

so that all complexes in $\operatorname{Irr}^0 G$ are obtained exactly once, and $\operatorname{supp} \bar{A}_{N_1,N_2}$ is the closure of the conjugacy class of su where s is a semisimple element in G with $Z^0(s)$ isomorphic to, or doubly covered by $SO_{N_1}(k) \times SO_{N_2}(k)$ and u is a certain unipotent element in $Z^0(s)$; moreover, $\bar{A}'_{N_1,N_1}, \bar{A}''_{N_1,N_2}, \bar{A}'''_{N_1,N_2}$ (if defined) have the same support as $\bar{A}_{N_1,N_2}$. We have $s = e$ if and only if N_1 or N_2 is zero.

(d) $G = SO_{2n}(k)$, $\chi \neq 1$ $(n \geqslant 2)$. If $n \not\equiv 2 \pmod 4$, then $(\operatorname{Irr}^0 G)^\chi$ is empty. Assume now that $n \equiv 2 \pmod 4$. To each ordered pair (N_1, N_2) of even squares such that $2n = N_1 + N_2$ one can associate:

(i) one complex $A_{N_1,N_2} \in (\operatorname{Irr}^0 G)^\chi$, if N_1 or N_2 is zero,

(ii) two complexes $A_{N_1,N_2}, A'_{N_1,N_2} \in (\operatorname{Irr}^0 G)^\chi$, if $N_1 > 0$, $N_2 > 0$,

so that all complexes in $(\operatorname{Irr}^0 G)^\chi$ are obtained exactly once, and $\operatorname{supp} A_{N_1,N_2}$ is the closure of the conjugacy class of su where s is a semisimple element in G with $Z^0(s)$ isomorphic to $SO_{N_1}(k) \times SO_{N_2}(k)$ and u is a certain unipotent element in $Z^0(s)$; moreover A'_{N_1,N_2} (if defined) has the same support as A_{N_1,N_2}. If z is the nontrivial element of $\mathscr{Z}_G$, then $t_z^* A_{N_1,N_2} = A_{N_2,N_1}$ and $t_z^* A'_{N_1,N_2} = A'_{N_2,N_1}$, if $N_1 > 0$, $N_2 > 0$. Since $n \not\equiv 0 \pmod 4$, we have $N_1 \neq N_2$ and $\operatorname{supp} t_z^* A_{N_1,N_2} \neq \operatorname{supp} A_{N_1,N_2}$, $\operatorname{supp} t_z^* A'_{N_1,N_2} \neq \operatorname{supp} \bar{A}'_{N_1,N_2}$. We have $s = e$ if and only if $N_1 = 0$ and $s = z$ if and only if $N_2 = 0$.

(e) $G = \operatorname{Spin}_m(k)$ $(m \geqslant 3)$, χ faithful, hence $m \not\equiv 0 \pmod 4$. To each ordered pair (N_1, N_2) of triangular numbers, with N_1 even, such that $m = N_1 + N_2$, one can associate two complexes $A_{N_1,N_2}, A'_{N_1,N_2} \in (\operatorname{Irr}^0 G)^\chi$, so that all complexes in $(\operatorname{Irr}^0 G)^\chi$ are obtained exactly once. If $N_1 > 0$ and $N_2 > 1$ (resp. $N_2 = 1$) we have $\operatorname{supp} A_{N_1,N_2} = \operatorname{supp} A'_{N_1,N_2} =$ closure of the conjugacy class of su where s is a semisimple element in G such that $Z(s)$ is doubly covered by $\operatorname{Spin}_{N_1}(k) \times \operatorname{Spin}_{N_2}(k)$ (resp. $Z(s) \cong \operatorname{Spin}_{N_1}(k)$) and u is a certain unipotent element in $Z(s)$. For $N_1 = 0$, we have $\operatorname{supp} A_{0,N_2} =$ closure of a unipotent class of G and $\operatorname{supp} A'_{0,N_2}$ (and $\operatorname{supp} A_{N_2,0}$, $\operatorname{supp} A'_{N_2,0}$ for m

even) are obtained from supp A_{0,N_2} by left translation by nontrivial elements in $\mathscr{Z}_G$. The action of $\mathscr{Z}_G$ on $(\mathrm{Irr}^0 G)^\chi$ by t_z^* is free.

(f) $G = \frac{1}{2}\,\mathrm{Spin}_m(k)$ (half-spin group), $\chi \neq 1$, $m \equiv 0 \pmod 4$ $(m \geqslant 12)$. To each unordered pair (N_1, N_2) of even triangular numbers such that $m = N_1 + N_2$, once can associate

(i) two complexes A_{N_1,N_2}, $A'_{N_1,N_2} \in (\mathrm{Irr}^0 G)^\chi$ if $N_1 \neq N_2$,

(ii) four complexes A_{N_1,N_1}, A'_{N_1,N_1}, A''_{N_1,N_1}, $A'''_{N_1,N_1} \in (\mathrm{Irr}^0 G)^\chi$ if $N_1 = N_2$

so that all complexes in $(\mathrm{Irr}^0 G)^\chi$ are obtained exactly once. If $N_1 > 0$ and $N_2 > 0$, we have supp $A_{N_1,N_2} =$ supp A'_{N_1,N_2} $(=$ supp $A''_{N_1,N_2} =$ supp A'''_{N_1,N_2}, if defined$)$ = closure of the conjugacy class of su where s is a semisimple element in G such that $Z^0(s)$ is isogenous to $\mathrm{Spin}_{N_1}(k) \times \mathrm{Spin}_{N_2}(k)$, and u is a certain unipotent element in $Z^0(s)$. For $N_1 = 0$, we have supp $A_{0,N_2} =$ closure of a unipotent class of G and supp $A_{N_2,0}$ is obtained from supp A_{0,N_2} by left translation by the nontrivial element $z \in \mathscr{Z}_G$. The action of $\mathscr{Z}_G$ on $(\mathrm{Irr}^0 G)^\chi$ by t_z^* is free.

The results in this subsection will be used in 23.3–23.7 to prove the cleanness of G and, in some cases, the parity condition under an inductive assumption.

LEMMA 23.3. *Let $G = PSp_{2n}(k)$ $(n \geqslant 1)$ and assume that* (17.8.4), (17.8.5) *hold for all $Sp_{2n'}(k)$, $n' < n$. Then G is clean and for any $A \in \hat{G}$ we have $\varepsilon_A = \hat{\varepsilon}_A = 1$.*

Proof. Let L be a Levi subgroup of a proper parabolic subgroup of G. Then $L/\mathscr{Z}_L^0$ is a product of $PGL_{n_i}(k)$'s and possibly a copy of $PSp_{2n'}(k)$, $n' < n$. Using our assumption and 18.5, 17.11, 17.16 we see that (17.8.4) holds for $L/\mathscr{Z}_L^0$ hence also for L, by 17.10. It is then enough to check that any cuspidal character sheaf A of G is clean and satisfies $\varepsilon_A = \hat{\varepsilon}_A = 1$. The equality $\hat{\varepsilon}_A = 1$ follows from 23.2(a): if $\mathrm{Irr}^0 G$ is nonempty, then n is even. Assume now that the support of A is the closure of a unipotent class of G. By 23.2(a) there cannot be more than one A with this property. Hence A is clean, by 7.9. If A is not of this type, then its support is the closure of the conjugacy class of an element whose semisimple part has centralizer doubly covered by a product of two symplectic groups for which our inductive assumption applies. Using 7.11(d), it again follows that A is clean. Thus A is clean in all cases. We now check the parity conditions for A, using 18.3. Let $\mathscr{L}$ be such that $A \in \hat{G}_{\mathscr{L}}$. From (17.12.4), we see that $W'_{\mathscr{L}}$ is restricted in the following way. We have

(i) $W'_{\mathscr{L}} = W_{\mathscr{L}}$ of type $D_r \times C_{r'}$ $(r + r' = n,\ r \geqslant 2,\ r' \geqslant 1)$ or D_n, or

(ii) $W'_{\mathscr{L}} = W_{\mathscr{L}} = W$.

It follows that the number N in 18.3 is equal to n. Using 23.2(a), it follows that N is always even. The assumption (18.3.1) in 18.3 is satisfied by the rationality of representations of the Hecke algebras of $W'_{\mathscr{L}}$ in (i), (ii). Hence, from 18.3 we can deduce that $\varepsilon_A = \hat{\varepsilon}_A$ and our statement is proved.

LEMMA 23.4. *Let $G = Sp_{2n}(k)$ $(n \geqslant 1)$, and let $\chi: \mathscr{Z}_G \to \bar{Q}_l^*$ be the non-trivial character. Assume that* (17.8.4), (17.8.5) *hold for all $Sp_{2n'}(k)$, $n' < n$. Then G is clean and for any $A \in \hat{G}^\chi$, we have $\varepsilon_A = \hat{\varepsilon}_A = -1$.*

Proof. Let L be a Levi subgroup of a proper parabolic subgroup of G. Then L is a product of $GL_{n_i}(k)$'s and possibly a copy of $Sp_{2n'}(k)$, $n' < n$. As in 23.3, we see that (17.8.4) holds for L. From 23.3, it follows that any cuspidal character sheaf of G with trivial $\mathscr{Z}_G$-action is clean. It is then enough to check that any cuspidal character sheaf $A \in \hat{G}^\chi$ is clean and satisfies $\varepsilon_A = \hat{\varepsilon}_A = -1$. The equality $\hat{\varepsilon}_A = -1$ follows from 23.2(b): if $(\mathrm{Irr}^0 G)^\chi$ is nonempty, then n is odd. if the support of A is the closure of a unipotent class of G, then A is uniquely determined (23.2(b)) and hence is clean by 7.9. The case where the support of A is a central element times the closure of a unipotent class is reduced to the previous case, using t_z^*. If A is not of this type, then we may apply to it 7.11(d) and our inductive hypothesis (as in 23.3) and we see again that A is clean. Thus A is clean in all cases. We now check the parity condition for A, using 18.3. Let $\mathscr{L}$ be such that $A \in \hat{G}^\chi$. From (17.12.4), we see that $W'_{\mathscr{L}}$ is restricted in the following way. We have

(i) $W_{\mathscr{L}}$ of type $D_r \times C_{r'}$ $(r + r' = n,\ r \geqslant 2,\ r' \geqslant 1)$, or D_n and $\Omega_{\mathscr{L}}$ of order 2 acting nontrivially on the D_r-component, or

(ii) $W_{\mathscr{L}}$ of type C_{n-1}, $\Omega_{\mathscr{L}}$ of order 2.

It follows that the number N in 18.3 is $n - 1$. Using 22.3(b) it follows that N is always even. The assumption (18.3.1) in 18.3 is satisfied, by the rationality of representations of the Hecke algebras of $W'_{\mathscr{L}}$ in (i), (ii) above. Hence, from 18.3, we can deduce that $\varepsilon_A = \hat{\varepsilon}_A$, and our statement is proved.

LEMMA 23.5. *Let $G = PSO_m(k)$ $(m \geqslant 3)$ and assume that* (17.8.4), (17.8.5) *hold for all $SO_{m'}(k)$, $m' < m$. Then G is clean and for any $A \in \hat{G}$ we have $\varepsilon_A = \hat{\varepsilon}_A = 1$.*

The proof is very similar to that of 23.3 (it uses 23.2(c) instead of 23.2(a)) and will be omitted. We note only that if $A \in \hat{G}_{\mathscr{L}}$ is cuspidal, then, by (17.12.4), $W'_{\mathscr{L}} = W_{\mathscr{L}}$ must be one of the following.

(i) If $m = 2n + 1$, then $W_{\mathscr{L}}$ is of type $B_r \times B'_r$ $(r + r' = n,\ r \geqslant 1,\ r' \geqslant 1)$ or of type B_n.

(ii) If $m=2n$, then $W_{\mathscr{L}}$ is of type $D_r \times D_{r'}$ ($r+r'=n$, $r \geqslant 2$, $r' \geqslant 2$) or $W_{\mathscr{L}} = W$.

LEMMA 23.6. *Let $G = SO_{2n}(k)$ $(n \geqslant 2)$ and let $\chi: \mathscr{Z}_G \to \bar{Q}_l^*$ be the non-trivial character. Assume that* (17.8.4), (17.8.5) *hold for all $SO_{2n'}(k)$, $n'<n$. Then G is clean and for any $A \in \hat{G}^\chi$, we have $\varepsilon_A = \hat{\varepsilon}_A = 1$.*

The proof is very similar to that of 23.3 or 23.4. We shall only indicate the proof of the parity condition for a cuspidal $A \in \hat{G}^\chi$. We shall use again 18.3. Note that if $A \in \hat{G}^\chi_{\mathscr{L}}$ is cuspidal then, by (17.12.4), $W'_{\mathscr{L}}$ must be of the following form.

(i) $W_{\mathscr{L}}$ is of type $D_r \times D_{r'}$ ($r+r'=n$, $r \geqslant 2$, $r' \geqslant 2$), $\Omega_{\mathscr{L}}$ of order 2 acting nontrivially on each factor D_r, $D_{r'}$, or

(ii) $W_{\mathscr{L}}$ is of type D_{n-1} (if $n \geqslant 3$), $\Omega_{\mathscr{L}}$ of order 2 acting nontrivially on $W_{\mathscr{L}}$.

It follows that the number N in 18.3 is $n-2$. Using 23.2(d), we see that N is always even. The assumption (18.3.1) in 18.3 is satisfied by the rationality of representations of the Hecke algebras of $W'_{\mathscr{L}}$ in (i), (ii) above. Hence, from 18.3, we can deduce that $\varepsilon_A = \hat{\varepsilon}_A$, as desired.

LEMMA 23.7. *Let $G = \mathrm{Spin}_m(k)$ $(m \geqslant 3)$, or $G = \frac{1}{2}\mathrm{Spin}_m(k)$ $(m \geqslant 12$, $m \equiv 0 \pmod 4)$. Assume that* (17.8.4), (17.8.5) *hold for all* $\mathrm{Spin}_{m'}(k)$, $m'<m$. *Then G is clean.*

The proof is entirely similar to that of 23.4; it will be omitted.

Note that we are not able to prove a parity condition for G, as we did in the other cases (23.3–23.6); the difficulty is that we cannot apply 18.3 since the groups $W_{\mathscr{L}}$ which appear may contain factors A_n with nontrivial action of $\Omega_{\mathscr{L}}$, and there is no simple rationality statement for the representations of the corresponding Hecke algebras.

23.8. We now consider a general G, an $\mathscr{L} \in \mathscr{S}(T)$ and a character $\chi: \mathscr{Z}_G/\mathscr{Z}_G^0 \to \bar{Q}_l^*$.

(23.8.1) Assume that G satisfies $(17.8.3)_\chi$ and G' satisfies $(17.8.3)_{\chi'}$. (Here G' is another group like G, and $\chi': \mathscr{Z}_{G'}/\mathscr{Z}_{G'}^0 \to \bar{Q}_l^*$ is a character.) Then $G \times G'$ satisfies $(17.8.3)_{\chi \times \chi'}$ where $\chi \times \chi': \mathscr{Z}_{G \times G'}/\mathscr{Z}^0_{G \times G'} \to \bar{Q}_l^*$ is the product of χ, χ'.

(This is a refinement of 17.11; the proof is left to the reader.)

(23.8.2) Let $\bar{G} = G/\mathscr{Z}_G^0$ so that $\mathscr{Z}_{\bar{G}} = \mathscr{Z}_G/\mathscr{Z}_G^0$, and let $\bar{\chi}$ be the corresponding character of $\mathscr{Z}_{\bar{G}}$ defined by χ. If $\bar{G}$ satisfies $(17.8.3)_{\bar{\chi}}$, then G satisfies $(17.8.3)_\chi$.

(This is proved just as in 17.10.)

Note that for any $A \in \hat{G}^{\chi}_{\mathscr{L}}$, any $E \in \hat{W}'_{\mathscr{L}}$ and any character ϕ of $W'_{\mathscr{L}}/W_{\mathscr{L}}$ we have

$$(A : R^{\mathscr{L}}_{E \otimes \phi}) = \phi(x)^{-1}(A : R^{\mathscr{L}}_{E}). \tag{23.8.3}$$

Indeed, from (16.2.9), 11.9, and 17.18(b) we see that

$$\begin{aligned}(A : R^{\mathscr{L}}_{E \otimes \phi}) &= \sum_{y \in W_{\mathscr{L}}} c_{y^{-1}, E \otimes \phi}(A : R^{\mathscr{L}}_{\alpha_y}) = \sum_{y \in xW_{\mathscr{L}}} c_{y^{-1}, E \otimes \phi}(A : R^{\mathscr{L}}_{\alpha_y}) \\ &= \phi(x)^{-1} \sum_{y \in xW_{\mathscr{L}}} c_{y^{-1}, E}(A : R^{\mathscr{L}}_{\alpha_y}) = \phi(x)^{-1}(A : R^{\mathscr{L}}_{E}).\end{aligned}$$

It is easy to check that for any $\bar{m} \in \mathscr{M}(\mathscr{G}_{\mathscr{F}'})^{\chi}$ ($\mathscr{F}'$ family in $\hat{W}'_{\mathscr{L}}$) and E, ϕ as above, we have $(\bar{m}, m_{E \otimes \phi}) = \phi(x)^{-1}(\bar{m}, m_E)$.

(23.8.4) It follows that, assuming $\chi : \mathscr{Z}_G/\mathscr{Z}^0_G \to \bar{Q}_l$ is faithful, we have $(A : R^{\mathscr{L}}_E) = 0$ ($A \in \hat{G}^{\chi}_{\mathscr{L}}$, $E \in \hat{W}'_{\mathscr{L}}$) unless the restriction of E to $W_{\mathscr{L}}$ is irreducible (such E are said to be *nonsingular*).

In this case, we see also that the condition "$(A : R^{\mathscr{L}}_E) = \hat{\varepsilon}_A(\bar{m}_A, m_E)$ for all $A \in \hat{G}^{\chi}_{\mathscr{L}}$, $E \in \hat{W}'_{\mathscr{L}}$" in $(17.8.3)_{\chi}$ is equivalent to the condition "$(A : \tilde{R}^{\mathscr{L}}_E) = \hat{\varepsilon}_A |\Omega_{\mathscr{L}}| (\bar{m}_A, m_E)$ for all $A \in \hat{G}^{\chi}_{\mathscr{L}}$ and all nonsingular $E \in \hat{W}'_{\mathscr{L}}$", where $\tilde{R}^{\mathscr{L}}_E = \sum_{\phi} \phi(x) R^{\mathscr{L}}_{E \otimes \phi}$ and ϕ runs over all characters of $W'_{\mathscr{L}}/W_{\mathscr{L}} = \Omega_{\mathscr{L}}$.

LEMMA 23.9. *Let $\pi : G' \to G$ be a surjective homomorphism with finite kernel Γ, where G, G' are connected semisimple groups over k. Let χ: $\mathscr{Z}_G \to \bar{Q}^*_l$ be a character and let $\chi' : \mathscr{Z}_{G'} \to \bar{Q}_l$ be the composition of χ with the map $\mathscr{Z}_{G'} \to \mathscr{Z}_G$ induced by π.*

(a) *If the parity condition $\varepsilon_A = \hat{\varepsilon}_A$ is satisfied for all $A \in \hat{G}^{\chi}$, then it is also satisfied for all $A' \in \hat{G}'^{\chi'}$.*

(b) *Assume that* $\mathrm{Irr}^0(G)^{\chi} \subset \hat{G}^{\chi}$. *Then* $\mathrm{Irr}^0(G')^{\chi'} \subset \hat{G}'^{\chi'}$.

(c) *Assume that any cuspidal $A \in \hat{G}^{\chi}$ is clean. Then any cuspidal $A' \in \hat{G}'^{\chi'}$ is clean.*

(d) *Assume that $(17.8.3)_{\chi'}$ holds for G' and that $t^*_z A' \neq A'$ for any $A' \in \hat{G}'^{\chi'}$ and any $z \in \Gamma$, $z \neq e$. Assume also that Γ is cyclic of order m.*

Then $(17.8.3)_{\chi}$ holds for G.

Proof. The statements (a), (b), (c) are proved by the arguments in 17.16. (Note that if $A' \in \mathrm{Irr}^0(G')^{\chi'}$ then $\pi_* A'$ is a direct sum of complexes in $\mathrm{Irr}^0(G)^{\chi}$.) We now prove (d). Let T be a maximal torus of G and let $T' = \pi^{-1}(T)$. Let $\mathscr{L} \in \mathscr{S}(T)$ and let $\mathscr{L}' = \pi^* \mathscr{L}$. For any $A' \in \hat{G}'^{\chi'}$, the direct image $\pi_* A'$ is a direct sum of irreducible perverse sheaves on G, each one with multiplicity one (since Γ is cyclic). These irreducible perverse sheaves are in fact in $\hat{G}^\chi$, (see 17.16). Conversely, if $A \in \hat{G}^\chi$, then the inverse image $\pi^* A$ is a direct sum of character sheaves in $\hat{G}'^{\chi'}$, each one with multiplicity one. For $A' \in \hat{G}'^{\chi'}$ and $A \in \hat{G}^\chi$ we denote by $f_{A,A'}$ the multiplicity of A in $\pi_* A'$ or, equivalently, the multiplicity of A' in $\pi_* A$. Then $f_{A',A}$ is always 0 or 1. There is a natural action of Γ on $\hat{G}'^{\chi'}$ by $z: A' \to t_z^* A'$ $(z \in \Gamma)$, see 17.17, and a natural action of $\hat{\Gamma} = \mathrm{Hom}(\Gamma, \bar{Q}_l^*)$ on $\hat{G}^\chi$ by $\alpha: A \to A \otimes \mathscr{E}_\alpha$. (Here, $\alpha \in \hat{\Gamma}$ and $\mathscr{E}_\alpha$ is the local system on G associated to the principal Γ-bundle $G' \to G$ and to α.) We have a 1–1 correspondence between the set of Γ-orbits on $\hat{G}'^{\chi'}$ and the set of $\hat{\Gamma}$-orbits on $\hat{G}^\chi$: the orbit of $A' \in \hat{G}'^{\chi'}$ corresponds to the orbit of $A \in \hat{G}^\chi$ precisely when $f_{A,A'} = 1$. When two such orbits correspond, the number of objects in one orbit times the number of objects in the other orbit is always equal to $m = |\Gamma|$. In our case, Γ acts freely on $\hat{G}'^{\chi'}$, by assumption. It follows that π_* defines a bijection between $\hat{G}'^{\chi'}$, modulo the action of Γ, and $\hat{G}^\chi$, and also:

(23.9.1) a bijection between $\hat{G}'^{\chi'}_{\mathscr{L}'}$, modulo the action of Γ, and $\hat{G}^\chi_{\mathscr{L}}$.

