Let $k = \bar{k}$ and G be a connected red. gp$/k$

Fix $\ell \neq \mathrm{char}(k)$ and $E = \overline{\mathbb{Q}}_\ell$ our coefficients.

Fix $\xi \in \hat{X} := X^*(T) \otimes (\mathbb{Z}_{(p)}/\mathbb{Z})$, a Kummer local system.

Recall we have $C_{\xi,w} \in D(\frac{G}{G})$; it is given by

$A_{\xi,w} \in \mathrm{Perv}(U \backslash \overline{BwB}/U)$ and

$$A_{\xi,w} \dashrightarrow E \otimes A_{\xi,w}(\cdot) \dashleftarrow \tilde{A}_{\xi,w} \qquad C_{\xi,w} = r_* \tilde{A}_{\xi,w}$$

$$G \xleftarrow{pr_2} G \times G \to G/B \times G \xrightarrow{r} G$$

We have $C_{\xi,w} = \bigoplus_{A \in \hat{G}(\xi)} A(-i)^{\oplus [A : {}^pH^i(C_{\xi,w})]}$. We would like to classify such A, which is closely related to determine $[A : {}^pH^i(C_{\xi,w})]$.

Method: Using some standard trick, may assume $k = \overline{\mathbb{F}_q}$.

Replacing q by q^n if necessary, we may assume G and T split$/\mathbb{F}_q$, d = order of ξ divides $q-1$ so that $\mathrm{Gal}(T \xrightarrow{d} T) \cong \mu_d(T)$ is constant $\Rightarrow$ ξ can be realized as $T^F \xrightarrow{\xi} E^*$.

Also may assume all $A \in \hat{G}(\xi)$ are defined$/\mathbb{F}_q$, i.e. $F^* A \cong A$. Since we showed that A is admissible, and all admissible complexes are IC extensions of local systems of finite monodromy, we can choose $\varphi : F^*A \xrightarrow{\sim} A$ pure of wt. 0.

For any $g \in G^F$, denote by $\chi_{A,\varphi}(g) := Tr(\varphi_g : A|_g)$

$$:= \sum_i (-1)^i Tr(\varphi_g : \mathcal{H}^i(A)|_g)$$

<u>Thm 1</u> Assume G is <u>clean</u> (see below). Then for A, A' as above

$$\langle \chi_{A,\varphi}, \chi_{A',\varphi'} \rangle_{G^F} := |G^F|^{-1} \sum_{g \in G^F} \chi_{A,\varphi}(g) \chi_{A',\varphi'}(g) = \begin{cases} 0 & \text{if } A' \not\cong DA \\ q^{-\dim G} & \text{else} \end{cases}$$

Say G is <u>clean</u> if for any Levi $L \subset G$, all cuspidal perverse sheaves on L are clean. i.e. they are of the form $IC(\overline{\Sigma_i}, \mathcal{L})$ whose restriction to $\overline{\Sigma_i} \setminus \Sigma_i$ is zero, for an isolated stratum $\Sigma_i \subset L$.

<u>Fact</u> Any connected reductive G is clean

This was proved in CS4 & CS5 by Lusztig (case-by-case) for G of type A, B, C, D and exceptional types when p is good. By further work of Lusztig and Ostrik, all G are clean.

Ex $G = SL_2$. We have a cuspidal perverse sheaf A and $\varphi : F^*A \xrightarrow{\sim} A$

s.t. $\chi_{A,\varphi}(g) = \begin{cases} \pm q^{-1} & \text{if } g \in G^F \text{ is reg unip} \\ 0 & \text{else} \end{cases}$ since A is clean.

Let $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G^F$ be reg. unip.. Then half of regular unip. elements

are of the form $\mathrm{Ad}(g)u$, $g \in G^F / Z_G(u)^F$. We have

$$\left| G^F / Z_G(u)^F \right| = \frac{|G^F|}{|(Z_G(u)^\circ)^F| \cdot |\pi_0(Z_G(u))^F|} = \frac{1}{2q}|G^F| \quad \text{because } Z_G(u)^\circ \text{ is unipotent.}$$

Hence the total number of unip elements in the geometric orbit is

$\frac{1}{q}|G^F|$ and thus $\langle \chi_{A,\varphi}, \chi_{A,\varphi} \rangle = \frac{1}{q^3}$.

Meanwhile, $A_{\xi,w}$ corresponds to explicit function

$$c_w \in \mathcal{H}'_\xi(q^{\frac{1}{2}}) \overset{:= \mathcal{H}'_\xi \otimes_{\mathbb{Z}[t,t^{-1}]} E(q^{\frac{1}{2}})}{\cong} E(q^{\frac{1}{2}})\left[(B^F, -\xi) \backslash G^F / (B^F, \xi) \right]^{\text{s.c.}}$$ and let $\gamma_{\xi,w}$ be the

G^F / B^F-average of the function, which corresponds to $c_{\xi,w}$.

<u>Lemma</u>[CS3, Cor. 13.7] For $w, x \in W'_\xi := \mathrm{Stab}_W(\xi)$

$$\langle \gamma_{\xi,w}, \gamma_{-\xi,x} \rangle_{G^F} := |G^F|^{-1} \sum_{g \in G^F} \gamma_{\xi,w}(g)\, \gamma_{-\xi,x}(g)$$

is equal to the trace of $f \mapsto q^{\frac{1}{2}(\ell(w)+\ell(w'))} c_{\xi,x^{-1}} f\, c_{\xi,w}$

on $\mathcal{H}'_\xi(q^{\frac12}) := E[(B^F,-\xi)\backslash G^F/(B^F,\xi)] = \mathrm{End}_{G^F}(G^F/(B^F,\xi))$

The algebra $\mathcal{H}'(t) := \mathcal{H}' \otimes E(t)$ is semisimple, with its irreducible modules parametrized by $\mathrm{Irr}(W'_\xi)$; we denote by $M(t)$ the irred $\mathcal{H}'(t)$-module corresp. to $M \in \mathrm{Irr}(W'_\xi)$ (so $M(1) = M$). Recall we have $\mathcal{H}'/[\mathcal{H}',\mathcal{H}'] \xrightarrow{\tau} E(t)[\hat{G}(\xi)]$.

