Let G be a connected reductive gp $/ k = \bar{k}$.

Let $L \subset \underset{\underset{L \ltimes U_P}{\shortparallel}}{P} \subset G$ be a Levi and parabolic.

We look at

$$\begin{array}{ccccccc} L & \xleftarrow{\alpha} & G \times P & \xrightarrow{\beta} & G \times_P P & \xrightarrow{\delta} & G \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \bar{p} & \longleftarrow & (g, p) & \longmapsto & (g, p) & \longmapsto & g p g^{-1} \end{array}$$

$\forall\, K \in \mathrm{Perv}^L(L)$, $\exists\, \tilde{K} \in \mathrm{Perv}^G(G \times_P P)$ s.t.

$$\alpha^* K[2 \dim U_P] = \alpha^* K[\dim G - \dim L] = \beta^* \tilde{K}.$$

We define

$$\mathrm{ind}_P^G K := \delta_* \tilde{K}$$

Since δ is proper, $\mathrm{ind}_P^G K$ is semisimple if K is, and ${}^p H^i(\mathrm{ind}_P^G K)$ are G-equiv.

__Fact__ $Q = M \cdot U_Q$ another parabolic with $\begin{matrix} M \subset L \\ Q \subset P \end{matrix}$. Then

$$\mathrm{ind}_Q^G = \mathrm{ind}_P^G \circ \mathrm{ind}_{Q \cap L}^L$$

Prop 7.1.7. Let $K_1 \in \mathrm{Perv}^G(G)$, $K \in \mathrm{Perv}^L(L)$ and suppose $\mathrm{res}_P^G K_1$ has perverse degree ≤ 0. Then

$$\mathrm{Hom}_{DG}(K_1, \mathrm{ind}_P^G K) = \mathrm{Hom}_{DL}(\mathrm{res}_P^G K_1, K)$$

Lemma 7.2.5 Let $K_1 \in \mathrm{Perv}(G)^G$ be s.s.

$\mathrm{res}_P^G K_1 = \bigoplus K_\lambda[n_\lambda]$ where $K_\lambda \in \mathrm{Perv}(L)^L$ are s.s.

$\mathrm{ind}_P^G K_\lambda \in \mathrm{Perv}(G)^G$. Then $\mathrm{res}_P^G K_1$ has perv. deg ≤ 0

Let $\Sigma_1 \subset L$ be an isolated class, i.e. Σ_1 is the preimage under $L \twoheadrightarrow L/Z(L)^0$ of a conjugacy class in $L/Z(L)^0$ and, for any/every $x \in \Sigma_1$, we have

$Z(Z_L(x_s))^0 = Z(L)^0$ (x_s = semisimple part of x). Let

$$\Sigma_r := \{ x \in \Sigma_1 \mid Z_G(x_s)^0 \subset L \}$$

$$Y = Y_{(L,\Sigma)} := \mathrm{Ad}(G)(\Sigma_r).$$

We now look at the big commutative diagram

smooth smooth proper

$$
\begin{tikzcd}
 & G \times \Sigma_r \arrow[dl, "\alpha_1"'] \arrow[r, "\beta_1"] \arrow[d, phantom, "\cap"] & G \times_L \Sigma_r \arrow[d, "\wr"] \arrow[dr, "\delta_1"] & \\
\Sigma_r \arrow[d, "\text{dense open}"'] \arrow[dr, phantom, "\square"] & G \times (\Sigma_r \cdot U_P) \arrow[l, "\alpha_2"'] \arrow[r, "\beta_2"] \arrow[d] \arrow[dr, phantom, "\square"] & G \times_P (\Sigma_r \cdot U_P) \arrow[r, "\delta_2"] \arrow[d] & Y \arrow[d] \\
\overline{\Sigma_r} \arrow[d, phantom, "\cap" description, "\text{closed}"'] \arrow[dr, phantom, "\square"] & G \times (\overline{\Sigma} \cdot U_P) \arrow[l, "\alpha"'] \arrow[r, "\beta"] \arrow[d, phantom, "\cap"] \arrow[dr, phantom, "\square"] & G \times_P \overline{\Sigma} \cdot U_P \arrow[r, "\delta"] \arrow[d, phantom, "\cap"] & \overline{Y} \arrow[d, phantom, "\cap"] \\
L \arrow[d, phantom, "\ni"] & G \times P \arrow[l, "\alpha"'] \arrow[r, "\beta"] \arrow[d, phantom, "\ni"] & G \times_P P \arrow[r, "\delta"] \arrow[d, phantom, "\ni"] & G \arrow[d, phantom, "\ni"] \\
p & (g, p) \arrow[l, mapsto] \arrow[r, mapsto] & (g, p) \arrow[r, mapsto] & gpg^{-1}
\end{tikzcd}
$$

$\forall\ \mathcal{L} \in Loc(\overline{\Sigma})^L$, $\exists\ \tilde{\mathcal{L}} \in Loc(G \times_L \Sigma_r)^G$ s.t. $\alpha_1^* \mathcal{L} = \beta_1^* \tilde{\mathcal{L}}$.

The map δ_1 is Galois with Galois gp $W(L, \Sigma) := N_G(L, \Sigma)/L$.

Hence $(\delta_1)_* \tilde{\mathcal{L}}$ is semisimple if $\mathcal{L}$ is.

Let $K = IC(\overline{\Sigma_1}, \mathcal{L})$; every cuspidal K on L is of this form.

But we don't assume K cuspidal.