Since G', $\mathscr{L}'$ satisfies $(17.8.3)_{\chi'}$, Γ must act freely on $\coprod_{\mathscr{F}'} \mathscr{M}(\mathscr{G}_{\mathscr{F}'})^{\chi'}$ by $z: (x, \sigma) \to (x, \sigma \otimes \sigma_z)$ (see 23.8); here, $\mathscr{G}_{\mathscr{F}'}$ are defined in terms of $W'_{\mathscr{L}'}$, and $\mathscr{F}'$ runs over all families in $\hat{W}'_{\mathscr{L}'}$. If $\mathscr{F}'$ is such a family, then $\mathscr{F}'$ corresponds to a family $\mathscr{F}$ in $\hat{W}_{\mathscr{L}'} = \hat{W}_{\mathscr{L}}$, well defined up to the action of $\Omega_{\mathscr{L}'}$. (Recall that $W'_{\mathscr{L}'} = \Omega_{\mathscr{L}'} \cdot W_{\mathscr{L}'}$.) We have $\mathscr{G}_{\mathscr{F}'} = \Omega_{\mathscr{L}',\mathscr{F}} \cdot \mathscr{G}_{\mathscr{F}}$ where $\Omega_{\mathscr{L}',\mathscr{F}}$ is the stabilizer of $\mathscr{F}$ in $\Omega_{\mathscr{L}'}$ and $\mathscr{G}_{\mathscr{F}}$ is defined as in 17.5, in terms of $\mathscr{F} \subset \hat{W}_{\mathscr{L}'} = \hat{W}_{\mathscr{L}}$. We have a commutative diagram

$$\begin{array}{ccc} \Omega_{\mathscr{L}'} & \hookrightarrow & \mathrm{Hom}(\mathscr{Z}_{G'}, \bar{Q}_l^*) \\ \uparrow & & \uparrow \\ \Omega_{\mathscr{L}} & \hookrightarrow & \mathrm{Hom}(\mathscr{Z}_G, \bar{Q}_l^*), \end{array} \tag{23.9.2}$$

where the horizontal maps are given by (11.8.1); the right vertical map is composition with π, so it carries χ to χ'.

(23.9.3) From the definitions, it follows that there is an induced imbedding $\Omega_{\mathscr{L}'}/\Omega_{\mathscr{L}} \hookrightarrow \mathrm{Hom}(\mathscr{Z}_{G'}, \bar{Q}_l^*)/\mathrm{Hom}(\mathscr{Z}_G, \bar{Q}_l^*) = \mathrm{Hom}(\Gamma, \bar{Q}_l^*)$.

We must construct a bijection

$$\hat{G}^\chi_{\mathscr{L}} \leftrightarrow \coprod_{\mathscr{F}'} \mathscr{M}(\mathscr{G}_{\mathscr{F}'})^\chi, \tag{23.9.4}$$

where $\mathscr{G}_{\mathscr{F}'}$ are defined in terms of $W'_{\mathscr{L}}$ and $\mathscr{F}'$ runs over all families in $\hat{W}'_{\mathscr{L}}$. We may assume that there exists $x_0 \in \Omega_{\mathscr{L}}$ which is mapped to χ by (11.8.1);

otherwise, both sets in (23.9.4) are empty and there is nothing to prove. Instead of defining (23.9.4) directly, we shall first define a bijection

$$\left(\coprod_{\mathscr{F}'} \mathscr{M}(\mathscr{G}_{\mathscr{F}'})^{\chi'}\right)\Big/\Gamma \leftrightarrow \coprod_{\mathscr{F}'} \mathscr{M}\left(\mathscr{G}_{\bar{\mathscr{F}}'}\right)^{\chi} \tag{23.9.5}$$

and then define (23.9.4) as composition of $(\coprod_{\mathscr{F}'} \mathscr{M}(\mathscr{G}_{\mathscr{F}'})^{\chi'})/\Gamma \leftrightarrow (\hat{G}'^{\chi'}_{\mathscr{L}'})/\Gamma$ (obtained from $(17.8.3)_{\chi'}$) with (23.9.1) and with (23.9.5). Let $\mathscr{F}$ be a family in $\hat{W}_{\mathscr{L}'} = \hat{W}_{\mathscr{L}}$ which is x_0-stable (i.e., stable under conjugation by x_0), let $\mathscr{F}'$, $\bar{\mathscr{F}}'$ be the corresponding families in $W'_{\mathscr{L}'}$, $W'_{\mathscr{L}}$, respectively, and let $\Omega_{\mathscr{L}',\mathscr{F}}, \Omega_{\mathscr{L},\mathscr{F}}$ be the stabilizers of $\mathscr{F}$ in $\Omega_{\mathscr{L}'}, \Omega_{\mathscr{L}}$, respectively. Consider the element $(x_0, 1) \in \mathscr{M}(\mathscr{G}_{\mathscr{F}'})^{\chi'}$; here 1 denotes the unit representation of $Z_{\mathscr{G}_{\mathscr{F}'}}(x_0) = \Omega_{\mathscr{L}',\mathscr{F}} Z_{\mathscr{G}_{\mathscr{F}}}(x_0)$. We know that the Γ-orbit of $(x_0, 1)$ consists of $|\Gamma| = m$ elements. This means that the restrictions of σ_z ($z \in \Gamma$, see 23.8) to $\Omega_{\mathscr{L}',\mathscr{F}}$ are distinct; however, the restrictions of σ_z to $\Omega_{\mathscr{L},\mathscr{F}}$ are all trivial. It follows that $\Omega_{\mathscr{L}',\mathscr{F}}/\Omega_{\mathscr{L},\mathscr{F}}$ has at least m distinct characters. Hence the index of $\Omega_{\mathscr{L},\mathscr{F}}$ in $\Omega_{\mathscr{L}',\mathscr{F}}$ is at least m. Using (23.9.3), we see that in the imbedding $\Omega_{\mathscr{L}',\mathscr{F}}/\Omega_{\mathscr{L},\mathscr{F}} \subset \Omega_{\mathscr{L}'}/\Omega_{\mathscr{L}}$ we have $m \leqslant |\Omega_{\mathscr{L}',\mathscr{F}}/\Omega_{\mathscr{L},\mathscr{F}}| \leqslant |\Omega_{\mathscr{L}'}/\Omega_{\mathscr{L}}| \leqslant m$ hence this imbedding is an equality. It follows that the $\Omega_{\mathscr{L}'}$-orbit of $\mathscr{F}$ is the same as the $\Omega_{\mathscr{L}}$-orbit of $\mathscr{F}$. Hence we have a natural bijection between the set of families $\mathscr{F}'$ in $\hat{W}'_{\mathscr{L}'}$ such that $\mathscr{M}(\mathscr{G}_{\mathscr{F}'})^{\chi'} \neq \varnothing$ and the set of families $\bar{\mathscr{F}}'$ in $\hat{W}'_{\mathscr{L}'}$ such that $\mathscr{M}(\mathscr{G}_{\bar{\mathscr{F}}'})^{\chi} \neq \varnothing$. Hence to define (23.9.5) it is enough to construct for each $\mathscr{F}, \mathscr{F}', \bar{\mathscr{F}}'$ as above, a bijection

$$\mathscr{M}(\mathscr{G}_{\mathscr{F}'})^{\chi'}/\Gamma \leftrightarrow \mathscr{M}(\mathscr{G}_{\bar{\mathscr{F}}'})^{\chi}. \tag{23.9.6}$$

Let $x_1 \in \mathscr{G}_{\mathscr{F}}$; consider the lement $(x_0 x_1, 1) \in \mathscr{M}(\mathscr{G}_{\mathscr{F}'})^{\chi'}$; here 1 denotes the unit representations of $Z_{\mathscr{G}_{\mathscr{F}'}}(x_0 x_1)$. We know that the Γ-orbit of $(x_0 x_1, 1)$ consists of $|\Gamma| = m$ elements. This means that the restrictions of σ_z ($z \in \Gamma$) to $Z_{\mathscr{G}_{\mathscr{F}'}}(x_0 x_1)$ are distinct; on the other hand, the restriction of σ_z to $Z_{\mathscr{G}_{\bar{\mathscr{F}}'}}(x_0 x_1)$ are trivial. It follows that $Z_{\mathscr{G}_{\mathscr{F}'}}(x_0 x_1)/Z_{\mathscr{G}_{\bar{\mathscr{F}}'}}(x_0 x_1)$ has at least m elements; it is contained in $\mathscr{G}_{\mathscr{F}'}/\mathscr{G}_{\bar{\mathscr{F}}'} = \Omega_{\mathscr{L}',\mathscr{F}}/\Omega_{\mathscr{L},\mathscr{F}}$ which has exactly m elements. Hence the imbedding $Z_{\mathscr{G}_{\mathscr{F}'}}(x_0 x_1)/Z_{\mathscr{G}_{\bar{\mathscr{F}}'}}(x_0 x_1) \subset \mathscr{G}_{\mathscr{F}'}/\mathscr{G}_{\bar{\mathscr{F}}'}$ is an equality. It follows that we have a canonical bijection between the set of $\mathscr{G}_{\mathscr{F}'}$-conjugacy classes of elements in $\mathscr{G}_{\mathscr{F}'}$ of form $x_0 x_1$ ($x_1 \in \mathscr{G}_{\mathscr{F}}$), and the set of $\mathscr{G}_{\bar{\mathscr{F}}'}$-conjugacy classes of elements in $\mathscr{G}_{\bar{\mathscr{F}}'}$ of form $x_0 x_1$ ($x_1 \in \mathscr{G}_{\mathscr{F}}$). Moreover, for each such x_1, the group $Z_{\mathscr{G}_{\mathscr{F}'}}(x_0 x_1)/Z_{\mathscr{G}_{\bar{\mathscr{F}}'}}(x_0 x_1)$ is cyclic of order m, and by assumption, Γ acts freely on the set of irreducible representations of $Z_{\mathscr{G}_{\mathscr{F}'}}(x_0 x_1)$, via $\otimes\, \sigma_z$. It follows that restriction of representations defines a bijection between the set of Γ-orbits on the set of irreducible representations of $Z_{\mathscr{G}_{\mathscr{F}'}}(x_0 x_1)$ on the one hand, and the set of irreducible representations of $Z_{\mathscr{G}_{\bar{\mathscr{F}}'}}(x_0 x_1)$ on the other hand. This gives rise to the bijection (23.9.6) and hence to the bijection (23.9.5).

Now let $A \in \hat{G}^{\chi}_{\mathscr{L}}$ and let $E \in \hat{W}'_{\mathscr{L}}$. Let $\bar{m}_A = (x_0 x_1, \sigma) \in \mathscr{M}(\mathscr{G}_{\mathscr{F}'})^{\chi}$ be the element corresponding to A under (23.9.5) and let m_E be as in 17.8. Let $A' \in \hat{G}'^{\chi'}_{\mathscr{L}'}$ be such that $\pi_* A' = A$ and let $E'_1, ..., E'_m$ be the irreducible representations of $W'_{\mathscr{L}'}$ whose restrictions to $W_{\mathscr{L}}$ are E. Let $\bar{m}_{A'} = (x_0 x_1, \sigma') \in \mathscr{M}(\mathscr{G}_{\mathscr{F}'})^{\chi'}$ be the element corresponding to A' under $(17.8.3)_{\chi'}$ and let $m_{E'_i}$ be as in 17.8. Note that $\hat{\varepsilon}_{A'} = \hat{\varepsilon}_A$. We have

$$\begin{aligned}(A : R_E^{\mathscr{L}}) &= (\pi_* A' : R_E^{\mathscr{L}}) = (A' : \pi^* R_E^{\mathscr{L}}) \\ &= \sum_{i=1}^{m} (A' : R_{E'_i}^{\mathscr{L}'}) = \sum_{i=1}^{m} \hat{\varepsilon}_{A'} \{\bar{m}_{A'}, m_{E'_i}\} \\ &= \sum_{i=1}^{m} \hat{\varepsilon}_{A'} \{(x_0 x_1, \sigma'), m_{E'_i}\}.\end{aligned}$$

Using the definitions and the fact that σ is the restriction of σ', we see that the last expression is equal to $\hat{\varepsilon}_A\{(x_0 x_1, \sigma), m_E\}$. The verification of the fact that the bijection (23.9.5) is compatible with the action of $\mathscr{Z}_G/\mathscr{Z}_G^0$ (see 23.8) is left to the reader. This completes our proof.

LEMMA 23.10. *Assume that G is a connected semisimple group over k and let $\chi: \mathscr{Z}_G \to \bar{\mathbb{Q}}_l^*$ be a faithful character. Assume that $(17.8.3)_\chi$ holds for G. Then $t_z^* A \neq A$ for any $A \in \hat{G}^{\chi}$ and any $z \in \mathscr{Z}_G$, $z \neq e$.*

Proof. Using $(17.8.3)_\chi$ we see that it is enough to prove the following statement. "Let $\mathscr{L} \in \mathscr{S}(T)$, let $x_0 \in \Omega_{\mathscr{L}}$ be such that x_0 is mapped to χ by (11.8.1), let $\mathscr{F} \in \hat{W}_{\mathscr{L}}$ be a family stable under conjugation by x_0, let $x_1 \in \mathscr{G}_{\mathscr{F}}$, and let σ be an irreducible representation of $Z_{\Omega_{\mathscr{L}}\mathscr{G}_{\mathscr{F}}}(x_0 x_1)$. Then $\sigma \otimes \sigma_z$ is not isomorphic to σ for any $z \in \mathscr{Z}_G$, $z \neq e$ (see 23.8)." It is enough to show that $\sigma_z(x_0 x_1) \neq 1$ for all $z \neq e$, or equivalently that $\sigma_z(x_0) \neq 1$ for all $z \in \mathscr{Z}_G$, $z \neq e$. By the definition of x_0, σ_z, we have $\sigma_z(x_0) = \chi(z)$ and we have $\chi(z) \neq 1$ for all $z \in \mathscr{Z}_G$, $z \neq e$ since χ is faithful.

LEMMA 23.11. *Assume that $G \neq \{e\}$ is a semisimple connected group over k, and let $\chi: \mathscr{Z}_G \to \bar{\mathbb{Q}}_l^*$ be a faithful character. Let $G_1, G_2, ..., G_r$ be the set of almost simple, closed normal subgroups $\neq \{e\}$ of G, and let $\chi_i: \mathscr{Z}_{G_i} \to \bar{\mathbb{Q}}_l^*$ be the restriction of χ to $\mathscr{Z}_{G_i}$ $(1 \leqslant i \leqslant r)$. Assume that $(17.8.3)_{\chi_i}$ holds for G_i, for all i. Then $(17.8.3)_\chi$ holds for G.*

Proof. We may assume that $r \geqslant 2$ and that the result is true for $r-1$. Let G'_2 be the subgroup of G generated by $G_2, ..., G_r$, let $G' = G_1 \times G'_2$ and let $\pi: G' \to G$ be defined by $\pi(g_1, g'_2) = g_1 \cdot g'_2$. From our assumption, it follows that, if χ'_2 is the restriction of χ to $\mathscr{Z}_{G'_2}$ (a subgroup of $\mathscr{Z}_G$), then $(17.8.3)_{\chi'_2}$ holds for G'_2. From (23.8.1) it follows that $(17.8.3)_{\chi'}$ holds for G' where $\chi': \mathscr{Z}_{G'} \to \bar{\mathbb{Q}}_l^*$ is the composition of χ with $\pi: \mathscr{Z}_{G'} \to \mathscr{Z}_G$. (We have

$\mathscr{Z}_{G'} = \mathscr{Z}_{G_1} \times \mathscr{Z}_{G'_2}$ and $\chi'(a, b) = \chi_1(a)\,\chi'_2(b)$, for $a \in \mathscr{Z}_{G_1}$, $b \in \mathscr{Z}_{G'_2}$.) Let Γ be the kernel of π. Then $\Gamma = \{(a, b) \in \mathscr{Z}_{G_1} \times \mathscr{Z}_{G'_2} \mid a = b^{-1}\}$ is isomorphic to $\mathscr{Z}_{G_1} \cap \mathscr{Z}_{G'_2} \subset \mathscr{Z}_G$ hence it is cyclic since $\chi_1 : \mathscr{Z}_{G_1} \to \bar{Q}_l^*$ is faithful. Let $A' \in \hat{G}'^{\chi'}$, and let $z = (a, a^{-1}) \in \Gamma$, $z \neq e$ $(a \in \mathscr{Z}_{G_1} \cap \mathscr{Z}_{G'_2})$. We want to show that $t_z^* A' \neq A'$. We have $A' = A'_1 \boxtimes A'_2$ where $A'_1 \in \hat{G}_1^{\chi_1}$, $A'_2 \in \hat{G}_2'^{\chi'_2}$, $t_z^* A' = t_a^* A'_1 \boxtimes t_{a^{-1}}^* A'_2$. To show that $A' \neq t_z^* A'$, it is enough to show that $A'_1 \neq t_a^* A'_1$, and this follows from 23.10 since χ_1 is a faithful character of $\mathscr{Z}_{G_1}$. (Note that $\mathscr{Z}_{G_1}$ is a subgroup of $\mathscr{Z}_G$.) It remains to apply 23.9. The lemma is proved.

23.12. We shall now fix G and $\chi : \mathscr{Z}_G \to \bar{Q}_l^*$ as in (a), (b), (c), (d), (e), or (f) below and we shall make an inductive assumption as indicated.

(a) $G = PSp_{2n}(k)$ $(n \geqslant 2)$, $\chi = 1$. Assume that (17.8.4), (17.8.5) hold for all $Sp_{2n'}(k)$, $1 \leqslant n' < n$, and that (17.8.3) holds for all $PSp_{2n'}(k)$, $1 \leqslant n' < n$.

(b) $G = Sp_{2n}(k)$ $(n \geqslant 2)$, $\chi \neq 1$. Assume that (17.8.4), (17.8.5) hold for all $Sp_{2n'}(k)$, $1 \leqslant n' < n$, and that $(17.8.3)_{\chi'}$ holds for all $Sp_{2n'}(k)$, $1 \leqslant n' < n$, where χ' is the nontrivial character of the centre, (see 23.8).

(c) $G = PSO_m(k)$ $(m \geqslant 4)$, $\chi = 1$. Assume that (17.8.4), (17.8.5) hold for all $SO_{m'}(k)$, $3 \leqslant m' < m$, and that (17.8.3) holds for all $PSO_{m'}(k)$, $3 \leqslant m' < m$.

(d) $G = SO_{2n}(k)$ $(n \geqslant 3)$, $\chi \neq 1$. Assume that (17.8.4), (17.8.5) hold for all $SO_{2n'}(k)$, $2 \leqslant n' < n$, and that $(17.8.3)_{\chi'}$ holds for all $SO_{2n'}(k)$, $2 \leqslant n' < n$, where χ' is the nontrivial character of the centre (see 23.8).

(e_1) $G = \mathrm{Spin}_m(k)$ $(m \geqslant 5,\ m$ odd$)$, $\chi \neq 1$.. Assume that (17.8.4), (17.8.5) hold for all $\mathrm{Spin}_{m'}(k)$, $3 \leqslant m' < m$ and that $(17.8.3)_{\chi'}$ holds for all $\mathrm{Spin}_{m'}(k)$ $(3 \leqslant m' < m,\ m'$ odd$)$ where χ' is the nontrivial character of the centre.

(e_2) $G = \mathrm{Spin}_m(k)$ $(m \geqslant 10,\ m \equiv 2 \pmod 4)$, χ faithful. Assume that (17.8.4), (17.8.5) hold for all $\mathrm{Spin}_{m'}(k)$, $3 \leqslant m' < m$ and that $(17.8.3)_{\chi'}$ holds for all $\mathrm{Spin}_{m'}(k)$ $(6 \leqslant m' < m,\ m' \equiv 2 \pmod 4)$ where χ' is any faithful character of the centre.

(f) $G = \frac{1}{2}\mathrm{Spin}_m(k)$ $(m \geqslant 12,\ m \equiv 0 \pmod 4)$, $\chi \neq 1$. Assume that (17.8.4), (17.8.5) hold for all $\mathrm{Spin}_{m'}(k)$, $3 \leqslant m' < m$, that $(17.8.3)_{\chi'}$ holds for all $\frac{1}{2}\mathrm{Spin}_{m'}(k)$ $(8 \leqslant m' < m,\ m' \equiv 0 \pmod 4)$ where χ' is the nontrivial character of the centre and that $(17.8.3)_{\chi''}$ holds for all $SO_{m'}(k)$ $(6 \leqslant m' < m,\ m' \equiv 2 \pmod 4)$ where χ'' is the nontrivial character of the centre.

LEMMA 23.13. *Let G, χ be as in 23.12. Then:*

(a) *If L is a Levi subgroup of a proper parabolic subgroup of G, and $\chi_1 : \mathscr{Z}_L/\mathscr{Z}_L^0 \to \bar{Q}_l^*$ is a faithful character, then $(17.8.3)_{\chi_1}$ holds for L.*

(b) *Let* $\mathscr{L} \in \mathscr{S}(T)$ *be such that for some* $w \in W$, *and some* $I \subsetneq S$, $\mathscr{L}' = w^*\mathscr{L}$ *satisfies* $W'_{\mathscr{L}'} = W'_{\mathscr{L}',I}$. *Then* (17.8.3) *holds for* $G, \mathscr{L}$.

Proof. (a) Using (23.8.2) we may replace L by $\bar{L} = L/\mathscr{Z}_L^0$ and χ_1 by the corresponding character of $\mathscr{Z}_{\bar{L}}$. We shall apply 23.11 to $\bar{L}$. This is applicable, in view of the inductive assumptions on G (see 23.12) and the following statement:

(23.13.1) Any almost simple group of type A satisfies $(17.8.3)_\chi$ for any character χ of the centre.

(This statement can be extracted from the proof of 18.5.)

(b) We may assume that $w = e$ or $w = s$, a simple reflection. The case $w = s$ reduces to the case $w = e$ by the method of 14.15. Assume now that $w = e$, so that $\mathscr{L}' = \mathscr{L}$. Then the method of 17.12 shows that (b) for χ, G is a consequence of (a) for χ_1, L_I. Here, χ, χ_1 are related by $\chi = \chi_1 \circ \phi$, where $\phi \colon \mathscr{Z}_G \to \mathscr{Z}_{L_I}/\mathscr{Z}_{L_I}^0$ is the natural (surjective) homomorphism; ϕ must in fact be an isomorphism, otherwise, $\hat{G}_{\mathscr{L}}^\chi$ is empty. To be able to apply the method of 17.12, we need to know that G is clean. But this has been verified in 23.3–23.7. This completes the proof.

23.14. Let G, χ be as in 23.12(a), (b), (c), (d), (e_1), (e_2), (f). To prove $(17.8.3)_\chi$ for $G, \mathscr{L}$ ($\mathscr{L} \in \mathscr{S}(T)$) we may assume by 23.13(b) that $\mathscr{L}$ is restricted in the following way:

(a) $\mathscr{L}$ is as in 23.3(i), (ii).

(b) $\mathscr{L}$ is as in 23.4(i), (ii).

(c) $\mathscr{L}$ is as in 23.5(i), (ii).