<u>Thm 3</u> [CS?, Thm 14.9] [MS, Thm 11.2.4] $\exists$ function $c : \hat{G}(\xi) \times W'_\xi \to E$ s.t. $\forall u \in \mathcal{H}'$ we have

$$\tau(u) = \sum_{\substack{A \in \hat{G}(\xi) \\ M \in \mathrm{Irr}(W'_\xi)}} c(A,M)\, Tr(u : M(t))\, [A]. \qquad (*)$$

The proof has two steps. The first step is to use Thm 2 and Lemma 2. Write $u(q^{\frac12}) \in E[(B^F,\xi)\backslash G^F/(B^F,\xi)]$ to be the specialization of u under $t \mapsto q^{\frac12}$. Then $(*)$ + Thm 1 gives

$$\langle u(q^{1/2}),\, u'(q^{\frac12})\rangle = q^{\dim G} \sum_{A,M,M'} c(A,M)\, c(DA,M')\, Tr(u : M(q^{\frac12}))\, Tr(u' ; M'(q^{\frac12}))$$

Prop 4 [CS3, Prop. 17.12][MS, Cor. 11.7.8] For $M, M' \in \mathrm{Irr}(W_\xi')$ we have

$$\sum_{A \in \hat{G}(\xi)} c(A, M) \cdot c(A, M') = \begin{cases} 1 & \text{if } M^* \cong M' \\ 0 & \text{else} \end{cases}$$

Step 2: Argue that $c(A, M) = f(q^{\frac{1}{2}})$ for some $f \in \mathbb{Q}^{ab}[t, t^{-1}]$ independent of f, and $c(A, M^*) = \overline{c(A, M)}$

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§2-sided cells.

Recall we had canonical basis $(c_w)_{w \in W_\xi'}$ for $\mathcal{H}'$.

There is $(h_{xyz})_{x,y,z \in W_\xi'}$, $h_{xyz} \in \mathbb{Z}_{\geq 0}[t^{\pm 1}]$, s.t.

$c_x c_y = \sum_z h_{xyz}(t) c_z$. Whenever $h_{xyz} \neq 0$ we declare $z \leq_L y$ and $z \leq_R x$. Then $\leq_L$ and $\leq_R$ are transitive.

$\leq_L$ and $\leq_R$ together generate a relation $\leq_{LR}$, and we say $x \sim_{LR} y$ if $x \leq_{LR} y$ and $y \leq_{LR} x$. in which case we say x and y are in the same 2-sided cell.

For any $x \in W'_\xi$, $\{C_y \mid y \leq_{LR} x\}$ span some 2-sided ideal $I'_x \trianglelefteq \mathcal{H}'$. For $M \in \mathrm{Irr}(W'_\xi)$ we say $M \leq_{LR} x$ if M is annihilated by I'_x. We say $M \sim_{LR} x$ if $M \leq_{LR} x$ and $M \not\leq_{LR} x$ if $y \lneq_{LR} x$.

For $M, M' \in \mathrm{Irr}(W'_\xi)$, we say $M \sim_{LR} M'$ if $\exists x \in W'_\xi$ s.t. $M \sim_{LR} x$, $M' \sim_{LR} x$.

Thm [CS3, Thm. 16.6] [MS, Thm 11.7.2] Suppose $c(A, M) \neq 0$ and $c(A, M') \neq 0$. Then $M \sim_{LR} M'$.

Cor $\exists$ surjection $\hat{G}(\xi) \twoheadrightarrow \mathrm{Irr}(W'_\xi)/\sim_{LR}$

Rmk In [CS5, Thm 23.1] $c(A,M)$ are given explicitly via (non-abelian) Fourier transform; this seems to be a case-by-case proof.

Ex $G = GL_2$, $\xi = 0$. $W'_\xi = W_\xi = W$ is of type A_1.

We have $\mathcal{H}' = \mathbb{Z}[t, t^{-1}][C_1] / \langle C_1(C_1 - t^2 - 1) \rangle$. We have

$(\mathrm{triv})(t) = \mathcal{H}'_t/(c_1 - t^2 - 1)$ and $(\mathrm{sgn})(t) = \mathcal{H}'_t/c_1$. Also put $c_e = 2$.

Then

$\mathrm{Tr}(c_e, (\mathrm{triv})(t)) = 2$	$\mathrm{Tr}(c_e, (\mathrm{sgn})(t)) = 1$	
$\mathrm{Tr}(c_1, (\mathrm{triv})(t)) = t^2+1$	$\mathrm{Tr}(c_e, (\mathrm{sgn})(t)) = 0$	

$$\langle \gamma_e, \gamma_e \rangle_{G^F} = 2, \quad \langle \gamma_e, \gamma_1 \rangle_{G^F} = q+1, \quad (\gamma_1, \gamma_1) = (q+1)^2$$

$\exists A_{\mathrm{triv}}, A_{\mathrm{sgn}} \in \hat{G}(1)$ s.t.

$$[C_e] = [A_{\mathrm{sgn}}] + [A_{\mathrm{triv}}]$$

$$[C_1] = (t + t^{-1})[A_{\mathrm{triv}}]$$

$$c(A_i, M_j) = \delta_{ij}, \quad i,j \in \{\mathrm{triv}, \mathrm{sgn}\}$$