<u>Thm 7.2.2.</u> ([CSO, Prop. 4.5]) $\quad \mathrm{ind}_P^G K = IC(\overline{Y}, (\delta_1)_* \tilde{\mathcal{L}})$

pf: We have $\alpha_1^* \mathcal{L} = \beta_1^* \tilde{\mathcal{L}} \Rightarrow \alpha_2^* \mathcal{L} = \beta_2^* \tilde{\mathcal{L}}$

Let $\tilde{K} := IC(G \times_P (\overline{\Sigma} \cdot U_P), \tilde{\mathcal{L}})$. Since α, β are smooth

with connected nonempty fibers we have $\alpha^* K[2 \dim U_P] = \beta^* \tilde{K}$.

Also $\tilde{K}|_{G \times_P (\Sigma_{r} \cdot U_P)} = \tilde{\mathcal{L}}[\dim Y]$

and $\delta_* \tilde{K}|_Y = (\delta_2)_* \tilde{K}|_{G \times_P \Sigma_{r} U_P} = (\delta_2)_* \tilde{\mathcal{L}}[\dim Y] = (\delta_1)_* \tilde{\mathcal{L}}[\dim Y]$

The goal is to show

$$\mathrm{ind}_P^G K = \delta_* \tilde{K} = IC(\overline{Y}, (\delta_1)_* \tilde{\mathcal{L}})$$

It suffices to prove that $\mathrm{ind}_P^G K = j_* \tilde{K}$ satisfies the dimension inequality for intermediate ext. from Y to $\bar{Y}$, (and the same for $D(\mathrm{ind}_P^G K) = \mathrm{ind}_P^G(D(K))$), i.e.

$$\dim \operatorname{supp} H^i(j_* \tilde{K}) < -i \quad \text{when } i > -\dim Y.$$

We have $\overline{\Sigma_1} = \bigsqcup S_j$ where $S_j \subset L$ are $L \times Z(L)^\circ$-orbits.

This is a finite union, w.r.t. which K is constructible.

Let $X = G \times_P (\overline{\Sigma_1} \cdot U_P)$ and $X_j = G \times_P (S_j \cdot U_P)$, so that

$X = \bigsqcup X_j$. Suppose $S_0 = \Sigma_1$ and $X_0 = G \times_P (\Sigma_1 \cdot U_P)$.

We have K is constructible w.r.t. $\overline{\Sigma_1} = \bigsqcup S_j$

$\Rightarrow \alpha^* K = \beta^* K$ is ——— $G \times (\overline{\Sigma_1} U_P) = \bigsqcup G \times (S_j U_P)$

$\Rightarrow \tilde{K}$ is ——— $X = \bigsqcup X_j$

Suppose $g \in \operatorname{supp} H^i(\delta_* \tilde{K})$, i.e.

$$H^i(\delta_* \tilde{K})_g = H^i(\delta^{-1}(g), \tilde{K}) \neq 0.$$

$$\Rightarrow H^i(\delta^{-1}(g) \cap X_j, \tilde{K}) \neq 0 \text{ for some } j.$$

$$\Rightarrow \exists p+q = i \text{ i.e. } H^p(\delta^{-1}(g) \cap X_j, H^q(\tilde{K})) \neq 0.$$

$$\Rightarrow \begin{cases} p \leq 2\dim(\delta^{-1}(g) \cap X_j) \\ q \leq -\dim X_j, \text{ with } < \text{ if } j \neq 0 \end{cases}$$

$$\Rightarrow i \leq 2\dim(\delta^{-1}(g) \cap X_j) - \dim X_j \quad \text{for some } j.$$
$< \text{ if } j \neq 0.$

The theorem now follows from

<u>Lemma 7.2.3.</u> (i) $\forall j \neq 0$

$$\dim \{ g \in Y \mid \dim(\delta^{-1}(g) \cap X_j) > \tfrac{1}{2}(i + \dim X_j) \} < -i$$

(ii) For $j = 0$

$$\dim \{ g \in Y \mid \dim(\delta^{-1}(g) \cap X_0) > \tfrac{1}{2}(i + \dim Y) \} < -i \quad \text{when } i + \dim Y > 0.$$

pf Recall $X_j = G \times_P (S_j \cdot U_P)$. Consider

$$Z_j := \{ (g, x_1 P, x_2 P) \in G \times G/P \times G/P \mid x_1^{-1} g x_1,\ x_2^{-1} g x_2 \in S_j \cdot U_P \}.$$

The fiber of $Z_j \xrightarrow{pr_1} \overline{Y}$ is $(\delta^{-1}(g) \cap X_j)^2$. Hence

that (i) is valid $\forall i \iff \dim Z_j \le \dim X_j = \dim S_j + \dim G - \dim L$

This is shown in Prop. 6.3.2.

Similarly, that (ii) is valid when $i + \dim Y > 0$

$\iff \dim Z_0 \le \dim X_0 = \dim Y$ and that all components

of Z_0 of dim equal to $\dim Y$ maps dominantly to $\overline{Y}$.

(Also shown in Prop. 6.3.2)

Ex $L = T$, $P = B$, the only $S_j = S_0 = T$. $X_0 = G \times_B B \cong G$.

$Y = G^{rs}$, $\overline{Y} = G$.

$$Z_0 = \{ (g, x_1 B, x_2 B) \mid x_1^{-1} g x_1,\ x_2^{-1} g x_2 \in B \}$$

$$\cong \{ (g, B_1, B_2) \in G \times \mathcal{B} \times \mathcal{B} \mid g \in B_1 \cap B_2 \}$$