(d) $\mathscr{L}$ is as in 23.6(i), (ii).

(e_1) (i) $W_{\mathscr{L}}$ is of type $B_r \times B_r \times A_{r'}$ $(2r + r' + 1 = \frac{1}{2}(m-1),\ r \geqslant 1,\ r' \geqslant 1)$ with $\Omega_{\mathscr{L}}$ of order 2 switching the two B_r-factors, and (if $r' \geqslant 2$) acting nontrivially on the $A_{r'}$-factors, or

(ii) $W_{\mathscr{L}}$ is of type $A_{r'}$ $(r' + 1 = \frac{1}{2}(m-1))$, with $\Omega_{\mathscr{L}}$ of order 2 acting nontrivially (if $r' \geqslant 2$) on $W_{\mathscr{L}}$ or

(iii) $W_{\mathscr{L}}$ is of type $B_r \times B_r$ $(2r = \frac{1}{2}(m-1),\ r \geqslant 1)$, with $\Omega_{\mathscr{L}}$ of order 2 switching the two B_r-factors.

(e_2) (i) $W_{\mathscr{L}}$ is of type $D_r \times D_r \times A_{r'}$ $(2r + r' + 1 = \frac{1}{2}m,\ r \geqslant 2,\ r' \geqslant 1)$, with $\Omega_{\mathscr{L}}$ cyclic of order 4 with the generator corresponding to χ (see (11.8.1)) switching the two D_r-factors, and (if $r' \geqslant 2$) acting nontrivially on the $A_{r'}$-factor (the square of the generator acts nontrivially on each D_r-factor), or

(ii) $W_{\mathscr{L}}$ is of type $A_{r'}$ $(r' + 3 = \frac{1}{2}m)$, with $\Omega_{\mathscr{L}}$ cyclic of order 4 with generator corresponding to χ (see (11.8.1)) acting nontrivially on $W_{\mathscr{L}}$.

(f) (i) $W_{\mathscr{L}}$ is of type $D_r \times D_r \times A_{r'}$ ($2r + r' + 1 = \frac{1}{2}m$, $r \geqslant 2$, $r' \geqslant 1$), with $\Omega_{\mathscr{L}}$ of order 2 switching the two D_r-factors and (if $r' \geqslant 2$) acting nontrivially on the $A_{r'}$-factor, or

(ii) $W_{\mathscr{L}}$ is of type $A_{r'}$ ($r' + 1 = \frac{1}{2}m$), with $\Omega_{\mathscr{L}}$ of order 2 acting nontrivially on $W_{\mathscr{L}}$, or

(iii) $W_{\mathscr{L}}$ is of type $D_r \times D_r$ ($2r = \frac{1}{2}m$), with $\Omega_{\mathscr{L}}$ of order 2 switching the two D_r-factors.

In the cases (a), (b), (c), (d), the parity condition $\varepsilon_A = \hat{\varepsilon}_A$ is satisfied (see 23.3–23.6), hence for each family $\mathscr{F} \subset \hat{W}'_{\mathscr{L}}$, the subset $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$ of $\hat{G}_{\mathscr{L},\mathscr{F}}$ is well defined, (see (17.13.2), 23.0).

In the cases (e_1), (e_2), (f), the parity condition is not yet known. We shall define $\hat{G}^{\chi,+} = \{A \in \hat{G}^{\chi} \mid \varepsilon_A = (-1)^{\operatorname{rank} G}\}$ and $\hat{G}^{\chi,+}_{\mathscr{L}} = \hat{G}^{\chi,+} \cap \hat{G}^{\chi}_{\mathscr{L}}$. If $A \in \hat{G}^{\chi,+}$, then the parity condition $\varepsilon_A = \hat{\varepsilon}_A$ is satisfied. Indeed, if A is cuspidal, then $\hat{\varepsilon}_A = (-1)^{\operatorname{rank} G}$ (by (15.4.7), 15.5, (15.11.2)) and $\varepsilon_A = (-1)^{\operatorname{rank} G}$, by assumption; if A is noncuspidal then the equality $\varepsilon_A = \hat{\varepsilon}_A$ follows from the inductive assumptions on G in 23.12 and from the conservation of ε_A, $\hat{\varepsilon}_A$ by induction (15.12). We could also define $\hat{G}^{\chi,-} = \{A \in \hat{G}^{\chi} \mid \varepsilon_A = (-1)^{\operatorname{rank}(G)+1}\}$, $\hat{G}^{\chi,-}_{\mathscr{L}} = \hat{G}^{\chi,-} \cap \hat{G}^{\chi}_{\mathscr{L}}$; the parity condition for $A \in \hat{G}^{\chi,-}$ is not yet known at this stage in the proof. We shall also attach a sign $+$ or $-$ to any nonsingular $E \in \hat{W}'_{\mathscr{L}}$. (Recall that E is said to be nonsingular, if its restriction to $W_{\mathscr{L}}$ is irreducible, see 23.8.) Let c be the two-sided cell of $W'_{\mathscr{L}}$ such that $E \sim_{LR} z$ for some $z \in c$. From (16.2.9) we see that there exists $y \in xW_{\mathscr{L}} \cap c$ (where $x \in \Omega_{\mathscr{L}}$ corresponds to χ under (11.8.1)) such that $\alpha_y \neq 0$. One checks that $l(y) - a(y)$ (mod 2) is independent of the choice of y; it depends only on of c and x. (This is a result of the same type as 17.20.) We say that E (or the corresponding two-sided cell c, or the corresponding family in $\hat{W}'_{\mathscr{L}}$) is of $+$ type (resp. $-$ type) if $l(y) - a(y) + \operatorname{rank}(G)$ is even (resp. odd). If $y \in c$ is as above ($y \in xW$, $\alpha_y \neq 0$) then from 17.18 we see that

(23.14.1) $R^{\mathscr{L}}_{\alpha_y}$, $R^{\mathscr{L}}_{A_y}$ are $\mathbb{Z}$-linear combinations of character sheaves $A \in \hat{G}^{\chi+}_{\mathscr{L}}$ (resp. $A \in \hat{G}^{\chi-}_{\mathscr{L}}$) if c is of $+$ type (resp. of $-$ type).

Since the parity condition $\varepsilon_A = \hat{\varepsilon}_A$ is known for $A \in \hat{G}^{\chi,+}_{\mathscr{L}}$, we see that the proof of the disjointness theorem 16.6 can be carried out and yields the conclusions (a) and (b) of 16.6 provided that we assume that w in 16.6(a) is in a 2-sided cell of $+$ type of $W'_{\mathscr{L}}$ and in $xW_{\mathscr{L}}$, A is in $\hat{G}^{\chi+}_{\mathscr{L}}$, and E, E' are nonsingular of $+$ type. From (23.14.1), we see also that if $E \in \hat{W}'_{\mathscr{L}}$ is nonsingular of $-$ type and $A \in \hat{G}^{\chi,+}_{\mathscr{L}}$, then $(A : R^{\mathscr{L}}_E) = 0$. For each family $\mathscr{F}$ of $+$ type in $\hat{W}'_{\mathscr{L}}$ (corresponding to the two-sided cell c), we define $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$ to be the set of all $A \in \hat{G}^{\chi,+}_{\mathscr{L}}$ such that $(A : R^{\mathscr{L}}_E) \neq 0$ for some $E \in \mathscr{F}$. Then the sets $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$ form a partition of $\hat{G}^{\chi,+}_{\mathscr{L}}$ (cf. (17.13.2)). (Note that, at this stage, we cannot define an analogous partition of $\hat{G}^{\chi,-}_{\mathscr{L}}$.)

(23.14.2) We shall say that a family $\mathscr{F} \subset \hat{W}'_{\mathscr{L}}$ of $+$ type (or in cases (a), (b), (c), (d), an arbitrary family $\mathscr{F} \subset \hat{W}'_{\mathscr{L}}$) satisfies $(17.8.3)_{\chi,\mathscr{F}}$ if there exists a bijection $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}} \leftrightarrow \mathscr{M}(\mathscr{G}_{\mathscr{F}})^{\chi}$ $(A \leftrightarrow \bar{m}_A)$ such that $(A : R^{\mathscr{L}}_{E}) = \hat{\varepsilon}_A\{\bar{m}_A, m_E\}$ for all $A \in \hat{G}_{\mathscr{L}}$ and all $E \in \hat{W}'$ and which is compatible with the actions of $\mathscr{Z}_G$ (see 23.8).

23.15. Let $G, \chi, \mathscr{L}$ be as in 23.14(a), (b), (c), (d), (e_1), (e_2), or (f). Let $\mathscr{F} \subset \hat{W}'_{\mathscr{L}}$ be a family; in cases (e_1), (e_2), (f) we assume that $\mathscr{F}$ is of $+$ type. Assume that there exists $I \subsetneq S$ such that either $\mathscr{F}$ or $\mathscr{F} \otimes \varepsilon$ (ε as in (12.9.7)) is smoothly induced from a family of $W'_{\mathscr{L},I}$ (see 17.13). Using 23.13(a) and the method of proof of (17.13.7), we see that $(17.8.3)_{\chi,\mathscr{F}}$ holds. More generally, assume that there exists $I \subsetneq S$ and $w \in W$ such that $w^*\mathscr{L} = \mathscr{L}'$ and conjugation by w carries $\mathscr{F} \subset \hat{W}'_{\mathscr{L}}$ to a family $\mathscr{F}_1 \subset \hat{W}'_{\mathscr{L}'}$ such that $\mathscr{F}_1$ or $\mathscr{F}_1 \otimes \varepsilon$ is smoothly induced from a family of $W'_{\mathscr{L}',I}$. Then the method of 17.15 reduces us to the previous case and shows again that $(17.8.3)_{\chi,\mathscr{F}}$ holds. For given $\mathscr{L}$, there is at most one $\mathscr{F} \subset \hat{W}'_{\mathscr{L}}$ for which $I \subset S$ and $w \in W$ as above cannot be found. (Such $\mathscr{F}$ is said to be a cuspidal family.)

23.16. Using [6, 8.1], we have the following description of cuspidal families. Let $W_{\mathscr{L}} = \prod_i W^i_{\mathscr{L}}$ be the decomposition of $W_{\mathscr{L}}$ into a product of irreducible Weyl groups. The condition that a cuspidal family exists in $W'_{\mathscr{L}}$ is the following:

(i) if $W^i_{\mathscr{L}}$ is of type B_r or C_r, then $r = d^2 + d$ for some $d \geqslant 1$,
(ii) if $W^i_{\mathscr{L}}$ is of type D_r, then $r = d^2$ for some $d \geqslant 2$,
(iii) if $W^i_{\mathscr{L}}$ is of type A_r, then $r + 1 = \frac{1}{2}(d^2 + d)$ for some $d \geqslant 2$.

If these conditions are satisfied then the cuspidal family consists of the irreducible representations of $W'_{\mathscr{L}}$ whose restrictions to $W_{\mathscr{L}}$ contain an irreducible representation $E = \otimes_i E^i$, $(E^i \in \hat{W}'_{\mathscr{L}})$ where

(i) E^i belongs to the family of $W^i_{\mathscr{L}}$ described in 22.4, if $W^i_{\mathscr{L}}$ is of type B or C,

(ii) E^i belongs to the family of $W^i_{\mathscr{L}}$ described in 22.5, if $W^i_{\mathscr{L}}$ is of type D,

(iii) E^i has symbol $[1, 3, 5, \ldots, 2d-1]$ (see [6, 4.4]) if $W^i_{\mathscr{L}}$ is of type $A_{(1/2)(d^2+d)-1}$.

In the cases (e_1), (e_2), (f) the cuspidal family $\mathscr{F}$ in $\hat{W}'_{\mathscr{L}}$ (if it exists) is automatically of $+$ type. We shall prove this, for example, in the case 23.14(e_1)(i), when $W_{\mathscr{L}}$ is of type $B_r \times B_r \times A_{r'}$ $r = d^2 + d \geqslant 2$, $r' = \frac{1}{2}(d'^2 + d') - 1 \geqslant 2$, and $\Omega_{\mathscr{L}}$ is of order 2 acting nontrivially on $B_r \times B_r$ and on $A_{r'}$. Let $y \in W' - W$ be in the two-sided cell corresponding to $\mathscr{F}$.

We have $\tilde{l}(y)-a(y)\equiv a_{E'}+A_{E'} \pmod 2$, where E' is the irreducible representation of the $A_{r'}$-component of $W_{\mathscr{L}}$ with symbol $[1, 3, 5,..., 2d-1]$ (cf. [6, (7.6.3)]). Since $E'\simeq E'\otimes\text{sign}$, we have $A_{E'}=-a_{E'}+r'(r'+1)/2$ (see [6, (5.11.5)]). Hence $\tilde{l}(y)-a(y)\equiv (r'(r'+1)/2) \pmod 2$. Let x be the generator of $\Omega_{\mathscr{L}}$. Then $l(y)=\tilde{l}(y)+l(x)$, rank $G=2r+r'+1$, and it remains to show that $l(x)\equiv r'+1+(r'(r'+1)/2) \pmod 2$; this is left to the reader.

23.17. Let G, χ, $\mathscr{L}$ be as in 23.14(a), (b), (c), or (d) and let $\mathscr{F}$ be a cuspidal family in $\hat{W}'_{\mathscr{L}}$. We want to prove that $(17.8.3)_{\chi,\mathscr{F}}$ holds. First, we consider the case 23.14(b). Thus $G=Sp_{2n}(k)$ $(n\geqslant 1)$, $\chi\colon \mathscr{Z}_G\to\bar{Q}_l^*$ is nontrivial; we assume that $\mathscr{L}$ is as in 23.4(i), that is, $W_{\mathscr{L}}$ is of type $D_r\times C_{r'}$ $r=d^2\geqslant 4$, $r'=d'^2+d'\geqslant 2$, $r+r'=n$, and $\Omega_{\mathscr{L}}$ is of order 2 acting nontrivially on the D_r-factor. Let V_1 (resp. V_2) be the F_2-vector space of all subsets of even cardinality of $\{0, 1, 2,..., 2d-1\}$ (resp. of $\{0, 1, 2,..., 2d'\}$), and let $V=V_1\oplus V_2$. Any representation E in $\mathscr{F}$ is of form $E_1\otimes E_2$ where E_1 is an irreducible representation of the D_r-factor extended by $\Omega_{\mathscr{L}}$ and E_2 is an irreducible representation of the $B_{r'}$-factor.

As in 22.5, we make E_1 correspond to a point $v_1\in V_1$ and as in 22.4, we make E_2 correspond to a point $v_2\in V_2$. We attach to E the point $(v_1, v_2)\in V$. This defines an injective map $\mathscr{F}\to V$ whose image $\tilde{V}$ is just $\tilde{V}_1\times\tilde{V}_2$ where $\tilde{V}_1\subset V_1$, $\tilde{V}_2\subset V_2$ are defined as in 22.5, 22.4. We write E_y for the representation in $\mathscr{F}$ corresponding to $y\in\tilde{V}$. The symplectic forms on V_1, V_2 in 22.5, 22.4 give rise by direct sum to a symplectic form on V with 1-dimensional radical Rad V. Let $\xi_0\colon \text{Rad } V\to F_2$ be the unique linear form $\neq 0$.

In our case, the statement $(17.8.3)_{\chi,\mathscr{F}}$, which we are trying to prove, can be reformulated as follows. Let $X=\{\eta\in\text{Hom}(V, F_2) : \eta \mid \text{Rad } V=\xi_0\}$.

(23.17.1) *There exists a bijection* $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}\leftrightarrow X\times\{0, 1\}$ $(A_{\eta,0}\leftrightarrow(\eta, 0)$, $A_{\eta,1}\leftrightarrow(\eta, 1))$ *such that* $(A_{\eta,i}\colon R^{\mathscr{L}}_{E_y})=-2^{-(d+d')}(-1)^{\eta(y)}$ *for all* $y\in\tilde{V}$, *all* $\eta\in X$ *and for* $i\in\{0, 1\}$, *and such that*

$$t_z^*A_{\eta,0}=A_{\eta,1},\qquad t_z^*A_{\eta,1}=A_{\eta,0},$$

where z is the nontrivial element in $\mathscr{Z}_G$.

Let $e_1, e_2,..., e_{2d+2d'-1}$ be the basis of V such that $e_1, e_2,..., e_{2d-1}$ is the canonical basis of V_1 defined in 2.5 and $e_{2d}, e_{2d+1},..., e_{2d+2d'-1}$ is the canonical basis of V_2 defined in 2.4. Let $\mathscr{T}(V)$ be the collection of maximal isotropic subspaces of V associated to the basis $e_1, e_2,..., e_{2d+2d'-1}$ as in [6, 9.1]. Using [6, (9.5.2)] we see that for any $C\in\mathscr{T}(V)$ and any linear form $\xi\colon C\to F_2$ such that $\xi \mid \text{Rad } V=\xi_0$, there exists $w\in W'_{\mathscr{L}}-W_{\mathscr{L}}$ such that $\sum_{y\in C}(-1)^{\xi(y)}E_y=-\alpha_w$, see (16.2.7). (The minus sign in front of α_w

comes from the fact that $l(w)=\bar{l}(w)+1 \pmod 2$.) Applying 16.6(a), we see that

(23.17.2) $\rho_{c,\xi} =^{\text{def}} \sum_{y\in C}(-1)^{\xi(y)+1} R^{\mathscr{L}}_{E_y}$ is a linear combination of character sheaves in $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$ with integral, $\geqslant 0$ coefficients.

If (C', ξ') is another pair like (C, ξ) we have, using 14.13:

(23.17.3) $(\rho_{C,\xi} : \rho_{C',\xi'})=$ twice the number of $\eta \in X$ such that $\eta \mid C=\xi$, $\eta \mid C'=\xi'$.

We note that for any $y\in \tilde{V}$ and any $C\in \mathscr{T}(V)$ containing y, we have (by (23.17.2))

$$2^{d+d'-1}(R^{\mathscr{L}}_{E_y}-R^{\mathscr{L}}_{E_{y+v}})=\sum_{\xi}(-1)^{\xi(y)+1}\rho_{C,\xi} \tag{23.17.4}$$

sum over all $\xi\in \operatorname{Hom}(C, F_2)$ extending ξ_0. (Here v is the generator of Rad V.) In particular, the right-hand side of (23.17.4) is independent of C; it depends only on y. For $y=0$, we obtain that

$$h_C=\sum_{\xi}\rho_{C,\xi}=-2^{d+d'-1}(R^{\mathscr{L}}_{E_0}-R^{\mathscr{L}}_{E_v}) \qquad (C\in\mathscr{T}(V)) \tag{23.17.5}$$

(sum over all $\xi\in \operatorname{Hom}(C, F_2)$ extending ξ_0) is independent of C. We denote it by h. We now borrow a part of the argument in [6, 9.2]. Let C_1 be the subspace of V spanned by Rad V and by $e_1, e_3, e_5,\ldots$; let C_2 be the subspace of V spanned by Rad V and by $e_2, e_4,\ldots$. Then $C_1, C_2\in\mathscr{T}(V)$. Let $\eta\in X$ and let $\xi_1 : C_1\to F_2$, $\xi_2 : C_2\to F_2$ be the restrictions of η. From (23.17.2) we see that $(\rho_{C_1,\xi_1} : \rho_{C_2,\xi_2})=2$. Using now (23.17.2), we see that either

(α) there are exactly two objects in $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$ which appear in both $\rho_{C_1,\xi_1}, \rho_{C_2,\xi_2}$ (they must appear with coefficient 1 in both $\rho_{C_1,\xi_1}, \rho_{C_2,\xi_2}$) or

(β) there is exactly one object (say A) in $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$ which appears in both $\rho_{C_1,\xi_1}, \rho_{C_2,\xi_2}$ (it must appear with coefficient 1 in one of them and with coefficient 2 in the other).

Assume that (β) holds; we shall reach a contradiction as follows. If $\xi_2' : C_2'\to F_2$ extends ξ_0 and is different from ξ_2 then $(\rho_{C_2,\xi_2} : \rho_{C_2,\xi_2'})=0$ by (23.17.3) hence A doesn't appear in $\rho_{C_2,\xi_2'}$, (see (23.17.2)), hence A appears in h_{C_2} (see (23.17.5)) with the same coefficient as in ρ_{C_2,ξ_2}. Similarly A appears in h_{C_1} with the same coefficient as in ρ_{C_1,ξ_1}. Since we have assumed that (β) holds it follows that $h_{C_1}\neq h_{C_2}$, a contradiction.

Thus (α) holds; we denote by $A_{\eta,0}, A_{\eta,1}$ the two objects in $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$ defined by (α). The previous argument shows that $h=\sum_{\eta\in X}(A_{\eta,0}+A_{\eta,1})$ is multiplicity free. Moreover, any $A \in \hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$ must appear in it. Indeed, we have $(A : R^{\mathscr{L}}_{E_y}) \neq 0$ for some $y \in \tilde{V}$. Moreover, by (23.8.4), we have

$$(A : R^{\mathscr{L}}_{E_{y+v}}) = -(A : R^{\mathscr{L}}_{E_y}), \tag{23.17.6}$$

hence $(A : R^{\mathscr{L}}_{E_y} - R^{\mathscr{L}}_{E_{y+v}}) \neq 0$. Using (23.17.4), we see that $(A : \rho_{C,\xi}) \neq 0$ for some $C \in \mathscr{T}(V)$ and some ξ. By (23.17.2), we then have $(A : \rho_{C,\xi}) > 0$ and using (23.17.5) we have $(A : h) > 0$, as claimed. Thus, we have $h = \sum A$ (sum over all $A \in \hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$).

Now, let $D = \{\eta', \eta''\}$ be a two element subset of X such that $\eta' + \eta''$ is equal to the inner product with the standard basis element e_i of V and $\eta'(e_i) = \eta''(e_i) = 0$. As in the proof of 22.6, the collection of such subsets (for fixed i) parametrizes (in a 1–2 fashion) certain character sheaves of the Levi subgroup L_i of a parabolic subgroup of G. Applying induction to those character sheaves, and following this by truncation (= projection onto the subspace of $\mathscr{K}_0(G)$ spanned by $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$) we obtain, as in the proof of 22.6, some elements ρ_D, $\tilde{\rho}_D \in \mathscr{K}_0(G)$ which are linear combinations with integral, $\geqslant 0$ coefficients of character sheaves in $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$ and have the properties (23.17.7)–(23.17.9) below.

$$(\rho_D : \rho_{C,\xi}) = (\tilde{\rho}_D : \rho_{C,\xi}) = \text{number of } \eta \in D \text{ such that } \eta \mid C = \xi$$
$$(\text{for } C \in \mathscr{T}(V),\ \xi \in \operatorname{Hom}(C, F_2) \text{ extending } \xi_0) \tag{23.17.7}$$

$$t_z^* \rho_D = \tilde{\rho}_D,\ t_z^* \tilde{\rho}_D = \rho_D, \text{ where } z \text{ is the generator of } \mathscr{Z}_G, \tag{23.17.8}$$

$$\sum_D (\rho_D + \tilde{\rho}_D) = 2^{d+d'-2}(-R^{\mathscr{L}}_{E_0} + R^{\mathscr{L}}_{E_v} - R^{\mathscr{L}}_{E_{e_i}} + R^{\mathscr{L}}_{E_{e_i+v}}) \tag{23.17.9}$$

(here D runs over all subsets of X as above, corresponding to a fixed e_i).

We shall now explain (23.17.8) and (23.17.9), which did not appear in 22.6. The two character sheaves of L_i parametrized by D are related to each other by $t_{z_1}^*$ for z_1 an element in $\mathscr{Z}_{L_i} - \mathscr{Z}^0_{L_i}$. Since induction and truncation are compatible with the operations t^*, we obtain (23.17.8). The left-hand side of (23.17.9) is obtained by induction and truncation from an element which plays for L_i the same role as h (above) plays for G. That element can be expressed in terms of $R^{?}_{?}$ as in (23.17.5), and hence the corresponding induction and truncation can be computed from 15.7(a); we then obtain the right-hand side of (23.17.9). Using (23.17.4) for $y = 0$ and e_i we see that the right-hand side of (23.17.9) equals $\sum_\xi \rho_{C,\xi}$ where C is a fixed subspace in $\mathscr{T}(V)$ containing e_i and ξ runs over all elements in $\operatorname{Hom}(C, F_2)$ which extend ξ_0 and vanish on e_i. The last sum is a partial sum of (23.17.5) hence it is multiplicity free since h is so. Thus, the left-hand side of (23.17.9) is multiplicity free. It follows that for each D in the sum, $\rho_D + \tilde{\rho}_D$ is multiplicity free. In particular, ρ_D and $\tilde{\rho}_D$ are disjoint in the sense that no character sheaf can appear in both of them. Combining this with (23.17.8), we see that if A is a character sheaf appearing in ρ_D or $\tilde{\rho}_D$, then $t_z^* A \neq A$.

Using (23.17.7) we see just as in [6, p. 272] that if $D=\{\eta', \eta''\}$ is as above, then both ρ_D and $\tilde{\rho}_D$ are of the form $A_{\eta',i}+A_{\eta'',j}$ for suitable $i \neq j$ in $\{0, 1\}$. Since ρ_D, $\tilde{\rho}_D$ are disjoint, we may arrange notations so that

$$\rho_D = A_{\eta',0}+A_{\eta'',0}, \qquad \tilde{\rho}_D = A_{\eta',1}+A_{\eta'',1}. \tag{23.17.10}$$

Now $A_{\eta',0}$, $A_{\eta',1}$ are characterized by the fact that they are the only character sheaves which appear in both ρ_{C_1,ξ_1}, ρ_{C_2,ξ_2}, where $\xi_1=\eta' \mid C_1$, $\xi_2=\eta' \mid C_2$; moreover, by (17.17.3), both ρ_{C_1,ξ_1}, ρ_{C_2,ξ_2} are t_z^*-invariant. It follows that the 2-element set $\{A_{\eta',0}, A_{\eta',1}\}$ is t_z^*-invariant. As we have seen, $A_{\eta',0}$ and $A_{\eta',1}$ are not fixed by t_z^*. It follows that

$$t_z^* A_{\eta',0}=A_{\eta',1}, \qquad t_z^* A_{\eta',1}=A_{\eta',0}. \tag{23.17.11}$$

These identities hold for any η' which is contained in a D as above, hence for all $\eta' \in X$, except possibly for one. Using again (17.17.3) we deduce that the identity

$$(A_{\eta,0} : R_{E_y}^{\mathscr{L}}) = (A_{\eta,1} : R_{E_y}^{\mathscr{L}}) \tag{23.17.12}$$

holds for all $y \in \tilde{V}$ and for all $\eta \in X$, except possibly for a single η. Assume now that there exists η_0, y_0 which violate (23.17.12). Then η_0 is uniquely determined, and from (23.17.4), (23.17.6) we see that there exists C, ξ such that $(A_{\eta_0,0} : \rho_{C,\xi}) \neq (A_{\eta_0,1} : \rho_{C,\xi})$. But $\rho_{C,\xi}$ is multiplicity free, since h is, (see (23.17.5)). Hence one of the numbers $(A_{\eta_0,0} : \rho_{C,\xi})$, $(A_{\eta_0,1} : \rho_{C,\xi})$ must be 0, and the other must be 1. For any $\eta \in X$, $\eta \neq \eta_0$ we have $(A_{\eta,0} : \rho_{C,\xi}) = (A_{\eta,1} : \rho_{C,\xi})$. It follows that $(\sum_{\eta \in X}(A_{\eta,0}+A_{\eta,1}) : \rho_{C,\xi})$ is odd. On the other hand, this equals $(h : \rho_{C,\xi}) = \text{even}$, by (23.17.3). This contradiction shows that (23.17.12) holds for all $\eta \in X$ and for all $y \in \tilde{V}$.

We can therefore define a function $[\eta, y]$ on $X \times \tilde{V}$ with values in F_2 by

$$(A_{\eta,0} : R_{E_y}^{\mathscr{L}}) = (A_{\eta,1} : R_{E_y}^{\mathscr{L}}) = -(-1)^{[\eta,y]} 2^{-(d+d')}.$$

(The fact that $(A : R_{E_y}^{\mathscr{L}}) = \pm 2^{-(d+d')}$, for $A \in G_{\mathscr{L},\mathscr{F}}^{\chi}$, follows from (23.17.4) and (23.17.6).) Here are three properties of the function $[\eta, y]$.

(23.17.13) For any $C \in \mathscr{T}(V)$ and any $\eta \in X$, the function $C \to F_2$, $(y \to [\eta, y])$ is F_2-linear.

(This follows from (23.17.4) and (23.17.6).)

(23.17.14) For any $D=\{\eta', \eta''\} \subset X$ as above (with respect to e_i) and any $y \in \tilde{V}$, we have

$$(-1)^{[\eta',y]}+(-1)^{[\eta'',y]} = (-1)^{\eta'(y)}+(-1)^{\eta''(y)}.$$

(This follows from (23.17.10), (23.17.7), (23.17.4), and (23.17.6).)

$$\sum_{\eta \in X} (-1)^{[\eta, y]} = \sum_{\eta \in X} (-1)^{\eta(y)}. \tag{23.17.15}$$

(This is proved as follows:

$$\begin{aligned}
-\sum_{\eta \in X} (-1)^{[\eta, y]} \cdot 2^{-(d+d'-1)} &= \left(\sum_{\eta \in X} (A_{\eta,0} + A_{\eta,1}) : R^{\mathscr{L}}_{E_y} \right) = (h : R^{\mathscr{L}}_{E_y}) \\
&= -2^{d+d'-1} (R^{\mathscr{L}}_{E_0} - R^{\mathscr{L}}_{E_v} : R^{\mathscr{L}}_{E_y}) \\
&= \begin{cases} -2^{d+d'-1} & \text{if } y = 0 \\ 2^{d+d'-1} & \text{if } y = v \\ 0 & \text{if } y \neq 0, v \end{cases} \\
&= 2^{-(d+d'-1)} \sum_{\eta \in X} (-1)^{\eta(y)}.)
\end{aligned}$$

We now apply [6, 3.3] and deduce that $[\eta, y] = \eta(y)$ for all $\eta \in X$ and all $y \in \tilde{V}$. Thus (23.17.1) is verified, except for the statement on the action of t_z^*, which is known (see (23.17.11)) for all η except possibly for a single $\eta = \eta_0$, defined by $\eta_0(e_i) = 1$ for all i. (This η_0 is in X if and only if d is odd.) To check this remaining case (for d odd), we first note that $A_{\eta_0,0}$ and $A_{\eta_0,1}$ are cuspidal. (Using 18.2, as in the proof of 19.3(a), this statement is reduced to the statements (22.4.1), (22.5.2); the multiplicities of $A_{\eta_0,0}$, $A_{\eta_0,1}$ in $R^{\mathscr{L}}_{E_y}$ are already known.) Using 23.2(b) it now follows that t_z^* does not leave $A_{\eta_0,0}$ or $A_{\eta_0,1}$ invariant. Hence it must interchange them. Thus (23.17.1) is verfied. Hence $(17.8.3)_{\chi,\mathscr{F}}$ is also verified in our case; we have at the same time obtained a lower bound for the number of cuspidal character sheaves in $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$: this number is $\geqslant 2$ if d is odd. In the case where $\mathscr{L}$ is as in 23.4(ii), i.e., $W_{\mathscr{L}}$ is of type C_{n-1} $(n = d^2 + d + 1)$, and $\Omega_{\mathscr{L}}$ is of order 2 we see in an entirely similar way that $(17.8.3)_{\chi,\mathscr{F}}$ holds and that there are at least 2 cuspidal character sheaves in $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$.

The same method applies in the cases 2.14(a), (b), (c), (d). We thus obtain that $(17.8.3)_{\chi,\mathscr{F}}$ holds in each of these cases. (In the cases (a), (c) the proof is simpler than in the other cases; it is along the same lines as the proof in 22.6, 22.7.) We also obtain in each of these cases a lower bound for the number of cuspidal character sheaves in $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$.

23.18. Now let G, χ, $\mathscr{L}$ be as in 23.14 (e_1), (e_2), or (f) and let $\mathscr{F}$ be a cuspidal family in $\hat{W}'_{\mathscr{L}}$. (As we have seen in 23.16, $\mathscr{F}$ is automatically of + type.) We want to prove that $(17.8.3)_{\chi,\mathscr{F}}$ holds. First, we consider the case 23.14(e_2). Thus $G = \mathrm{Spin}_m$, $m \equiv 2 \pmod 4$, $m \geqslant 10$, and $\chi: \mathscr{Z}_G \to \bar{Q}_l^*$ is faithful. We assume that $\mathscr{L}$ is as in 23.14(e_2)(i), in particular, $W_{\mathscr{L}}$ is of type $D_r \times D_r \times A_{r'}$, $(2r + r' + 1 = \frac{1}{2}m)$, $r = d^2 \geqslant 4$, $r' = \frac{1}{2}(d'^2 + d') - 1 \geqslant 2$ and $\Omega_{\mathscr{L}}$ is cyclic of order 4, with a fixed generator x corresponding under (11.8.1) to χ.

Let V be the F_2-vector space of all subsets of even cardinality of $\{0, 1, 2, \dots, 2d-1\}$. V has a natural symplectic form, a natural basis $e_1, \dots, e_{2d-1}$ and a natural subset $\tilde{V}$, as in 22.5. Let $\mathscr{T}(V)$ be the corresponding family of maximal isotropic subspaces of V, defined as in [6, 9.1]. We now attach to each nonsingular (see 23.8) representation $E \in \mathscr{F}$ a point in $\tilde{V} \times F_2$ as follows. We can write $W_{\mathscr{L}} = W^1 \times W^1 \times W^2$, where W^1 (resp. W^2) is a Weyl group of type D_r (resp. $A_{r'}$). The restriction of E to $W_{\mathscr{L}}$ can be written as $E^1 \boxtimes E^1 \boxtimes E^2$ where E^1 (resp. E^2) is an irreducible representation of W^1 (resp. W^2).

Here E^1 is in the family $\mathscr{F}_0$ (with notations of 22.5) and E^2 is independent of E (it has symbol $[1, 3, 5, \dots, 2d'-1]$, see [6, 4.4]). The action of the generator $x \in \Omega_{\mathscr{L}}$ on E is given by

$$e_1 \otimes e_1' \otimes e_2 \to \alpha(e_1') \otimes e_1 \otimes \beta_0(e_2);$$

here $\beta_0 \colon E^2 \to E^2$ is the involution defined as in 17.2(b) and $\alpha \colon E^1 \to E^1$ satisfies $\alpha^2 = \pm 1$. If $\alpha^2 = 1$, then (E^1, α) is a representation of W_1, the semidirect product of W^1 with the cyclic group of order 2 acting on W_1 by a nontrivial graph automorphism; it then corresponds as in 22.5 to a point $y \in \tilde{V}$ and we associate $(y, 0) \in \tilde{V} \times F_2$ to E. If $\alpha^2 = -1$, then $(E^1, \sqrt{-1}\alpha)$ is a representation of $\tilde{W}_1$ which again corresponds as in 22.5 to a point $y \in \tilde{V}$ and we associate $(y, 1) \in \tilde{V} \times F_2$ to E. (Here, $\sqrt{-1}$ is a fixed square root of -1 in $\bar{\mathbb{Q}}_l$.) This gives a bijection $E_{y,i} \leftrightarrow (y, i)$ between the set of nonsingular representations in $\mathscr{F}$ and the set $\tilde{V} \times F_2$. Let ξ_0: Rad $V \to F_2$ be the unique linear form $\neq 0$. In our case, the statement $(17.8.3)_{\chi, \mathscr{F}}$ which we want to prove can be reformulated as follows. Let $X = \{\eta \in \operatorname{Hom}(V, F_2) \colon \eta \mid \operatorname{Rad} V = \xi_0\}$.

(23.18.1) *There exists a bijection $\hat{G}^{\chi}_{\mathscr{L}, \mathscr{F}} \leftrightarrow X \times \{0, 1, 2, 3\}$ $(A_{\eta, j} \leftrightarrow (\eta, j))$, such that*

$$(A_{\eta, j} : R^{\mathscr{L}}_{E_{y,i}}) = -2^{-(d+1)}(-1)^{\eta(y)}(\sqrt{-1})^i$$

for all $y \in \tilde{V}$, all $\eta \in X$, all $i \in \{0, 1\}$ and all $j \in \{0, 1, 2, 3\}$ and such that for any $\eta \in X$, t_z^* $(z \in \mathscr{Z}_G)$ act simply transitively on the set $\{A_{\eta j} \mid j \in \{0, 1, 2, 3\}\}$.

The proof is similar to the proof of (23.17.1) (but slightly more complicated). Let $C \in \mathscr{T}(V)$ and let $\xi \colon C \to F_2$ be a linear form extending ξ_0. Then there exists $w \in xW_{\mathscr{L}}$ such that $\sum_{y \in C}(-1)^{\xi(y)}(E_{y,0} - \sqrt{-1}\, E_{y,1}) = -\alpha_w$.

Applying 16.6(a) or rather its version discussed in 23.14, we see that

(23.18.2) $\rho_{C,\xi} =^{\text{def}} \sum_{y\in C}(-1)^{\xi(y)+1}(R^{\mathscr{L}}_{E_{y,0}} - \sqrt{-1}\, R^{\mathscr{L}}_{E_{y,1}})$ is a linear combination of character sheaves in $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$ with integral, $\geqslant 0$ coefficients.

If (C', ξ') is another pair like (C, ξ), we have, using 14.13:

(23.18.3) $(\rho_{C,\xi} : \rho_{C',\xi'})$ = four times the number of $\eta \in X$ such that $\eta \mid C = \xi$, $\eta \mid C' = \xi'$.

Next we note that for any $y \in \tilde{V}$ and any $C \in \mathscr{T}(V)$ containing y, we have

(23.18.4) $2^{d-1}(R^{\mathscr{L}}_{E_{y,0}} - R^{\mathscr{L}}_{E_{y+v,0}} - \sqrt{-1}\, R^{\mathscr{L}}_{E_{y,1}} + \sqrt{-1}\, R^{\mathscr{L}}_{E_{y+v,1}}) = \sum_{\xi}(-1)^{\xi(y)+1}\rho_{C,\xi}$, sum over all $\xi \in \text{Hom}(C, F_2)$ extending ξ_0. Here v is the generator of Rad V. It follows that

$$h_C = \sum_{\xi} \rho_{C,\xi} \tag{23.18.5}$$

(sum over all $\xi \in \text{Hom}(C, F_2)$ extending ξ_0) is independent of C. We denote it h. Define $C_1, C_2 \in \mathscr{T}(V)$ as in 23.17. Let $\eta \in X$ and let $\xi_1 : C_1 \to F_2$, $\xi_2 : C_2 \to F_2$ be the restrictions of η. We have $(\rho_{C_1,\xi_1} : \rho_{C_2,\xi_2}) = 4$, by (23.18.2). The argument in 23.17 shows that there are two possibilities:

(α) there are exactly four objects say $A_{\eta,j}$, $j \in \{0, 1, 2, 3\}$ in $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$ which appear in both ρ_{C_1,ξ_1}, ρ_{C_2,ξ_2} (they must appear with coefficient 1 in both) or

(β) there is exactly one object say A_η in $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$ which appears in both ρ_{C_1,ξ_1}, ρ_{C_2,ξ_2} (it must appear with coefficient 2 in both).

(A priori there are also other possibilities but they can be excluded as in 23.17.) Thus, some $\eta \in X$ satisfy (α) and other η satisfy (β); this gives a partition $X = X_\alpha \cup X_\beta$. We have

$$h = \sum_{\eta \in X_\alpha} (A_{\eta,0} + A_{\eta,1} + A_{\eta,2} + A_{\eta,3}) + \sum_{\eta \in X_\beta} 2A_\eta \tag{23.18.6}$$

and all $A \in \hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$ appear in this sum.

Now let $D = \{\eta', \eta''\}$ be a two element subset of X such that $\eta' + \eta''$ is equal to the inner product with e_i and $\eta'(e_i) = \eta''(e_i) = 0$. As in 23.17, to D we can associate, using induction and truncation, four objects ρ^1_D, ρ^2_D, ρ^3_D, ρ^4_D which are linear combinations with $\geqslant 0$ integral coefficients of character sheaves in $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$, which have properties analogous to (23.17.7)–(23.17.9). Thus,

(23.18.7) $(\rho^j_D : \rho_{C,\xi})$ = number of $\eta \in D$ such that $\eta \mid C = \xi$ for any $C \in \mathscr{T}(V)$ and $\xi \in \text{Hom}(C, F_2)$ extending ξ_0 and any $j = 1, 2, 3, 4$.

(23.18.8) t_z^* $(z \in \mathscr{Z}_G)$ permute transitively the four objects ρ_D^j, $j=1, 2, 3, 4$.

$$\sum_{\substack{D \\ 1 \leq j \leq 4}} \rho_D^j = \sum_{\xi} \rho_{C,\xi}, \tag{23.18.9}$$

where D runs over all subsets of X as above, corresponding to a fixed e_i, $C \in \mathscr{T}(V)$ contains e_i, and ξ runs over all elements in $\mathrm{Hom}(C, F_2)$ extending ξ_0 and vanishing on e_i.

Since the right-hand side of (23.18.9) is a partial sum of (23.18.5) or (23.18.6), it follows any $A \in \hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$ must appear in it with coefficient 0, 1 or 2. Hence any such A appears in $\rho_D^1 + \rho_D^2 + \rho_D^3 + \rho_D^4$ with coefficient 0, 1 or 2. Let $\eta \in X_\beta$; from the definition of A_η and the invariance of ρ_{C_1,ξ_1}, ρ_{C_2,ξ_2} under t_z^* it follows that $t_z^* A = A$ for all $z \in \mathscr{Z}_G$. Thus, if $A = A_\eta$ $(\eta \in X_\beta)$ appears in ρ_D^i for some i then from (23.18.8) it follows that A appears in each of $\rho_D^1, \rho_D^2, \rho_D^3, \rho_D^4$, hence it appears with coefficient ≥ 4 in $\rho_D^1 + \rho_D^2 + \rho_D^3 + \rho_D^4$, a contradiction. Thus each ρ_D^i involves only character sheaves of form $A_{\eta,j}$ $(\eta \in X_\alpha)$. As in 23.17, we then see that we can arrange notation so that

$$\rho_D^j = A_{\eta',j} + A_{\eta'',j} \qquad (j=1, 2, 3, 4) \tag{23.18.10}$$

and that $\eta', \eta'' \in X_\alpha$. Moreover, we see that

(23.18.11) t_z^* $(z \in \mathscr{Z}_G)$ permute transitively the four objects $\{A_{\eta',j} \mid j = 1, 2, 3, 4\}$.

This holds for any $\eta \in X$ except possibly for a single η defined (when d is odd) by $\eta_0(e_l) = e_l$ for all l. (All η $(\eta \neq \eta_0)$ are contained in some D as above.) In particular, we see that X_β is empty (if d is even) and it has at most one element: η_0 (if d is odd). From (23.18.11) we see that for any $y \in \tilde{V}$, any i, and any $\eta \in X$, $\eta \neq \eta_0$,

$$(A_{\eta,j} : R_{E_{y,i}}^{\mathscr{L}}) \text{ is independent of } j \ (j=0, 1, 2, 3). \tag{23.18.12}$$

Moreover, this inner product is of the form $\pm 2^{-(d+1)}(\sqrt{-1})^i$, as follows from (23.18.4) and (23.8.3). We can write in the form $-2^{-(d+1)}(-1)^{[\eta,y]}(\sqrt{-1})^i$. This defines a pairing $[\eta, y]$ with values in F_2. It is defined for all $\eta \in X$, $\eta \neq \eta_0$ and for all $y \in \tilde{V}$. We also define $[\eta_0, y]$ (when d is odd) and $y \in \tilde{V}$ by

$$-2^{-(d+1)}(-1)^{[\eta_0,y]}(\sqrt{-1})^i = \begin{cases} (A_{\eta_0,0} + A_{\eta_0,1} + A_{\eta_0,2} + A_{\eta_0,3} : R_{E_{y_i}}^{\mathscr{L}}) & \text{if } \ \eta_0 \in X_\alpha \\ 2(A_{\eta_0} : R_{E_{y,i}}^{\mathscr{L}}) & \text{if } \ \eta_0 \in X_\beta. \end{cases} \tag{23.18.13}$$

Then $[\eta, y]$ satisfies properties like (23.17.13)–(23.17.15). Using again [6, 3.3], we see that $[\eta, y] = \eta(y)$ for all $\eta \in X$ and all $y \in \tilde{V}$.

We now show that when d is odd, η_0 must be of type α. Assume that $\eta_0 \in X_\beta$. By (23.18.13) we have $(A_{\eta_0} : R^{\mathscr{L}}_{E_{y,i}}) = -2^{-(d+2)}(-1)^{[\eta_0, y]}(\sqrt{-1})^i$ for all $y \in \tilde{V}$ and for $i = 0, 1$. From this, we can show that A_{η_0} is cuspidal; using 18.2, this is reduced to the statement (22.5.2) and the statement below.

(23.18.14) Let E^2 be the irreducible representation of the Weyl group W^2 of type $A_{(1/2)(d'^2+d')-1}$ corresponding to the symbol $[1, 3, 5, ..., 2d'-1]$ (see [6, 4.4]). Then the character of E^2 vanishes on all elements of W^2 which have some eigenvalue -1 on the reflection representation of W^2. (See [23, proof of 9.4].)

Thus A_{η_0} is cuspidal. By the results in 23.2(e), the orbit of any complex in $(\mathrm{Irr}^0 G)^\chi$ under t_z^* $(z \in \mathscr{Z}_G)$ consists of four different objects. On the other hand, as we have seen earlier in the proof, A_{η_0} is invariant under all t_z^*. This is a contradiction. It follows that $\eta_0 \in X_\alpha$ hence $X = X_\alpha$. Hence $A_{\eta_0, j}$ $(i = 0, 1, 2, 3)$, are defined. We also see that $h = \sum_{\eta \in X, 0 \leqslant j \leqslant 3} A_{\eta, j}$.

We now show that (23.18.12), which is known to hold for $\eta \neq \eta_0$, holds also for $\eta = \eta_0$. Assume that this is not so. Using (23.18.4) and (23.8.3) we see that there exists C, ξ such that $(A_{\eta_0, j} : \rho_{C,\xi}) \neq (A_{\eta_0, j'} : \rho_{C,\xi})$ for some $j, j' \in \{0, 1, 2, 3\}$. These multiplicities must be 0 or 1. (Indeed, h is multiplicity free, hence $\rho_{C,\xi}$ is also, since it is a partial sum of h, see (23.18.5).) Then, among the multiplicities $(A_{\eta_0, j} : \rho_{C,\xi})$ $(j = 0, 1, 2, 3)$, at least one is 1 and all are 0 or 1. Hence $\sum_{0 \leqslant j \leqslant 3} (A_{\eta_0, j} : \rho_{C,\xi}) \not\equiv 0 \pmod 4$. Since (23.18.12) holds for $\eta \neq \eta_0$, we have $\sum_{0 \leqslant j \leqslant 3} (A_{\eta, j} : \rho_{C,\xi}) \equiv 0 \pmod 4$ for $\eta \neq \eta_0$. It follows that $\sum_{\eta \in X, 0 \leqslant j \leqslant 3} (A_{\eta, j} : \rho_{C,\xi}) \not\equiv 0 \pmod 4$. On the other hand the last sum equals $(h : \rho_{C,\xi})$ which is divisible by 4, by (23.18.3). This contradiction shows that (23.18.12) holds for all $\eta \in X$. We can therefore deduce from (23.18.13) that

$$(A_{\eta_0, j} : R^{\mathscr{L}}_{E_{y,i}}) = -2^{-(d+1)}(-1)^{\eta_0(y)}(\sqrt{-1})^i$$

for all $y \in \tilde{V}$, $i \in \{0, 1\}$, $j \in \{0, 1, 2, 3\}$. (Note that, as we have seen earlier, we have $[\eta_0, y] = \eta_0(y)$.)

Using this identity, together with (22.5.2) and (23.18.14), we see as above that $A_{\eta_0, j}$ is cuspidal for all $j \in \{0, 1, 2, 3\}$. By the results in 23.2(e), the orbit of any complex in $\mathrm{Irr}^0(G)^\chi$ under t_z^* $(z \in \mathscr{Z}_G)$ must consist of four different complexes; on the other hand, the set $\{A_{\eta_0, j} \mid j = 0, 1, 2, 3\}$ is invariant under all t_z^*. It follows that t_z^* $(z \in \mathscr{Z}_G)$ permute transitively the objects in this set. This completes the proof of (23.18.1). Hence $(17.8.3)_{\chi, \mathscr{F}}$ is also verified in our case; we have at the same time obtained a lower bound for the number of cuspidal character sheaves in $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$: this number is $\geqslant 4$, if d is odd.

In a similar (but somewhat simpler) way we see that $(17.8.3)_{\chi,\mathscr{F}}$ holds also when $\mathscr{L}$ is as in 23.14(e_2)(ii) and also in the cases 23.14(e_1), (f). We also obtain in each of these cases a lower bound for the number of cuspidal character sheaves in $\hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$.

23.19. Let G, χ be as in 23.12(a)–(f). In this subsection we shall show that the set of cuspidal character sheaves in $\hat{G}^{\chi}$ (which is contained in $\mathrm{Irr}^0(G)^{\chi}$, by 3.12)) coincides with $\mathrm{Irr}^0(G)^{\chi}$. We denote by α the number of cuspidal character sheaves in $\hat{G}^{\chi}$, by α' the number of elements in $\mathrm{Irr}^0(G)^{\chi}$; in the cases (e_1), (e_2), (f), we denote by α^+ the number of cuspidal character sheaves in $\hat{G}^{\chi,+}$. We have clearly $\alpha \leqslant \alpha'$, and it is enough to show that $\alpha = \alpha'$. We shall use the fact that 23.17, 23.18 provide a lower bound for α. We consider each case separately.

(a) $G = PSp_{2n}(k)$ $(n \geqslant 1)$, $\chi = 1$. By 23.2(a), we may assume that n is even.

Let $\mathscr{X}'$ be the set of all unordered pairs (N_1, N_2) of triangular numbers such that $n = N_1 + N_2$ in which each pair with $N_1 = N_2$ is repeated twice.

Let $\mathscr{X}$ be the set of all ordered pairs (M_1, M_2) such that M_1 is an even square, M_2 is twice a triangular number and $n = M_1 + M_2$, in which the pair with $M_1 = 0$ is repeated twice.

By 23.2(a), we have $\alpha' = |\mathscr{X}'|$ and by the method of 23.17, we see that $\alpha \geqslant |\mathscr{X}|$. We have a bijection $\mathscr{X} \xrightarrow{\sim} \mathscr{X}'$ defined by

$$(a^2, b^2 + b) \to (\tfrac{1}{2}(a+b)(a+b+1), \tfrac{1}{2}(a-b)(a-b-1)). \quad (23.19.1)$$

It follows that $\alpha \geqslant \alpha'$, hence $\alpha = \alpha'$.

(b) $G = Sp_{2n}(k)$ $(n \geqslant 1)$, $\chi \neq 1$. By 23.2(b), we may assume that n is odd.

Let $\mathscr{X}'$ be the set of all unordered pairs (N_1, N_2) of triangular numbers such that $n = N_1 + N_2$.

Let $\mathscr{X}$ be the set of all ordered paris (M_1, M_2) such that M_1 is an odd square, M_2 is twice a triangular number and $n = M_1 + M_2$.

By 23.2(b), we have $\alpha' = 2\,|\mathscr{X}'|$ and by 23.17 we see that $\alpha \geqslant 2\,|\mathscr{X}|$. We have a bijection $\mathscr{X} \xrightarrow{\sim} \mathscr{X}'$ defined by (23.19.1). It follows that $\alpha \geqslant \alpha'$, hence $\alpha = \alpha'$.

(c_1) $G = SO_{2n+1}(k)$ $(n \geqslant 1)$, $\chi = 1$. Let $\mathscr{X}'$ be the set of unordered pairs (N_1, N_2) of squares such that $2n + 1 = N_1 + N_2$, in which each pair (N_1, N_2) with $N_1 > 0$, $N_2 > 0$ is repeated twice.

Let $\mathscr{X}$ be the set of ordered pairs (M_1, M_2) such that M_1, M_2 are twice triangular numbers and $n = M_1 + M_2$.

By 23.2(c), we have $\alpha' = |\mathscr{X}'|$ and by the method of 23.17 we have $|\alpha| \geqslant |\mathscr{X}|$. We have a bijection $\mathscr{X} \cong \mathscr{X}'$ defined by

$$(a^2 + a, b^2 + b) \to ((a+b+1)^2, (a-b)^2).$$

It follows that $\alpha \geqslant \alpha'$, hence $\alpha = \alpha'$.

(c$_2$) $G = PSO_{2n}(k)$ $(n \geqslant 2)$, $\chi = 1$. By 23.2(c), we may assume that $n \equiv 0 \pmod 4$. Let $\mathscr{X}'$ be the set of unordered pairs (N_1, N_2) of even squares such that $2n = N_1 + N_2$ in which each pair (N_1, N_2) with $N_1 > 0$, $N_2 > 0$, $N_1 \neq N_2$ is repeated twice and each pair (N_1, N_2) with $N_1 = N_2$ is repeated four times.

Let $\mathscr{X}$ be the set of ordered pairs (M_1, M_2) of even squares with $n = M_1 + M_2$ in which each pair with $M_1 = 0$ or $M_2 = 0$ is repeated twice.

By 23.2(c), we have $\alpha' = |\mathscr{X}'|$ and by the method of 23.17 we have $|\alpha| \geqslant |\mathscr{X}|$. We have a bijection $\mathscr{X} \cong \mathscr{X}'$ defined by

$$(a^2, b^2) \to ((a+b)^2, (a-b)^2). \tag{23.19.2}$$

It follows that $\alpha \geqslant \alpha'$, hence $\alpha = \alpha'$.

(d) $G = SO_{2n}(k)$ $(n \geqslant 2)$, $\chi \neq 1$. By 23.2(d), we may assume that $n \equiv 2 \pmod 4$. Let $\mathscr{X}'$ be the set of unordered pairs (N_1, N_2) of even squares such that $2n = N_1 + N_2$, in which each pair (N_1, N_2) with $N_1 > 0$, $N_2 > 0$ is repeated twice.

Let $\mathscr{X}$ be the set of ordered pairs (M_1, M_2) of odd squares such that $n = M_1 + M_2$.

By 23.2(d), we have $\alpha' = 2\,|\mathscr{X}'|$ and by the method of 23.17 we have $\alpha \geqslant 2\,|\mathscr{X}|$. We have a bijection $\mathscr{X} \cong \mathscr{X}'$ defined by (23.19.2). It follows that $\alpha \geqslant \alpha'$, hence $\alpha = \alpha'$.

(e$_1$) $G = \mathrm{Spin}_{2n+1}(k)$ $(n \geqslant 2)$, $\chi \neq 1$.

Let $\mathscr{X}'$ be the set of unordered pairs (N_1, N_2) of triangular numbers such that $2n+1 = N_1 + N_2$.

Let $\mathscr{X}$ be the set of ordered pairs (M_1, M_2) of triangular numbers such that $n = 4M_1 + M_2$.

By 23.2(e), we have $\alpha' = 2\,|\mathscr{X}'|$ and by the method of 23.18 we have $\alpha^+ \geqslant 2\,|\mathscr{X}|$. We have a bijection $\mathscr{X} \cong \mathscr{X}'$ defined by

$$(\tfrac{1}{2}(a^2+a), \tfrac{1}{2}(b^2+b)) \to (\tfrac{1}{2}(2a+b+1)(2a+b+2), \tfrac{1}{2}(2a-b)(2a-b+1)).$$

It follows that $\alpha^+ \geqslant \alpha'$. Since $\alpha' \geqslant \alpha \geqslant \alpha^+$, we must have $\alpha = \alpha^+ = \alpha'$.

(e$_2$) $G = \mathrm{Spin}_{4n+2}(k)$ $(n \geqslant 2)$, χ faithful. Let $\mathscr{X}'$ be the set of unordered pairs (N_1, N_2) of even triangular numbers such that $4n+2 = N_1 + N_2$.

Let $\mathscr{X}$ be the set of ordered pairs (M_1, M_2) such that M_1 is an odd square, M_2 is a triangular number and $2n+1 = 2M_1 + M_2$.

By 23.2(e), we have $\alpha' = 4\,|\mathscr{X}'|$ and by 23.18 we have $\alpha^+ \geqslant 4\,|\mathscr{X}|$. We have a bijection $\mathscr{X} \simeq \mathscr{X}'$ defined by

$$(a^2, \tfrac{1}{2}(b^2+b)) \to (\tfrac{1}{2}(2a+b)(2a+b+1), \tfrac{1}{2}(2a-b)(2a-b-1)). \quad (23.19.3)$$

It follows that $\alpha^+ \geqslant \alpha'$. Since $\alpha' \geqslant \alpha \geqslant \alpha^+$, we must have $\alpha = \alpha^+ = \alpha'$.

(f) $G = \frac{1}{2}\,\mathrm{Spin}_{4n}(k)$ $(n \geqslant 3)$, $\chi \neq 1$. Let $\mathscr{X}'$ be the set of unordered pairs (N_1, N_2) of even triangular numbers such that $4n = N_1 + N_2$ and such that a pair (N_1, N_2) with $N_1 = N_2$ is repeated twice.

Let $\mathscr{X}$ be the set of ordered pairs (M_1, M_2) such that M_1 is an even square, M_2 is a triangular number and $2n = 2M_1 + M_2$, and such that a pair (M_1, M_2) with $M_1 = 0$ is repeated twice.

By 23.2(f), we have $\alpha' = 2\,|\mathscr{X}'|$ and by the method of 23.18 we have $\alpha^+ \geqslant 2\,|\mathscr{X}|$. We have a bijection $\mathscr{X} \simeq \mathscr{X}'$ defined by (23.19.3).

It follows that $\alpha^+ \geqslant \alpha'$. Since $\alpha' \geqslant \alpha \geqslant \alpha^+$, we must have $\alpha = \alpha^+ = \alpha'$.

23.20. Let G, χ be as in 23.12(e_1), (e_2), (f). We now prove that for any $A \in \hat{G}^\chi$, the parity condition $\varepsilon_A = \hat{\varepsilon}_A$ is satisfied. If A is not cuspidal, this follows from the inductive assumptions on G in 23.12. If A is cuspidal, then we have necessarily $A \in \hat{G}^{\chi,+}$ (by the equality $\alpha = \alpha^+$ proved in 23.19); for such A, the parity condition has already been noted in 23.14.

It follows that for any $\mathscr{L} \in \mathscr{S}(T)$ and any family $\mathscr{F} \subset \hat{W}'_{\mathscr{L}}$, we can consider the statement $(17.8.3)_{\chi,\mathscr{F}}$ just as in (23.14.2). (The restriction made in (23.14.2) that $\mathscr{F}$ is of + type can now be dropped.) We now show that $(17.8.3)_{\chi,\mathscr{F}}$ holds. As noted in 23.14, we may assume that $\mathscr{L}$ is as in 23.14(e_1), (e_2), (f). If $\mathscr{F}$ is of − type, then $\mathscr{F}$ is not a cuspidal family (see 23.16) so that the method of 23.15 applies and shows that $(17.8.3)_{\chi,\mathscr{F}}$ holds. If $\mathscr{F}$ is of + type we can assume, using 23.15, that $\mathscr{F}$ is cuspidal; in that case $(17.8.3)_{\chi,\mathscr{F}}$ holds by 23.18.

23.21. *Proof of Theorem* 23.1. When $G = \{e\}$, the theorem is obvious. We now assume that $\dim G \geqslant 1$ and that the theorem is already proved when G is replaced by a group of dimension $< \dim G$.

Using 17.10 and (23.8.2) we see that we may assume that G is semisimple. Let $G_1, G_2, ..., G_r$ be the set of almost simple, closed normal subgroups $\neq \{e\}$ of G. First, assume that $r \geqslant 2$. Let $\tilde{G} = G_1 \times G_2 \times \cdots \times G_r$ and let $\pi\colon \tilde{G} \to G$ be the finite covering defined by $(g_1, g_2, ..., g_r) \to g_1 g_2 \cdots g_r$. By the induction hypothesis, (17.8.4) and (17.8.5) hold for each G_i hence also for $\tilde{G}$ (see 17.11), and hence also for G, (see (17.16.3), (17.16.4)). Using the induction hypothesis for each G_i and 23.11, we see that $(17.8.3)_\chi$ holds for G, for any faithful character $\chi\colon \mathscr{Z}_G \to \bar{Q}_l^*$.

It remains to consider the case where G is almost simple.

When G is of type A, the theorem holds by 18.5 and (23.13.1).

When G is of type B, C, or D and $p=2$, the theorem holds by 22.6, 22.7.

Assume now that G is of type B, C, or D and $p \neq 2$. To prove the theorem we may assume in addition in 23.1(a) that $A \in \hat{G}^{\chi}$ and in 23.1(b) that $A \in \mathrm{Irr}^0(G)^{\chi}$, with χ faithful. (The case where χ is not faithful, is reduced to the case where χ is faithful, by 23.9(a), (b), (c), by replacing G by the quotient of G by the kernel of χ.) Under this additional assumption, the theorem holds for G, by 23.3–23.7, 23.17–23.20, using the inductive hypothesis.

When G is of type E_8, the theorem holds by 21.2, 21.4(a), 21.12.

When G is of type F_4, the theorem holds by 21.3, 21.4(a), 21.13.

When G is of type G_2, then theorem holds by 20.6.

When G is adjoint of type E_6 (resp. E_7), the theorem holds by 20.3(a) (resp. 20.3(c)).

Assume now that G is simply connected of type E_6 (resp. E_7). Then parts (a) and (b) of the theorem as well as the statement (17.8.3) hold for G by 20.3(b) (resp. 20.5). Let $\chi\colon \mathscr{Z}_G \to \bar{Q}_l^*$ be a nontrivial character. Using the inductive hypothesis and the method of 23.15, we see that $(17.8.3)_{\chi,\mathscr{F}}$ (defined just as in (23.14.2)) holds for $G, \mathscr{L}$, for any $\mathscr{L} \in \mathscr{S}(T)$ and any family $\mathscr{F} \subset \hat{W}'_{\mathscr{L}}$ except possibly when $\mathscr{F}$ is a cuspidal family. (The notion of cuspidal family is defined as in 23.15.) The statement $(17.8.3)_{\chi,\mathscr{F}}$ can be deduced from (17.8.3) for G, provided that the following statement is known.

(23.21.1) For any $A \in \hat{G}^{\chi}_{\mathscr{L},\mathscr{F}}$ (defined in 23.0) with $\mathscr{F}$ cuspidal, and any $z \in \mathscr{Z}_G$, $z \neq e$ we have $t_z^* A \neq A$.

If G is of type E_7, then $\mathscr{L}$ in (23.21.1) is uniquely determined (up to W-action); it satisfies: $W_{\mathscr{L}}$ of type E_6, $\Omega_{\mathscr{L}}$ of order 2 acting nontrivially on $W_{\mathscr{L}}$. Moreover, $\mathscr{F}$ is uniquely determined. In this case, (23.12.1) has been already proved (see (20.5.5)).

If G is of type E_6, then $\mathscr{L}$ in (23.21.1) is again uniquely determined (up to W-action); it satisfies: $W_{\mathscr{L}}$ of type D_4, $\Omega_{\mathscr{L}}$ of order 3 acting nontrivially on $W_{\mathscr{L}}$. Moreover, $\mathscr{F}$ is uniquely determined. From the description of $\mathrm{Irr}^0(G)$ in the proof of 20.3(b) we see that if $A \in \hat{G}^{\chi}_{\mathscr{L}}$ is cuspidal and $z \in \mathscr{Z}_G$, $z \neq e$ then A and $t_z^* A$ have different supports, hence $A \neq t_z^* A$. If $A \in \hat{G}^{\chi}_{\mathscr{L}}$ is noncuspidal then it is a summand of a complex induced from a parabolic subgroup of type $A_2 \times A_2$. As in the proof of (20.5.5) we see again that A and $t_z^* A$ have different supports, hence $A \neq t_z^* A$. Thus $(17.8.3)_{\chi}$ for G is verified. This completes the proof of the theorem.

24. Local Intersection Cohomology with Twisted Coefficients of the Closure of a Unipotent Class

24.0. The purpose of this section is to compute the local intersection cohomology of the closure of a unipotent class C of G with coefficient in an irreducible G-equivariant local system on C (in good characteristic), thus extending the results of Shoji [28] and Beynon–Spaltenstein [20]. To do so we shall borrow an idea of Shoji [28], which is to make use of the orthogonality relations for Green functions. We shall use here the orthogonality relations for the generalized Green functions (in Sections 9 and 10) and this will lead to stronger results. Throughout this section we assume that G satisfies the restrictions (23.0.1).

24.1. Let I be the set of all pairs $(C, \mathscr{E})$ where C is a unipotent class in G and $\mathscr{E}$ is an irreducible $\bar{Q}_l$-local system (given up to isomorphism) on C which is G-equivariant for the conjugation action of G. For each $i \in I$, we denote $K_{(i)} = IC(\bar{C}, \mathscr{E})$ [dim C] regarded as a perverse sheaf on the unipotent variety G_{un} of G, which is zero outside $\bar{C}$.

The set I has a natural preorder: given $i = (C, \mathscr{E})$, $i' = (C', \mathscr{E}')$ in I we say that $i' \leqslant i$ if $C' \subset \bar{C}$. We say that $i \sim i'$ if $C = C'$. We say that $i' < i$ if $C' \subsetneq \bar{C}$.

We define J to be the set of triples $(L, C_1, \mathscr{E}_1)$ up to G-conjugacy where L is a Levi subgroup of a parabolic subgroup of G, C_1 is a unipotent class of L_1 and $\mathscr{E}_1$ is an irreducible L-equivariant local system on C_1 such that $(\mathscr{Z}_L^0 \times C_1, 1 \boxtimes \mathscr{E}_1)$ is a cuspidal pair for L in the sense of [4, 2.4]. (Here $1 \boxtimes \mathscr{E}_i$ is the inverse image of $\mathscr{E}_1$ under $pr_2 \colon \mathscr{Z}_L^0 \times C_1 \to C_1$.) Given $j \in J$, we consider the perverse sheaf K_j on G defined in terms of $(L, \mathscr{Z}_L^0 \times C_1, 1 \boxtimes \mathscr{E}_1)$ in the same way as K was defined in (8.1.2) in terms of $(L, \Sigma, \mathscr{E})$.

According to [4, 6.5] the restriction of K_j to G_{un} is a direct sum of complexes of the form $K_{(i)}$ [dim $\mathscr{Z}_L^0$], where $i = (C, \mathscr{E}) \in I$; the various $i \in I$ which appear form a subset I_j of I and the subsets I_j $(j \in J)$ form a partition of I. Thus, we have a canonical surjective map $\tau \colon I \to J$ defined by $\tau(i) = j \Leftrightarrow i \in I_j$.

Let $i = (C, \mathscr{E}) \in I$ and let $j = \tau(i)$. Let A_i be an irreducible perverse sheaf on G which is a direct summand of K_j and satisfies

$$A_i \mid G_{\text{un}} = K_{(i)} [\dim \mathscr{Z}_L^0]. \tag{24.1.1}$$

Then (24.1.1) characterizes A_i up to isomorphism and we have

$$\mathscr{H}^a(A_i) \mid C = \begin{cases} \mathscr{E} & \text{if } a = a_0 \\ 0 & \text{if } a \neq a_0, \end{cases} \tag{24.1.2}$$

where $a_0 = -\dim C - \dim \mathscr{Z}_L^0$.

(24.1.3) Let $i \to i^*$ (resp. $j \to j^*$) be the involution on I (resp. J) defined by $(C, \mathscr{E}) \to (C, \mathscr{E}^\vee)$ (resp. $(L, C_1, \mathscr{E}_1) \to (L, C_1, \mathscr{E}_1^\vee)$) where $\mathscr{E}^\vee$, $\mathscr{E}_1^\vee$ denote the local systems dual to $\mathscr{E}$, $\mathscr{E}_1$. We have $\tau(i^*) = \tau(i)^*$ for all $i \in I$.

24.2. From now until the end of 24.6, we assume that k is an algebraic closure of the finite field F_q and that G has a fixed F_q rational structure compatible with the group structure, with Frobenius map $F: G \to G$. Then F acts naturally on I and J by $(C, \mathscr{E}) \to (F^{-1}C, F^*\mathscr{E})$, $(L, C_1, \mathscr{E}_1) \to (F^{-1}L, F^{-1}C_1, F^*\mathscr{E}_1)$; this action is compatible with $\tau: I \to J$. Hence τ induces a map $I^F \to J^F$ between the fixed point sets of F. Let $j \in J^F$. We shall represent (as we may) j by a triple $(L, C, \mathscr{E}_1)$ with L an F-stable Levi subgroup of an F-stable parabolic subgroup P of G, with $FC_1 = C_1$ and with $F^*\mathscr{E}_1 \approx \mathscr{E}_1$. We shall choose an isomorphism $\phi_j: F^*\mathscr{E}_1 \xrightarrow{\sim} \mathscr{E}_1$ which induces a map of finite order on the stalk of $\mathscr{E}_1$ at any F_q-rational point of C_1. This induces an isomorphism $F^*(1 \boxtimes \mathscr{E}_1) \xrightarrow{\sim} 1 \boxtimes \mathscr{E}_1$ over $\mathscr{Z}_L^0 C_1$ and (as in (8.3.1)) an isomorphism $\phi: F^*K_j \xrightarrow{\sim} K_j$, where K_j is as in 24.1.

We now consider $i \in I^F$ such that $\tau(i) = j$. Then we have $F^*A_i \approx A_i$ and our next objective is to define a particular isomorphism $\phi_{A_i}: F^*A_i \xrightarrow{\sim} A_i$.

Let $V_{A_i} = \operatorname{Hom}(A_i, K_j)$ be as in 10.1. It is an irreducible left $\mathscr{A}_j$-module (see 10.3) where $\mathscr{A}_j = \operatorname{End}(K_j)$. Let θ_w ($w \in W_j = N(L)/L$) be the canonical basis of $\mathscr{A}_j$ considered in the proof of 10.9; it satisfies the identities $\theta_w \theta_{w'} = \theta_{ww'}$ and $\iota(\theta_w) = \theta_{F^{-1}(w)}$ where $\iota: \mathscr{A}_j \to \mathscr{A}_j$ is as in 10.4. From the definition of simple reflections in $\mathscr{W}_j$ (see [4, 9.2(a)]) and from the fact that $FP = P$, we see that $F^{-1}: \mathscr{W}_j \to \mathscr{W}_j$ maps the set of simple reflections into itself, hence it is a Coxeter group automorphism of order, say, c, hence it defines a semidirect product $(\mathbb{Z}/cZ) \cdot \mathscr{W}_j$. For any isomorphism $\phi_{A_i}: F^*A_i \xrightarrow{\sim} A_i$ the corresponding map $\sigma_{A_i}: V_{A_i} \to V_{A_i}$ (see 10.4) is ι-semilinear and bijective. Hence it is equal to a scalar $\zeta \in \bar{Q}_l^*$ times the map defining the action of the standard generator of $\mathbb{Z}/c\mathbb{Z}$ in the preferred extension (17.2) of V_{A_i} to a $\mathbb{Z}/c\mathbb{Z} \cdot \mathscr{W}_j$-module. Replacing ϕ_{A_i} by a scalar multiple, we may achieve that $\zeta = 1$. This defines our choice of an isomorphism $\phi_{A_i}: F^*A_i \xrightarrow{\sim} A_i$. From the definition of preferred extension, it follows that with this choice of ϕ_{A_i} we have

$$\operatorname{Tr}(\theta_w \sigma_{A_i}, V_{A_i}) \in \mathbb{Z} \quad \text{and} \quad \operatorname{Tr}((\theta_w \sigma_{A_i})^{-1}, V_{A_i}) \in \mathbb{Z} \text{ for all } w \in \mathscr{W}_j. \qquad (24.2.1)$$

Having defined ϕ_{A_i}, we now define an isomorphism $\psi: F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ over C (where $i = (C, \mathscr{E})$) by the requirement that:

(24.2.2) Under (24.1.2), $q^{(a_0+r)/2}\psi$ corresponds to the map defined by $\phi_{A_i}: F^*\mathscr{H}^{a_0}(A_i) \to \mathscr{H}^{a_0}(A_i)$. (Here $a_0 = -\dim C - \dim \mathscr{Z}_L^0$ and $r = \dim \operatorname{supp} A_i$).

We now define, for ψ as in (24.2.2) a function $Y_i\colon G^F_{\mathrm{un}} \to \bar{Q}_l$ by

$$Y_i(g)=\begin{cases}\mathrm{Tr}(\psi, \mathscr{E}_g) & \text{if } g\in C^F\\ 0 & \text{if } g\notin C^F.\end{cases} \tag{24.2.3}$$

We shall need the following property of $\psi\colon F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ (in (24.2.2)).

(24.2.4) *For any $g\in C^F$, the map $\psi\colon \mathscr{E}_g \to \mathscr{E}_g$ is of finite order.*

For the proof, we shall introduce the varieties

$$Z_{g,j}=\{xP\in G/P \mid x^{-1}gx\in C_1 U_P\},$$
$$\bar{Z}_{g,j}=\{xP\in G/P \mid x^{-1}gx\in \bar{C}_1 U_P\},$$

so that $Z_{g,j}$ is open dense in $\bar{Z}_{g,j}$. We also define the local system $\mathscr{F}$ on $Z_{g,j}$ by the property that the inverse image of $\mathscr{F}$ under $\hat{Z}_{g,j}=\{x\in G \mid x^{-1}gx\in C_1U_p\}\to Z_{g,j}$ $(x\to xP)$, equals the inverse image of $\mathscr{E}_1$ under $\hat{Z}_{g,j}\to C_1$ ($x\to C_1$-component of $x^{-1}gx\in C_1U_p$). From the description of K_j given in 2.2, we see that $\mathscr{H}^a_g(K_j)=H^a(\bar{Z}_{g,j}, D)$ where D is a certain complex defined in terms of $IC(\bar{C}_1, \mathscr{E}_1)$; using 23.1 for L, we see that $IC(\bar{C}_1, \mathscr{E}_1)$ is $\mathscr{E}_1$ extended by 0 on $\bar{C}_1-C_1$ and it follows that $D=\mathscr{F}[r]$ extended by 0 on $\bar{Z}_{g,j}-Z_{g,j}$ (r as in (24.2.2)). It follows that $H^a(\bar{Z}_{g,j}, D)=H^a_c(Z_{g,j}, \mathscr{F}[r])$ hence

$$\mathscr{H}^a_g(K_j)=H^{a+r}_c(Z_{g,j}, \mathscr{F}). \tag{24.2.5}$$

The chosen isomorphism $F^*\mathscr{E}_1\xrightarrow{\sim}\mathscr{E}_1$ induces an isomorphism $\tilde{\phi}\colon F^*\mathscr{F}\xrightarrow{\sim}\mathscr{F}$ such that for any $z\in Z_{g,j}$ with $F^nz=z$, the map $\tilde{\psi}^n\colon \mathscr{F}_z\xrightarrow{\sim}\mathscr{F}_z$ has finite order. From [4, 1.2(b)] we see that $\dim Z_{g,j}\leqslant \frac{1}{2}(a_0+r)$. It follows that $\tilde{\phi}$ acts on $H^{a_0+r}_c(Z_{g,j}, \mathscr{F})$ as $q^{(a_0+r)/2}$ times a map of finite order. Hence ϕ acts on $\mathscr{H}^{a_0}_g(K_j)$ as $q^{(a_0+r)/2}$ times a map of finite order. In the isomorphism (10.4.1) the map ϕ on $\mathscr{H}^{a_0}_g(K_j)$ corresponds to the map $\phi_{A_i}\otimes\sigma_{A_i}$ on $\mathscr{H}^{a_0}_g(A_i)\otimes V_{A_i}$. It follows that ϕ_{A_i} acts on $\mathscr{H}^{a_0}_g(A_i)$ as $q^{(a_0+r)/2}$ times a map of finite order, and (24.2.4) follows.

An analogous proof gives the following statement.

(24.2.6) Let $\psi'\colon F^*\mathscr{E}^v\xrightarrow{\sim}\mathscr{E}^v$ be defined in terms of $(L, C_1, \mathscr{E}^v_1)$, ϕ^v_j: $F^*\mathscr{E}^v_1\xrightarrow{\sim}\mathscr{E}^v_1$ (=contragredient of $\phi_j\colon F^*\mathscr{E}_1\xrightarrow{\sim}\mathscr{E}_1$, above) and $(C, \mathscr{E}^v)$ in the same way as ψ was defined in terms of $(L, C_1, \mathscr{E}_1)$, $F^*\mathscr{E}_1\xrightarrow{\sim}\mathscr{E}_1$ and $(C, \mathscr{E})$. Then ψ' is the contragredient of ψ.

We shall also need the following statement.

(24.2.7) *The functions Y_i $(i\in I^F)$ (see 24.2.3) form a basis for the vector space $\mathscr{V}$ of G^F-invariant functions $G^F_{\mathrm{un}}\to\bar{Q}_l$.*

Let $g \in G_{\mathrm{un}}^F$, let C be its G-conjugacy class and let $\Gamma = Z(g)/Z^0(g)$. The G^F-orbits in C^F are in 1–1 correspondence with the set Γ modulo the equivalence relation $\gamma \sim \gamma_1^{-1}\gamma F(\gamma_1)$ $(\forall \gamma_1 \in \Gamma)$. The G-equivariant irreducible local systems $\mathscr{E}$ on C such that $F^*\mathscr{E} \approx \mathscr{E}$ are in 1–1 correspondence with the irreducible representations E of Γ for which there exists an isomorphism $\alpha_E: E \to E$ with $\alpha_E \circ \gamma = F(\gamma) \cdot \alpha_E$ for all $\gamma \in \Gamma$. Moreover, the matrix $(\mathrm{Tr}(\alpha_E \circ \gamma, E))$, indexed by (γ, E) (where γ are representatives for the $\sim$ classes on Γ and E are as above) is square and nonsingular. This implies that the functions Y_i, where $i \in I^F$ are of form $(C, \mathscr{E})$ with $\mathscr{E}$ variable, form a basis of the vector space of G^F-invariant functions $C^F \to \bar{Q}_l$; the statement (24.2.7) follows.

We now define for any $i = (C, \mathscr{E}) \in I^F$ a function $X_i \in \mathscr{V}$ by

$$X_i(g) = \sum_a (-1)^{a+a_0} \mathrm{Tr}(\phi_{A_i}, \mathscr{H}_g^a A_i)\, q^{-(a_0+r)/2}, \tag{24.2.8}$$

where a_0, r are as in (24.2.2).

From (24.2.7) we see that we can write uniquely

$$X_i = \sum_{i' \in I^F} P_{i',i} Y_{i'}, \ (P_{i',i} \in \bar{Q}_l). \tag{24.2.9}$$

From (24.1.1), (24.1.2), (24.2.2), and (24.2.3) we have

$$P_{i',i} = 0 \text{ if } i' \nleq i \qquad \text{or if} \quad i' \sim i, i' \neq i. \tag{24.2.10}$$

$$P_{i,i} = 1. \tag{24.2.11}$$

Let $\tilde{\phi}_{A_i}: F^*A_{i*} \simeq A_{i*}$ be defined as $q^{(a_0+r)}h^{-1} \circ \phi_{A_i}^v \circ F^*(h)$ where $\phi_{A_i}^v: F^*DA_i \simeq DA_i$ is the contragredient of $\phi_{A_i}: F^*A_i \simeq A_i$, $h: A_{i*} \simeq DA_i$ is an isomorphism and $F^*(h): F^*A_{i*} \simeq F^*DA_i$ is defined by h. Then $\tilde{\phi}_{A_{i*}}$ is not necessarily equal to $\phi_{A_{i*}}$. We define $\tilde{X}_i, \tilde{Y}_i \in \mathscr{V}$ by

$$\tilde{X}_i(g) = \sum_a (-1)^{a+a_0} \mathrm{Tr}(\tilde{\phi}_{A_{i*}}, \mathscr{H}_g^a A_{i*})\, q^{-(a_0+r)/2} \tag{24.2.12}$$

$$\tilde{Y}_i(g) = \begin{cases} \mathrm{Tr}(\psi^\vee, \mathscr{E}_g^\vee) & \text{if } g \in C^F \\ 0 & \text{if } g \notin C^F, \end{cases} \tag{24.2.13}$$

where ψ is as in (24.2.2).

We can again write

$$\tilde{X}_i = \sum_{i' \in I^F} \tilde{P}_{i',i} \tilde{Y}_{i'} \qquad (\tilde{P}_{i',i} \in \bar{Q}_l), \tag{24.2.14}$$

where

$$\tilde{P}_{i',i} = 0 \text{ if } i' \nleq i \qquad \text{or if} \quad i' \sim i, i' \neq i \tag{24.2.15}$$

and

$$\tilde{P}_{i,i}=1 \qquad \text{by (24.2.6) and (24.1.2).} \tag{24.2.16}$$

24.3. Consider the nonsingular bilinear form

$$(X, X')=\sum_{g\in G^F_{\mathrm{un}}} X(g)\,X'(g)$$

on $\mathscr{V}$.

Let

$$\lambda_{i,i'}=(Y_i, \tilde{Y}_{i'}). \tag{24.3.1}$$

Then

$$\lambda_{i,i'}=0 \qquad \text{unless} \quad i\sim i'. \tag{24.3.2}$$

Since (Y_i) is a basis of $\mathscr{V}$ (24.2.7) and similarly $(\tilde{Y}_i)$ is a basis of $\mathscr{V}$, we see that

(24.3.3) the matrix $(\lambda_{i,i'})_{(i,i')\in R\times R}$ is nonsingular for any equivalence class R (for $\sim$) in I.

Now let $i, i'\in I^F$ be such that $\tau(i)=\tau(i')=j=(L, C_1, \mathscr{E}_1)$.

For each $w\in \mathscr{W}_j=N(L)/L$, let $L^w=zLz^{-1}$ where $z^{-1}F(z)$ is a representative for w^{-1} in $N(L)$ and let

$$\omega_{i,i'}=|\mathscr{W}_j|^{-1}\sum_{w\in\mathscr{W}_j}\operatorname{Tr}((\theta_w\sigma_{A_i})^{-1}, V_{A_i})$$
$$\times\operatorname{Tr}(\theta_w\sigma_{A_{i'}}, V_{A_{i'}})\frac{|G^F|}{|\mathscr{Z}^{0F}_{L^w}|}\,q^{-\dim G}q^{-(a_0+a_0')/2}, \tag{24.3.4}$$

where $a_0=-\dim C-\dim \mathscr{Z}^0_L$, $a_0'=-\dim C'-\dim \mathscr{Z}^0_L$ and θ_w, σ_{A_i}, $\sigma_{A_i'}$ are as in 24.2. Using (24.2.1) we see that

$$\omega_{i',i}=\omega_{i,i'} \qquad \text{is a rational number} \tag{24.3.5}$$

If i, i' are such that $\tau(i)\neq\tau(i')$ we set $\omega_{i,i'}=0$.

We can now write the orthogonality relations (10.9.1) in the form

$$(X_i, \tilde{X}_{i'})=\omega_{i,i'} \qquad (i, i'\in I^F). \tag{24.3.6}$$

We note that the assumptions of (10.9.1) are satisfied, by Theorem 23.1.

We can now state the following result which extends results of Shoji [28] and Beynon–Spaltenstein [20].

THEOREM 24.4. *Recall that G is subject to the restriction* (23.0.1).

(a) $P_{i',i}=\tilde{P}_{i',i}$ *and* $\lambda_{i',i}=\lambda_{i,i'}$ *for all* i', i.

(b) *$(P_{i',i}, \lambda_{i',i})$ is the unique solution of the system of equations*

$$\begin{aligned}
\sum_{i'_1, i'_2 \in I^F} P_{i'_1,i_1} P_{i'_2,i_2} \lambda_{i'_1,i'_2} &= \omega_{i_1,i_2} && \forall i_1, i_2 \in I^F \\
P_{i,i} &= 1 && \forall i \in I^F \\
P_{i',i} &= 0 && \text{if} \quad i' \nleq i \text{ or if } i' \sim i,\ i' \neq i \\
\lambda_{i',i} &= 0 && \text{if} \quad i' \nsim i.
\end{aligned} \tag{24.4.1}$$

(c) *$P_{i',i}$ and $\lambda_{i',i}$ are rational numbers for all i', i*

(d) *$P_{i',i}$ and $\lambda_{i',i}$ are zero if $\tau(i') \neq \tau(i)$.*

Remark. We say that $i \in I$ is *uniform* if $\tau(i) = (T, \{e\}, \bar{Q}_l)$ where T is a maximal torus of G. In [28, 20] it is proved (assuming good characteristic) that $P_{i',i}$ for i', i uniform are determined by equations like (24.4.1) and that $P_{i',i} = 0$ if i is uniform and i' is not uniform. Note that in our theorem i', i are not necessarily uniform and that the characteristic is only subject to (23.0.1), in particular, for classical groups we allow $p = 2$. Moreover, in [28, 20] the $\lambda_{i',i}$ are assumed to be known in advance, while in our approach they are determined automatically by (24.4.1).

Proof. From (24.3.6), (24.2.14), (24.2.9), (24.3.1) we have

$$\sum_{i'_1, i'_2 \in I^F} P_{i'_1,i_1} \tilde{P}_{i'_2,i_2} \lambda_{i'_1,i'_2} = \omega_{i_1,i_2}, \qquad \text{for all} \quad i_1, i_2 \in I^F. \tag{24.4.2}$$

Consider for any integer δ the following two statements.

(A_δ) If $i' = (C', \mathscr{E}') \in I^F$, $\dim C' \leqslant \delta$ and $i \in I^F$, then $P_{i',i} = \tilde{P}_{i',i}$ is a rational number and it is zero unless $\tau(i) = \tau(i')$.

(B_δ) If $i' = (C', \mathscr{E}') \in I^F$, $\dim C' \leqslant \delta$ and $i \in I^F$, then $\lambda_{i',i} = \lambda_{i,i'}$ is a rational number and it is zero unless $\tau(i) = \tau(i')$.

It is clear that (A_δ), (B_δ) are true for $\delta < 0$. We now show that

If $\delta \geqslant 0$ and ($\mathrm{A}_{\delta-1}$), (B_δ) *are true, then* (A_δ) *is true.* (24.4.3)

Let $i' = (C', \mathscr{E}') \in I^F$, $\dim C' = \delta$, $i \in I^F$. We may assume that $i' < i$. We write Eq. (24.4.2) for $i_1 = a \sim i'$, $i_2 = i$. We may restrict the sum to those i'_1, i'_2 for which $i'_1 \sim i'_2$, see (24.3.2). We get

$$\sum_{i'_2 \sim i'} \tilde{P}_{i'_2,i} \lambda_{a,i'_2} = \omega_{a,i} - \sum_{\substack{i'_1 < i' \\ i'_2 \sim i'_1}} P_{i'_1,a} \tilde{P}_{i'_2,i} \lambda_{i'_1,i'_2}. \tag{24.4.4}$$

We also write Eq. (24.4.2) for $i_1 = i$, $i_2 = a \sim i'$.

$$\sum_{i'_2 \sim i'} P_{i'_2,i} \lambda_{i'_2,a} = \omega_{i,a} - \sum_{\substack{i'_1 \sim i' \\ i'_2 \sim i'_1}} P_{i'_2,i} \tilde{P}_{i'_1,a} \lambda_{i'_2,i'_1}. \tag{24.4.5}$$

Using $(\mathrm{A}_{\delta-1})$, $(\mathrm{B}_{\delta-1})$, and (24.3.5), we see that the right-hand sides of (24.4.4) and (24.4.5) are the same. Hence the left-hand sides are also the same. Using (B_δ) we see that

$$\sum_{i'_2 \sim i'} (P_{i'_2,i} - \tilde{P}_{i'_2,i})\, \lambda_{a,i'_2} = 0.$$

This holds for all $a \sim i'$.

Now, using (24.3.3) we see that $P_{i'_2,i} = \tilde{P}_{i'_2,i}$ for all $i'_2 \sim i'$.

Using $(\mathrm{A}_{\delta-1})$, $(\mathrm{B}_{\delta-1})$, and (24.3.5) we see that the right-hand side of (24.4.5) is a rational number. Hence $\sum_{i'_2 \sim i} P_{i'_2,i} \lambda_{i'_2,a}$ is a rational number. Using (24.3.3) and (B_δ) it follows that $P_{i'_2,i}$ is a rational number for all $i'_2 \sim i'$.

Assume now that $\tau(i) \neq \tau(i')$, and that a in (24.4.5) satisfies $\tau(a) = \tau(i')$.

From $(\mathrm{A}_{\delta-1})$, $(\mathrm{B}_{\delta-1})$, and the equality $\omega_{i,a} = 0$, we see that the right-hand side of (24.4.5) is zero. According to (B_δ), the matrix (24.3.3) consists of diagonal blocks according to the fibres of τ. Hence each of these diagonal blocks is invertible. It follows that $P_{i'_2,i} = 0$ for all $i'_2 \sim i'$, $\tau(i'_2) = \tau(i')$ and in particular that $P_{i',i} = 0$. Thus (24.4.3) is proved. We now show that

$$\textit{If } \delta \geqslant 0 \textit{ and } (\mathrm{A}_{\delta-1}), (\mathrm{B}_{\delta-1}) \textit{ are true, then } (\mathrm{B}_\delta) \textit{ is true.} \tag{24.4.6}$$

Let $i' = (C', \mathscr{E}') \in I^F$, $\dim C' = \delta$. $i \in I^F$, $i' \sim i$.

We write Eq. (24.4.2) with $i_1 = i'$, $i_2 = i$. We may restrict the sum to the i'_1, i'_2 such that $i'_1 \sim i'_2$; see (24.3.2). We get

$$\lambda_{i',i} = \omega_{i',i} - \sum_{\substack{i'_1 < i' \\ i'_2 < i}} P_{i'_1,i'} \tilde{P}_{i'_2,i} \lambda_{i'_1,i'_2}. \tag{24.4.7}$$

Using $(\mathrm{A}_{\delta-1})$, $(\mathrm{B}_{\delta-1})$, and (24.3.5) we see that the right-hand side of (24.4.7) is a rational number and is symmetric in i, i'.

When $\tau(i') \neq \tau(i)$, we see from $(\mathrm{A}_{\delta-1})$, $(\mathrm{B}_{\delta-1})$, and the vanishing of $\omega_{i',i}$ that the right-hand side of (24.4.7) is zero. Thus (24.4.6) is proved.

From (24.4.3) and (24.4.6) we see by induction on δ that (A_δ), (B_δ) are true for all δ. Thus (a), (c), (d) are proved.

The previous proof show also that (b) holds. This completes the proof.

24.5. We now fix $i = (C, \mathscr{E}) \in I^F$ and we consider the restriction of $\mathscr{H}^a(A_i)$ to a unipotent class C', $C' \subset \bar{C}$. This a local system on C' which can be decomposed as $\oplus_{\mathscr{E}'} (E^a_{\mathscr{E}'} \otimes \mathscr{E}')$, where $\mathscr{E}'$ runs over all irreducible G-equivariant local systems on C' and $E^a_{\mathscr{E}'}$ are finite-dimensional $\bar{Q}_l$-vector spaces. Let $\mathscr{E}'$ be such that $F^*\mathscr{E}' \approx \mathscr{E}'$ and let $\psi_{\mathscr{E}'}: F^*\mathscr{E}' \xrightarrow{\sim} \mathscr{E}'$ be defined as in (24.2.2). Then there is a unique isomorphism $\sigma_{\mathscr{E}'}: E^a_{\mathscr{E}'} \xrightarrow{\sim} E^a_{\mathscr{E}'}$ such that $\phi_{A_i}: F^*A_i \mid C' \xrightarrow{\sim} A_i \mid C'$ restricted to $E^a_{\mathscr{E}'} \otimes \mathscr{E}'$ is $\sigma_{\mathscr{E}'} \otimes \psi_{\mathscr{E}'}$. For each $\lambda \in \bar{Q}^*$

let $E_{\mathscr{E}'}^{a,\lambda}$ be the λ-generalized eigenspace of $\sigma_{\mathscr{E}'}$ on $E_{\mathscr{E}'}^{a}$. We then have for any $g' \in C'^F$ the equality

$$\sum_a (-1)^{a+a_0} \operatorname{Tr}(\phi_{A_i}, \mathscr{H}_{g'}^a A_i)\, q^{-(a_0+r)/2} = \sum_{\substack{\mathscr{E}',\lambda \\ F^*\mathscr{E}' \approx \mathscr{E}'}} \operatorname{Tr}(\psi_{\mathscr{E}'}, \mathscr{E}'_{g'}) \dim(E_{\mathscr{E}'}^{a,\lambda}) \cdot \lambda,$$

where a_0, r are defined in terms of i as in (24.2.2). Hence

$$P_{i',i} = \sum_{a,\lambda} (-1)^{a+a_0} \lambda q^{-(a_0+r)/2} \dim E_{\mathscr{E}'}^{a,\lambda}, \tag{24.5.1}$$

where $i' = (C', \mathscr{E}') \in I^F$. By a general result on eigenvalues of Frobenius, applied to $\phi_{A_i} q^{-(a_0+r)/2}\colon F^*IC(\bar{C}, \mathscr{E}) \to IC(\bar{C}, \mathscr{E})$ the numbers $\lambda q^{-(a_0+r)/2}$ must be algebraic integers. Hence $P_{i',i}$ is an algebraic integer. It is also a rational number (24.4(c)). Hence

$$P_{i',i} \text{ is an integer.} \tag{24.5.2}$$

PROPOSITION 24.6. *Assume that the characteristic of k is good for G. Let C be a unipotent class in G and let $\mathscr{E}$ be an irreducible G-equivariant local system on C. Let $\phi\colon F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ be an isomorphism which induces the identity map on the stalk of $\mathscr{E}$ at some point of C^F. Then for any $g \in \bar{C}^F$, $\phi\colon \mathscr{H}_g^a IC(\bar{C}, \mathscr{E})\circlearrowleft$ is a-pure in the following sense: its eigenvalues are algebraic numbers all of whose complex conjugates have absolute value $q^{a/2}$.*

Proof. In the case where $i = (C, \mathscr{E})$ is uniform (see the remark after Theorem 24.4), this is equivalent to a result of Springer [32]. Our proof, which is based on Deligne's theory, is very close to the proof [21] of the analogous statement for Schubert varieties. We shall replace G by its Lie algebra $\mathbf{g}$, C by the corresponding nilpotent orbit $\mathbf{c}$, g by an element $x \in \mathbf{c}^F$; we shall regard $\mathscr{E}$ as a local system on $\mathbf{c}$. As in [32], we use the following result of Spaltenstein [30]: there exists a 1-parameter subgroup $\lambda\colon k^* \to G$ and a linear subspace $\Sigma \subset \mathbf{g}$ such that

$$\operatorname{Ad}(\lambda(t))x = t^{-c}x \qquad \text{with} \quad c > 0$$

$\operatorname{Ad}\lambda(t)$ stabilizes Σ and its weights on Σ are of form $\xi(t) = t^b$, $b \geqslant 0$.

$$\dim \Sigma = \dim Z_G(x).$$

We may also assume that λ, Σ, and the weights in Σ are defined over F_q.

Then $\mathbf{S} = x + \Sigma$ is a transversal slice in $\mathbf{g}$ to the G-orbit of x; hence $\mathbf{S} \cap \bar{\mathbf{c}}$ is a transversal slice in $\bar{\mathbf{c}}$ to the G-orbit of x. It is then enough to show that $\phi\colon \mathscr{H}_x^a IC(\mathbf{S} \cap \bar{\mathbf{c}}, \mathscr{E})\circlearrowleft$ is a-pure for all a. For any $x' \in (\mathbf{S} \cap \bar{\mathbf{c}})^F$, $x' \neq x$, the G-orbit of x' has strictly bigger dimension than the G-orbit of x and $\mathscr{H}_{x'}^a IC(\mathbf{S} \cap \bar{\mathbf{c}}, \mathscr{E}) = \mathscr{H}_{x'}^a IC(\bar{\mathbf{c}}, \mathscr{E})$, hence we may assume by induction that

$\phi: \mathscr{H}^a_{x'} IC(\mathbf{S} \cap \bar{\mathbf{c}}, \mathscr{E}) \circlearrowleft$ is a-pure for all a. We may also assume that $x \notin \mathbf{c}$. Consider the action μ of k^* on $\mathbf{S}$ defined by $\mu(t)(y) = t^c \operatorname{Ad}(\lambda(t)y)$. If we regard $\mathbf{S}$ as a k-vector space (with origin at x) then the action μ is linear with weights $t \to t^{b+c}$, $b + c > 0$. Moreover $\mathbf{S} \cap \bar{\mathbf{c}}$, $\mathbf{S} \cap \mathbf{c}$ are μ-stable and $\mathscr{E} \mid \mathbf{S} \cap \mathbf{c}$ is equivariant for this action of k^*. Then the desired conclusion follows from the following statement.

(24.6.1) *Let Y_1 be a smooth irreducible locally closed F_q-subvariety of k^n invariant under the k^*-action*

$$(z_1, \ldots, z_n) \to (\lambda^{a_1} z_1, \ldots, \lambda^{a_n} z_n), \qquad \text{where} \quad a_1 > 0, \ldots, a_n > 0,$$

and let $Y = \bar{Y}_1$. Assume that $0 \notin Y_1$. Let $\mathscr{E}_1$ be a local system on Y_1 equivariant under this k^*-action, defined over F_q and pure of weight 0.

Assume that for all $y' \in Y^F - 0$, $\phi_1 : \mathscr{H}^a_{y'} IC(\bar{Y}, \mathscr{E}) \circlearrowleft$ is a-pure.

Then $\phi_1 : \mathscr{H}^a_0 IC(\bar{Y}, \mathscr{E}) \circlearrowleft$ is a-pure.

When $\mathscr{E} = \bar{Q}_l$ this is just [21, 4.5(b)], where it is deduced from the hard Lefschetz theorem of Deligne. The same proof applies when $\mathscr{E} \neq \bar{Q}_l$.

24.7. In this section, k is an arbitrary algebraically closed field.

Let $i = (C, \mathscr{E}) \in I$, $i' = (C', \mathscr{E}') \in I$, and let $j = \tau(i) = (L, C_1, \mathscr{E}_1)$, $a_0 = -\dim C - \dim \mathscr{Z}^0_L$, $\mathscr{W}_j = N(L)/L$, $\mathscr{X}_L =$ lattice of 1-parameter subgroups of $\mathscr{Z}^0_L$ (regarded as a $\mathscr{W}_j$-module).

We shall define $\Omega_{i,i'} \in Q(\mathbf{q})$ ($\mathbf{q}$ an indeterminate), as follows.

If $\tau(i') \neq \tau(i)$, we set $\Omega_{i,i'} = 0$.

If $\tau(i') = \tau(i)$, we set

$$\Omega_{i,i'} = |\mathscr{W}_j|^{-1} \sum_{w \in \mathscr{W}_j} \operatorname{Tr}(\theta_w^{-1}, V_{A_i}) \operatorname{Tr}(\theta_w, V_{A_{i'}})(\mathbf{q} - 1)^b \det(\mathbf{q} - w, \mathscr{X}_L)^{-1}$$
$$\times \sum_{y \in W} \mathbf{q}^{l(y)}\, \mathbf{q}^{(1/2)(\dim C + \dim C' - \dim G - b + 2 \dim \mathscr{Z}^0_L)},$$

where $b = \operatorname{rank} G$ and V_{A_i}, $V_{A_{i'}}$, θ_w are as in 24.2.

It is clear that $\Omega_{i,i'} = \Omega_{i',i}$. We can now state the following result which extends results of Shoji [28] and Beynon–Spaltenstein [20].

THEOREM 24.8. *Assume that k is any algebraically closed field whose characteristic is good for G.*

(a) *For any $i = (C, \mathscr{E}) \in I$ we have $\mathscr{H}^a A_i = 0$ if $a \not\equiv \dim \operatorname{supp} A_i$* (mod 2) *and $\mathscr{H}^a IC(\bar{C}, \mathscr{E}) = 0$ if a is odd.*

(b) *The system of equations*

$$\sum_{\substack{i'_1 \leqslant i_1 \\ i'_2 \leqslant i_2}} \Pi_{i'_1,i_1} \Pi_{i'_2,i_2} \Lambda_{i'_1,i'_2} = \Omega_{i_1,i_2}, \qquad \forall i_1, i_2 \in I.$$

$$\Lambda_{i'_1,i'_2} = 0 \qquad \text{if} \quad i'_1 \not\sim i'_2$$

$$\Pi_{i,i} = 1 \tag{24.8.1}$$

$$\Pi_{i',i} = 0 \qquad \text{if} \quad i' \sim i, i' \neq i$$

with unknowns $\Pi_{i',i}$ $(i' \leqslant i)$, $\Lambda_{i',i}$ $(i' \sim i) \in Q(\mathbf{q})$ has a unique solution. We have

$$\Pi_{i',i} = \sum_m (\mathscr{E}' : \mathscr{H}^{2m+a_0} A_i \mid C')\mathbf{q}^m$$
$$= \sum_m (\mathscr{E}' : \mathscr{H}^{2m} IC(\bar{C}, \mathscr{E}) \mid C')\mathbf{q}^m \tag{24.8.2}$$

where $i' = (C', \mathscr{E}')$, $i = (C, \mathscr{E})$.

(c) *We have*

$$\Pi_{i',i} = 0 \qquad \text{and} \qquad \Lambda_{i',i} = 0 \text{ if } \tau(i') \neq \tau(i).$$

(d) *$\Pi_{i',i}$, $\Lambda_{i',i}$, $\Omega_{i',i}$ are polynomials in $\mathbf{q}$.*

Proof. By general principles, we may assume that k is the algebraic closure of the finite field F_q. We consider an F_q-rational structure on G with Frobenius map F such that F acts trivially on I and such that there exists a maximal torus of G which is F_q-split. We shall also consider for each $s = 1, 2,\ldots$, the F_{q^s}-rational structure on G with Frobenius map F^s. It is clear that $\omega_{i,i'}$ defined in (24.3.4) with respect to F^s is just the value of $\Omega_{i,i'}$ at q^s. Hence the system of equations (24.4.1) (with respect to F^s) is just the system of equations obtained from (24.8.1) by specializing $\mathbf{q} = q^s$. The inductive method used to solve (24.4.1) can be also applied to (24.8.1) and it leads to a set of solutions $\Pi_{i',i}$, $\Lambda_{i',i}$ which are rational functions of $\mathbf{q}$ without pole at $\mathbf{q} = q^s$ $(s = 1, 2,\ldots)$. Moreover, we automatically have $\Pi_{i',i}(q^s) = P_{i',i}$ (with respect to F^s) and $\Lambda_{i',i}(q^s) = \lambda_{i',i}$ (with respect to F^s). By (24.5.2), $\Pi_{i',i}(q^s)$ is an integer for $s = 1, 2,\ldots$, hence $\Pi_{i',i}$ must be a polynomial in $\mathbf{q}$. By 24.4(c), $\lambda_{i',i}$ (with respect to F^s) is a rational number and from the definition (24.3.1) it is an algebraic integer; hence it is an integer. It follows that $\Lambda_{i',i}$ is a polynomial in $\mathbf{q}$. From (24.8.1) it now follows that $\Omega_{i',i}$ are polynomials in $\mathbf{q}$, hence (d).

We now prove (24.8.2). The second equality in (24.8.2) follows from (24.1.1) hence it is enough to show that

$$P_{i'i}(q^s) = \sum_{m \geqslant 1} (\mathscr{E}' : \mathscr{H}^{2m+a_0} A_i \mid C')\, q^{ms},$$

where $i=(C, \mathscr{E})$, $i'=(C', \mathscr{E}')$. (We write $P_{i',i}(q^s)$ for $P_{i',i}$ with respect to F^s.) Let $j=\tau(i)=(L, C_1, \mathscr{E}_1)$, $\phi_{A_i}: F^*A_i \xrightarrow{\sim} A_i$ be as in 24.2. From (24.1.1) we see that $\mathscr{H}^a_g(A_i)=\mathscr{H}^{a-a_0}_g IC(\bar{C}, \mathscr{E})$ for any $g \in \bar{C}$ and any a, where $a_0=-\dim C-\dim \mathscr{Z}^0_L$. From 24.6 it follows that there exists an isomorphism $\Phi: F^*IC(\bar{C}, \mathscr{E}) \xrightarrow{\sim} IC(\bar{C}, \mathscr{E})$ such that $\Phi: \mathscr{H}^{a-a_0}_g IC(\bar{C}, \mathscr{E})\circlearrowleft$ is $(a-a_0)$-pure (see 24.6) for all $g \in \bar{C}^F$ and all a.

Now by (24.1.1), ϕ_{A_i} defines also an isomorphism $F^*IC(\bar{C}, \mathscr{E}) \xrightarrow{\sim} IC(\bar{C}, \mathscr{E})$. By irreducibility of $IC(\bar{C}, \mathscr{E})$, there must exist $\alpha \in \bar{Q}_l$ such that $\phi_{A_i}=\alpha\Phi: \mathscr{H}^{a-a_0}_g IC(\bar{C}, \mathscr{E})\circlearrowleft$ for all $g \in \bar{C}^F$ and all a. Using (24.2.2) and (24.2.4), we see that $\alpha\Phi: \mathscr{H}^0_g IC(\bar{C}, \mathscr{E})\circlearrowleft$ is (a_0+r)-pure $(g \in C^F$, $r=\dim \operatorname{supp} A_i)$. Since $\mathscr{H}^0_g IC(\bar{C}, \mathscr{E}) \neq 0$ for $g \in C^F$ and Φ is 0-pure on it, we deduce that α is an algebraic number all of whose complex conjugates have absolute value $q^{(a_0+r)/2}$. It follows that

$$\phi_{A_i}: \mathscr{H}^0_g A_i\circlearrowleft \text{ is } (a+r)\text{-pure for all } g \in \bar{C}^F \text{ and all } a. \tag{24.8.3}$$

Let us write the equality (24.5.1) for F^s instead of F,

$$P_{i',i}(q^s)=\sum_{a,\lambda} (-1)^{a+a_0}\lambda^s q^{-(s/2)(a_0+r)} \dim E^{a,\lambda}_{\mathscr{E}'}. \tag{24.8.4}$$

By (24.8.3), a in the sum is uniquely determined by λ; hence there are no cancellations in the right-hand side of (24.8.4). On the other hand, as we have seen, $P_{i',i}(q^s)$ is a polynomial in q^s. Since a_0+r is even, it follows that each λ appearing in (24.8.4) must be an integral power of q and in fact, by (24.8.3), must be of form $q^{(a+r)/2}$ with $a+r$ even. Thus, we have

$$P_{i',i}(q^s)=\sum_{\substack{a \\ a \equiv r \pmod 2}} (\mathscr{E}': \mathscr{H}^a A_i \mid C')\, q^{(s/2)(a+r)} q^{-(s/2)(a_0+r)}$$

and $\mathscr{H}^0_g(A_i)=0$ if $a \not\equiv r \pmod 2$. This completes the proof of (a) and (b). Now (c) follows immediately from 24.4(d). The theorem is proved.

Remark 24.9. Solving the system of equation (24.8.1) is the same as decomposing the symmetric matrix $(\Omega_{i,i'})$ into a product of matrices ${}^t\Pi \cdot \Lambda \cdot \Pi$ where Π, Λ are block-matrices (with blocks defined by the equivalence classes for $\sim$ on I) and we want that: Π has an identity matrix in each diagonal block and 0 in each block below diagonal, Λ has 0 in each off-diagonal block.

24.10. From 24.8 we see that the polynomials $\Pi_{i',i}$ and $\Lambda_{i',i}$ can be explicitly computed (by induction as in (24.4.5), (24.4.7)) as soon as the polynomials $\Omega_{i',i}$ are known. The polynomials $\Omega_{i',i}$ are completely determined as soon as the generalized Springer correspondence has been explicitly determined. The generalized Springer correspondence has been described explicitly in [7, 26, 27, 19, 29, 4, 25, 31] except for two small gaps. One gap occurs for type E_8 in characteristic 2 which has in any case been excluded by (23.0.1). The other gap occurs for G almost simple simply connected of type E_6 in characteristic $\neq 3$ and $j = (L, C_1, \mathscr{E}_1)$ where $L/\mathscr{Z}_L^0$ is of type $A_2 \times A_2$, C_1 is the regular unipotent class of L and $\mathscr{E}_1$ is one of the two nontrivial L-equivariant local systems of rank 1 on C_1.

We wish to fill the gap in this case, assuming that the characteristic is $\neq 2, 3$. (The case of characteristic 2 is excluded by (23.0.1).)

For our j, the set $\tau^{-1}(j)$ consists of six elements of form $(C, \mathscr{E})$ where $\mathscr{E}$ is uniquely determined by C. We shall therefore designate these six elements by the corresponding notation for C. They are (with the notation of [31]): E_6, $E_6(a_1)$, A_5A_1, A_5, $2A_2A_1$, $2A_2$ (in decreasing order of dim C). The group $\mathscr{W}_j = N(L)/L$ is a Weyl group of type G_2. The generalized Springer correspondence attaches to each element of $\tau^{-1}(j)$ an irreducible representation of $\mathscr{W}_j$. According to Spaltenstein [31], to E_6 corresponds the unit representation of $\mathscr{W}_j$, to $2A_2$ corresponds the sign representation of $\mathscr{W}_j$ and to $E_6(a_1)$, $2A_2A_1$ correspond the other 2 one-dimensional representations of $\mathscr{W}_j$ (the precise correspondence is given in [31]). Then A_5A_1 and A_5 must correspond to the 2 two-dimensional irreducible representations ρ, ρ' of $\mathscr{W}_j$ (where ρ is the reflection representation) but the methods of [31] are insufficient to decide which of A_5A_1, A_5 corresponds to ρ and which one corresponds to ρ'. We can show that

$$A_5 \textit{ corresponds to } \rho \textit{ and } A_5A_1 \textit{ corresponds to } \rho'. \tag{24.10.1}$$

The method to prove this is as follows. Assume that the opposite is true: A_5 corresponds to ρ' and A_5A_1 corresponds to ρ. We can then use the algorithm (24.4.5), (24.4.7) to compute explicitly the polynomials $\Pi_{i',i}$, $\Lambda_{i',i}$, $i', i \in \tau^{-1}(j)$. For $i' = i =$ element denoted E_6, we find that $\Lambda_{i',i}$ is a polynomial in $\mathbf{q}$ whose value at $\mathbf{q} = q$ does not agree with the value $\lambda_{i',i}$ given by (24.3.1): the value of $\lambda_{i',i}$ is the number of F_q-rational points of C ($i = (C, \mathscr{E})$), for G defined over F_q, while $\Lambda_{i',i}(q)$ is strictly bigger than this number. This contradiction shows that (24.10.1) holds. We see in this way that, in our case, the polynomials $\Pi_{i',i}$ are described by the entries in the following table in which the rows (resp. columns) are indexed by the elements $i \in \tau^{-1}(j)$ (resp. by $i' \in \tau^{-1}(j)$)

	E_6	$E_6(a_1)$	A_5A_1	A_5	$2A_2A_1$	$2A_2$
E_6	1	0	0	q^3	q^6	q^6
$E_6(a_1)$		1	q	q	0	q^8
A_5A_1			1	1	q^5	q^5+q^7
A_5				1	q^3	q^3+q^7
$2A_2A_1$					1	1
$2A_2$						1

One can show that the gaps in the explicit determination of the generalized Springer correspondence in the remaining cases (E_8, $p=2$) and (E_6, $p=2$) can be removed in the same way provided that 24.4 holds in those cases.

COROLLARY 24.11. *Assume that the characteristic of k is good for G. Then for any character sheaf A of G, we have $\mathscr{H}^a A=0$ if $a \not\equiv \dim \operatorname{supp} A \pmod 2$.*

Proof. Let K be as in (8.1.2); since A may be taken to be a direct summand of K, it is enough to show that for any $g \in \operatorname{supp} K$, $\mathscr{H}^a_g K=0$ if $a \not\equiv \dim \operatorname{supp} K \pmod 2$. Using (8.8.5) (in which δ is even), for $g=su$ (Jordan decomposition) we are reduced to the analogous statement with G replaced by $Z^0_G(s)$ and g replaced by u. Thus, we may assume that g is unipotent. Then $\operatorname{supp} K$ contains some unipotent element hence the data (8.1.1) defining K must be $L, \Sigma, \mathscr{E}$, where $\Sigma = C_1 \cdot \mathscr{Z}^0_L$ and C_1 is a unipotent class of L. Moreover, the restriction of K to the unipotent variety of G depends only on the restriction of $\mathscr{E}$ to C_1. Hence we may assume that $\mathscr{E}$ is the inverse image under $C_1 \cdot \mathscr{Z}^0_L \to C_1$ of a G-equivariant local system on C_1. In this case, K is a direct sum of character sheaves A_i ($i \in I$) as in 24.1 (with the same support as K) and the equality $\mathscr{H}^a K=0$ for $a \not\equiv \dim \operatorname{supp} K \pmod 2$ follows from 24.8(a).

COROLLARY 24.12. *Assume that the characteristic of k is good for G. Let $\mathscr{L} \in \mathscr{S}(T)$ and let $w \in W'_L$. Then $\mathscr{H}^a \bar{K}^{\mathscr{L}}_w = 0$ if a is odd.*

Proof. We have $\bar{K}^{\mathscr{L}}_w = \oplus_i {}^pH^i \bar{K}^{\mathscr{L}}_w[-i]$ hence it is enough to show that $\mathscr{H}^{a-i}({}^pH^i \bar{K}_w)=0$ if a is odd. Now ${}^pH^i\bar{K}_w$ is a direct sum of character sheaves A such that $\dim \operatorname{supp} A \equiv i \pmod 2$ (by the parity condition $\varepsilon_A = \hat{\varepsilon}_A$; see 23.1) hence we are reduced to the statement that $\mathscr{H}^{a-i}(A)=0$ if a is odd and $\dim \operatorname{supp} A \equiv i \pmod 2$. But this is just 24.11.

25. CLASS FUNCTIONS ON A REDUCTIVE GROUP OVER A FINITE FIELD

25.1. In this section, we assume that $k=\bar{F}_q$, that G is defined over F_q and that $F: G \to G$ is the corresponding Frobenius map. We shall assume throughout this chapter that G satisfies the restriction (23.0.1). Let $\hat{G}(F_q)$ be the subset of $\hat{G}$ consisting of those character sheaves A for which there exists an isomorphism $F^*A \xrightarrow{\sim} A$. We shall select for each $A \in \hat{G}(F_q)$ an isomorphism $\phi_A: F^*A \xrightarrow{\sim} A$ with the following property: for any $y \in Y_{(L,\Sigma)}$ (where $\operatorname{supp} A = \bar{Y}_{(L,\Sigma)}$, see 3.11) such that $F^n y = y$, the eigenvalues of $\phi^n_A: \mathscr{H}^{-d}_y A \to \mathscr{H}^{-d}_y A$ ($d = \dim \operatorname{supp} A$) are of the form $q^{n(\dim G - d)/2}$ times a root of 1. (The existence of such ϕ_A follows from 14.2(a).)

THEOREM 25.2. *The characteristic functions* $\chi_{A,\phi_A}: G^F \to \bar{Q}_l$ *defined by*

$$\chi_{A,\phi_A}(g) = \sum_a (-1)^a \operatorname{Tr}(\phi_A, \mathscr{H}^a_g A), \qquad g \in G^F, \tag{25.2.1}$$

form a basis for the space of all class functions $G^F \to \bar{Q}_l$.

25.3. For the proof, we shall need a lemma. Before stating it, we recall some earlier notation. We denote by G_{un} the set of unipotent elements in G. Assume that we are given a Levi subgroup L of some parabolic subgroup of G, a unipotent class C of L and an L-equivariant irreducible local system $\mathscr{F}$ on C such that $(\mathscr{Z}^0_L \times C, 1 \boxtimes \mathscr{F})$ is a cuspidal pair for L in the sense of [4, 2.4]. Assume that $FL = L$ and that we are given an isomorphism $\phi_1: F^*\mathscr{F} \simeq \mathscr{F}$. Then the generalized Green function $Q_{L,G,C,\mathscr{F},\phi_1}: G^F_{\text{un}} \to \bar{Q}_l$ is well defined, see (8.3.1).

LEMMA 25.4. *The functions* $Q_{L,G,C,\mathscr{F},\phi_1}$ (*for various* $L, C, \mathscr{F}, \phi_1$ *as in* 25.3) *span the space* $\mathscr{V}$ *of* G^F-*invariant functions* $G^F_{\text{un}} \to \bar{Q}_l$.

Proof. Let C' be a unipotent class in G such that $FC' = C'$ and let $\mathscr{E}'$ be an irreducible G-equivariant local system on C' such that $F^*\mathscr{E}' \simeq \mathscr{E}'$. We choose an isomorphism $\psi: F^*\mathscr{E}' \simeq \mathscr{E}'$ and we define two functions $f_{C',\mathscr{E}'}$, $h_{C',\mathscr{E}'}$ on G^F_{un} by

$$f_{C',\mathscr{E}'}(g) = \begin{cases} \operatorname{Tr}(\psi, \mathscr{E}'_g) & \text{if } g \in C'^F \\ 0 & \text{if } g \notin C'^F \end{cases}$$

$$h_{C',\mathscr{E}'}(g) = \begin{cases} \Sigma(-1)^a \operatorname{Tr}(\psi, \mathscr{H}^a_g IC(\bar{C}', \mathscr{E}')) & \text{if } g \in \bar{C}'^F \\ 0 & \text{if } g \notin \bar{C}'^F \end{cases}$$

It is clear that these functions are in $\mathscr{V}$, that $f_{C',\mathscr{E}'}$ (for various $C', \mathscr{E}'$, as above) span $\mathscr{V}$ and that $h_{C',\mathscr{E}'} = \pm f_{C',\mathscr{E}'} +$ a linear combination of functions $f_{C'',\mathscr{E}''}$ with $C'' \subsetneq \bar{C}'$. By induction on $\dim C'$ we see that each $f_{C',\mathscr{E}'}$ is a linear combination of functions of form $h_{C',\mathscr{E}'}$. Hence the functions $h_{C',\mathscr{E}'}$ also span $\mathscr{V}$. Let $i = (C', \mathscr{E}') \in I$ (see 24.1) for $C', \mathscr{E}'$ as above, and let A_i be as in (24.1.1). Then there exists $\phi: F^*A_i \simeq A_i$. From (24.1.1) we see that the restriction of $\chi_{A_i,\phi}$ to G^F_{un} is equal up to a scalar factor to $h_{C',\mathscr{E}'}$. Hence the functions $\chi_{A_i,\phi} \mid G^F_{\text{un}}$ $(i \in I^F)$ span $\mathscr{V}$. From (10.4.5) and (10.6.1) we see that each such function $\chi_{A_i,\phi} \mid G^F_{\text{un}}$ is a linear combination of generalized Green functions. Hence the generalized Green functions span $\mathscr{V}$, as required.

25.5. *Proof of Theorem* 25.2. From the orthogonality relations (10.8.1) we see that the functions (25.2.1) are linearly independent. (The assumptions of 10.8 are verified by 23.1.) It remains to check that the functions (25.2.1) span the space of all class functions on G^F. First, we assume that G has simply connected derived group. We fix a semisimple element $s_0 \in G^F$. Its centralizer $Z_G(s_0)$ is connected. Let L_1 be the Levi subgroup of a parabolic subgroup of $Z_G(s_0)$ such that $FL_1 = L_1$. Let C_1 be a unipotent class of L_1 such that $FC_1 = C_1$ and let $\mathscr{E}_1$ be an L_1-equivariant irreducible local system on C_1 with a given isomorphism $\phi_1 : F^*\mathscr{E}_1 \xrightarrow{\sim} \mathscr{E}_1$.

Let $L = Z_G(\mathscr{Z}^0_{L_1})$. This is the Levi subgroup of a parabolic subgroup of G and $FL = L$. Let $\hat{C}$ be the unique conjugacy class of L containing $s_0 C_1$ and let $\Sigma = \mathscr{Z}^0_L \cdot \hat{C}$; then $F\hat{C} = \hat{C}$, $F\Sigma = \Sigma$. Let $\hat{\mathscr{E}}_1$ be an L-equivariant irreducible local system on $\hat{C}$ such that the inverse image of $\hat{\mathscr{E}}_1$ under $C_1 \to \Sigma$ ($u \to s_0 u$) is isomorphic to $\mathscr{E}_1$. Then $\hat{\mathscr{E}}_1$ is unique up to isomorphism and there is a unique isomorphism $\hat{\phi}_1 : F^*\hat{\mathscr{E}}_1 \xrightarrow{\sim} \hat{\mathscr{E}}_1$ such that $\mathrm{Tr}(\hat{\phi}_1, (\hat{\mathscr{E}}_1)_{s_0 u_1}) = \mathrm{Tr}(\phi_1, (\mathscr{E}_1)_{u_1})$, for any $u_1 \in C_1^F$.

Let θ: $\mathscr{Z}^{0F}_L \to \bar{Q}_l^*$ be a character. Then there exists a tame local system $\mathscr{G}^\theta$ of rank 1 on $\mathscr{Z}^0_L$ and an isomorphism ψ^θ: $F^*\mathscr{G}^\theta \xrightarrow{\sim} \mathscr{G}^\theta$ such that $\mathrm{Tr}(\psi^\theta, \mathscr{G}^\theta_z) = \theta(z)$ for all $z \in \mathscr{Z}^{0F}_L$.

Let π: $\mathscr{Z}^0_L \times \hat{C} \to \Sigma$ be the map given by multiplication in L. We set $\hat{\mathscr{E}}^\theta = \pi_*(\mathscr{G}^\theta \boxtimes \hat{\mathscr{E}}_1)$. This is a local system on Σ which inherits from $\psi^\theta \boxtimes \hat{\phi}_1$ an isomorphism $\hat{\psi}^\theta$: $F^*\hat{\mathscr{E}}^\theta \xrightarrow{\sim} \hat{\mathscr{E}}^\theta$.

Let K^θ be the perverse sheaf on G defined in terms of $(L, \Sigma, \hat{\mathscr{E}}^\theta)$ in the same way as K was defined in (8.1.2) in terms of $(L, \Sigma, \mathscr{E})$; let $\hat{\psi}$: $F^*K^\theta \xrightarrow{\sim} K^\theta$ be the isomorphism defined in terms of $\hat{\psi}^\theta$ in the same way as ϕ was defined in (8.1.3) in terms of ϕ_0. Let $\Gamma_1 = \{z \in \mathscr{Z}^0_L \mid zs_0$ is L-conjugate to $s_0\}$; it is a subgroup of $\mathscr{Z}^0_L$. We assume that $\theta \mid \Gamma_1^F \equiv 1$ and we compute the characteristic function $\chi_{K^\theta, \hat{\psi}} : G^F \to \bar{Q}_l$ (using 8.5) at any element $su \in G^F$ where s is semisimple and $u \in \mathscr{Z}_G(s)$ is unipotent.

We then take the sum over all characters θ: $\mathscr{Z}^{0F}_L \to \bar{Q}^*$ such that $\theta \mid \Gamma_1^F \equiv 1$ and we find

$$\sum_\theta \chi_{K^\theta, \hat{\psi}}(su)$$

$$= \begin{cases} |\mathscr{Z}^{0F}_1| \, Q_{L_1, Z_G(s_0), C_1, \mathscr{E}_1, \phi_1}(u) & \text{if } s = s_0 \\ 0 & \text{if } s \text{ is not } G^F\text{-conjugate to } s_0 \end{cases}$$

Let us define for any function f: $\{v \in Z_G(s_0)^F \mid v \text{ unipotent}\} \to \bar{Q}_l$ which is invariant under $Z_G(s)^F$-conjugacy a class function $\tilde{f}$: $G^F \to \bar{Q}_l$ by the requirement that

$$\tilde{f}(su) = \begin{cases} f(u) & \text{if } s = s_0 \\ 0 & \text{if } s \text{ is not } G^F\text{-conjugate to } s_0 \end{cases}$$

(s semisimple in G^F, $u \in Z_G(s)^F$ unipotent). Then

$$|\mathscr{Z}^{0F}_L|^{-1} \sum_\theta \chi_{K^\theta, \hat{\psi}} = \tilde{f}, \tag{25.5.1}$$

where

$$f(v) = Q_{L_1, Z_G(s_0), C_1, \mathscr{E}_1, \phi_1}(v). \tag{25.5.2}$$

The left-hand side of (25.5.1) is clearly contained in the $\bar{Q}_l$-vector space M spanned by all functions (25.2.1), hence so is $\tilde{f}$. By 25.4 applied to $Z_G(s_0)$, the functions f in (25.5.2) span the space of all $Z_G(s_0)^F$-invariant functions on unipotent elements in $Z_G(s_0)^F$. Hence the corresponding functions $\tilde{f}$ span the space of all G^F-invariant functions on G^F which are supported on elements with semisimple part conjugate to s_0. It follows that all such functions are in M. Since s_0 was arbitrary, we see that M is the space of all class functions on G^F.

We now drop the assumption that G has simply connected derived subgroup. We can find a connected reductive group G' over F_q, with simply connected derived subgroup, and a surjective homomorphism $\alpha: G' \to G$ defined over F_q whose kernel is a central torus $T_1 \subset G'$.

Then α defines a surjective homomorphism $G'^F \to G^F$. Hence for any class function $f: G^F \to \bar{Q}_l$ there exists a class function $f': G'^F \to \bar{Q}_l$ which is constant on the cosets of T_1^F in G'^F, and is such that

$$f(g) = \sum_{\substack{g' \in G'^F \\ \alpha(g') = g}} f'(g') \qquad \text{for all} \quad g \in G^F. \tag{25.5.3}$$

By the earlier part of the proof, the function f' is a linear combination $f' = \sum_{A'} c_{A'} \chi_{A', \phi_{A'}}$ where the functions $\chi_{A', \phi_{A'}}: G'^F \to \bar{Q}_l$ $(A' \in \hat{G}'(F_q))$, are defined as in (25.2.1) for G' instead of G. Using (25.5.3) we have

$$f(g) = \sum_{A'} c_{A'} \sum_{\substack{g' \in G'^F \\ \alpha(g') = g}} \chi_{A', \phi_{A'}}(g').$$

It remains to show that for each $A' \in \hat{G}'(F_q)$, the function on G^F

$$g \to \sum_{\substack{g' \in G'^F \\ \alpha(g') = g}} \chi_{A', \phi_{A'}}(g') \tag{25.5.4}$$

is a linear combination of functions of form (25.2.1). Given A' as above, there exists a tame local system $\mathscr{L}_1$ of rank 1 on T_1 and an isomorphism $\psi: F^*\mathscr{L}_1 \xrightarrow{\sim} \mathscr{L}_1$ with the properties (a), (b), (c) below. Let $\theta: T_1^F \to \bar{Q}_l^*$ be the character defined by $\theta(t) = \mathrm{Tr}(\psi, (\mathscr{L}_1)_t)$. Then

(a) $\chi_{A', \phi_{A'}}(tg') = \theta(t)\, \chi_{A', \phi_{A'}}(g')$ for all $g' \in G'^F$ and all $t \in T_1^F$.

(b) $\mathscr{L}_1 \approx \bar{Q}_l$ if and only if $\theta \equiv 1$.

(c) If $\mathscr{L}_1 \approx \bar{Q}_l$, then there exists a unique $A \in \hat{G}(F_q)$ such that $A' = \alpha^* A$.

First, assume that $\mathscr{L}_1 \not\approx \bar{Q}_l$. Then $\theta \not\equiv 1$ and using (a), we see that the function (25.5.4) is identically zero. Assume next that $\mathscr{L}_1 \approx \bar{Q}_l$ and let A be as in (c). Then the function (25.5.4) is a multiple of $\chi_{A,\phi_A}: G^F \to \bar{Q}_l$. This completes the proof.

THEOREM 25.6. *Let $A \in \hat{G}(F_q)$, $\phi_A: F^*A \xrightarrow{\sim} A$ be as in 25.1. Let $\phi_A^\vee$: $F^*DA \xrightarrow{\sim} DA$ be the contragredient isomorphism, and let $\phi'_A = q^{\dim G - d}\phi_A^\vee$, ($d = \dim \operatorname{supp} A = \dim \operatorname{supp} DA$). Then*

(a) *$\chi_{A,\phi_A}(g)$ is a cyclotomic integer for any $g \in G^F$.*

(b) *$\chi_{DA,\phi'_A}(g) = \overline{\chi_{A,\phi_A}(g)}$ ($g \in G^F$), where the bar denotes the automorphism of the maximal cyclotomic subfield of $\bar{Q}_l$ which maps each root of 1 to its inverse.*

Proof. It is known on general grounds that $\chi_{A,\phi_A}(g)$ is an algebraic integer, hence in (a) it is enough to show that $\chi_{A,\phi_A}(g)$ belongs to some cyclotomic field.

Let K, ϕ be defined in terms of $(L, \Sigma, \mathscr{E}, \phi_0)$ as in (8.1.3) such that ϕ_0 induces maps of finite order on the stalks of $\mathscr{E}$ at rational points of Σ; let K', ϕ' be defined similarly in terms of $(L, \Sigma, \mathscr{E}^\vee, \phi_0^\vee)$. Using (10.4.5), we see that (a), (b) would follow from the following statement:

(25.6.1) For any $g \in G^F$, $\chi_{K,\phi}(g)$ belongs to a cyclotomic field and $\chi_{K',\phi'}(g) = \overline{\chi_{K,\phi}(g)}$.

(In (10.4.5), we may assume that θ_w are chosen so that $\theta_w \circ \phi$ induces maps of finite order on the stalks of $\mathscr{E}$ at rational points of Σ. In (10.4.4), for $g \in Y_{L,\Sigma}^F$, the map $\theta_w \circ \phi$ on $\mathscr{H}_g^i K$ corresponds to the map $\phi_A \otimes (\theta_w \sigma_A)$ on $\mathscr{H}_g^i(A) \otimes V_A$. Hence $\theta_w \sigma_A: V_A \to V_A$ is $q^{-(\dim G - d)/2}$ times a map of finite order.)

Now let $(L, C, \mathscr{F}, \phi_1)$ be as in (8.3.1) and assume that ϕ_1 induces a map of finite order on the stalks of $\mathscr{F}$ at rational points of C. Using 8.5, we see that (25.6.1) is a consequence of the following statement:

(25.6.2) For any $g \in G_{\mathrm{un}}^F$, $Q_{L,C,\mathscr{F},\phi_1}(g)$ belongs to a cyclotomic field and $Q_{L,C,\mathscr{F}^\vee,\phi_1^\vee}(g) = \overline{Q_{L,C,\mathscr{F},\phi_1}(g)}$.

Using now (10.4.2), we can express $Q_{L,C,\mathscr{F},\phi_1}(g)$ in terms of the functions X_i ($i \in I^F$) in (24.2.8) and we see that (25.6.2) is a consequence of the following statement:

(25.6.3) For any $g \in G_{\mathrm{un}}^F$ and any $i \in I^F$, $X_i(g)$ belongs to a cyclotomic field, $\tilde{X}_i(g)$ belongs to a cyclotomic field and $\tilde{X}_i(g) = \overline{X_i(g)}$.

(Here, $\tilde{X}_i$ is as in (24.2.12).) Now, using (24.2.9) and (24.2.14) we see that (25.6.3) is a consequence of the following two statements:

(25.6.4) For any $g \in G^F_{\mathrm{un}}$, and any $i \in I^F$, $Y_i(g)$ belongs to a cyclotomic field and $\tilde{Y}_i(g) = \overline{Y_i(g)}$.

(25.6.5) For any $i', i \in I^F$, $P_{i',i} = \tilde{P}_{i',i}$ is a rational number.

Statement (25.6.4) is obvious from definitions (24.2.3), (24.2.13) of Y_i, $\tilde{Y}_i$.

Statement (25.6.5) is contained in 24.4(a) and (c). This completes the proof of the theorem.

COROLLARY 25.7. *Let $A, A' \in \hat{G}(F_q)$, $\phi_A : F^*A \cong A$, $\phi_{A'} : F^*A' \cong A'$ be as in* 25.1. *We have*

$$|G^F|^{-1} \sum_{g \in G^F} \overline{\chi_{A,\phi_A}(g)}\, \chi_{A',\phi_{A'}}(g) = \begin{cases} 0 & \text{if } A \neq A' \\ 1 & \text{if } A = A'. \end{cases}$$

(This should be understood as follows: we assume given a set of representatives of the isomorphism classes of character sheaves A such that $F^*A \approx A$ and for each A in this set we assume given ϕ_A as in 25.1. Then A, A' in the corollary are assumed to be in this set. Hence we have $A \approx A'$ if and only if $A = A'$ and then $\phi_A = \phi_{A'}$.)

Proof. This follows immediately from 10.8 and 25.6.

REFERENCES[1]

4. G. LUSZTIG, Intersection cohomology complexes on a reductive group, *Invent. Math.* **75** (1984), 205–272.
5. G. LUSZTIG, Character Sheaves, I, *Adv. in Math.* **56** (1985), 193–237.
6. G. LUSZTIG, Characters of reductive groups over a finite field, *in* "Ann. of Math. Studies 107," Princeton Univ. Press, Princeton, N. J., 1984.
7. T. A. SPRINGER, Trigonometric sums, Green functions of finite groups and representations of Weyl groups, *Invent. Math.* **36** (1976), 173–207.
13. G. LUSZTIG, Character Sheaves, II, *Adv. in Math.* **57** (1985), 226–265.
17. G. LUSZTIG, Character Sheaves, III, *Adv. in Math.* **57** (1985), 266–315.
19. D. ALVIS AND G. LUSZTIG, On Springer's correspondence for simple groups of type E_n ($n = 6, 7, 8$), *Math. Proc. Cambridge Philos. Soc.* **92** (1982), 65–72.
20. W. M. BEYNON, N. SPALTENSTEIN, Green functions of finite Chevalley groups of type E_n ($n = 6, 7, 8$), *J. Algebra* **88** (1984), 584–614.
21. D. KAZHDAN, AND G. LUSZTIG, Schubert varieties and Poincaré duality, in "Proc. Symp. Pure Math.," Vol. 36, pp. 185–203, Amer. Math. Soc., Providence, R. I., 1980.
22. G. LUSZTIG, Green polynomials and singularities of unipotent classes, *Advan. in Math.* **42** (1981), 169–178.
23. G. LUSZTIG, Irreducible representations of finite classical groups, *Invent. Math.* **43** (1977), 125–175.

[1] References [4–7, 13, 17] are from earlier parts of this series.

24. G. Lusztig, Character Sheaves, IV, *Adv. in Math.* **59** (1986), 1–63.
25. G. Lusztig and N. Spaltenstein, On the generalized Springer correspondence for classical groups, *in* "Algebraic Groups and Related Topics," (R. Hotta, Ed.), Advanced Studies in Pure Mathematics 6, North Holland Kinokunia, 1985.
26. T. Shoji, On the Springer representations of Weyl groups of classical algebraic groups, *Comm. Algebra* **7** (1979), 1713–1745; 2027–2033.
27. T. Shoji, On the Green polynomials of a Chevalley group of type F_4, *Comm. Algebra* **10** (1982), 505–543.
28. T. Shoji, On the Green polynomials of classical groups, *Invent. Math.* **74** (1983), 239–264.
29. N. Spaltenstein, Appendix, *Math. Proc. Cambridge Philos. Soc.* **92** (1982), 73–78.
30. N. Spaltenstein, Existence of good transversal slices to nilpotent orbits in good characteristic, *J. Fac. Sci. Univ. Tokyo Sect. I(A)* **31** 283–286.
31. N. Spaltenstein, On the generalized Springer correspondence for exceptional groups, *in* "Algebraic Groups and Related Topics," (R. Hotta, Ed.), Advanced Studies in Pure Mathematics 6, North Holland, Kinokunia, 1985.
32. T. A. Springer, A purity result for fixed point varieties in flag manifolds, *J. Fac. Sci. Univ. Tokyo Sect. I(A)* **31** 271–282.