Character Sheaves: Preliminaries

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1. Kummer local systems

Let T be an algebraic torus over $\Bbbk = \overline{\Bbbk}$, $X = \mathrm{Hom}(T, \mathbb{G}_m)$, $p \stackrel{*}{=} \mathrm{char}\Bbbk$ (1 if $\mathrm{char}\Bbbk = 0$) , and $E = \overline{\mathbb{Q}_\ell}$ or $\mathbb{C}$ (coefficient field). Let

$$\hat{X} = \hat{X}(T) = \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} X \Big/ 1 \otimes_{\mathbb{Z}} X$$

Note that ${}^a\!/_b \otimes \alpha$ has b torsion and since $p \not| b$ we see that $\hat{X}$ has no $p-$torsion.

Definition 1.1. *A rank 1 local system $\mathcal{L}$ on T is Kummer if $\exists n \neq 0$ coprime to p s.t. $\mathcal{L}^{\otimes n} \cong \underline{E}$.*

Let $(m, p) = 1$, then we have the following SES of group schemes[1]

$$0 \to {}_mT \to T \xrightarrow{(\cdot)^m} T \to 0$$

where on $R-$points $(\cdot)^m(R) : T(R) \ni r \mapsto r^m$ and ${}_mT(R) = \{r \in T(R) |\, r^m = 1\}$. Fix isomorphism ψ between roots of unity coprime to p in $\Bbbk$ with same thing in E [2]. Given $\alpha \in X$, define $\chi_{\alpha,m} :{}_mT(\Bbbk) \to E^*$ by

$$\chi_{m,\alpha}(\Bbbk)(z) = \psi(\alpha(z))$$

$(\cdot)^m$ is an étale Galois covering and we have $\mathrm{Gal}((\cdot)^m) \cong_m T(\Bbbk)$ as all deck transformations are of the form $t \mapsto \zeta t$ where $\zeta \in_m T(\Bbbk)$. Since $\pi_1^{et}(T, e) \twoheadrightarrow \mathrm{Gal}((\cdot)^m)$, $\chi_{\alpha,m}$ gives an étale local system $\mathcal{L}_{\alpha,m}$ on T.

Lemma 1.2. *(i)* $\mathcal{L}_{nm,n\alpha} \cong \mathcal{L}_{m,\alpha}$

(ii) $\mathcal{L}_{m,\alpha} \cong \mathcal{L}_{n,\beta} \iff m\beta - n\alpha \in mnX$

(iii) $\mathcal{L}_{m,\alpha}$ *is trivial* $\iff \alpha \in mX$

Proof. (i) Since $n\alpha(t) = \alpha(t^n)$, the following diagram commutes

$$\begin{tikzcd} \mathrm{Gal}((\cdot)^{nm}) \arrow[rr, "n\alpha"] \arrow[dr, "\zeta\mapsto\zeta^n"'] & & \mathbb{G}_m \\ & \mathrm{Gal}((\cdot)^m) \arrow[ur, "\alpha"'] & \end{tikzcd} \tag{1}$$

The transition maps in $\pi_1^{\acute{e}t}(T, e)$ are given as such. A map $f : X \to Y$ of Galois covers over S induces the natural map $\mathrm{Gal}(X/S) \to \mathrm{Gal}(Y/S)$ by restriction to the fiber over $s \in S$, $X(s) \to Y(s)$ as $\mathrm{Gal}(X/S) \cong X(s)$. Apply this to

$$\begin{tikzcd} T \arrow[rr, "(\cdot)^n"] \arrow[dr, "(\cdot)^{nm}"'] & & T \arrow[dl, "(\cdot)^m"] \\ & T & \end{tikzcd}$$

[1] In the étale topology

[2] via Hensel's lemma

we find that the map to $\mathrm{Gal}((\cdot)^m)$ in Eq. (1) is the transition map and so in the inverse limit $\alpha, n\alpha$ gives us the same representation. ∎

Dividing (ii) by mn, we see that $\mathcal{L}_{n,\alpha}$ only depends on the class of $\xi = m^{-1} \otimes \alpha \in \hat{X}$ so write $\mathcal{L}_\xi = \mathcal{L}_{m,\alpha} (= \mathcal{L}_{\xi,T})$.

Remark.

$$\mathcal{L}_\xi \otimes \mathcal{L}_\eta = \mathcal{L}_{\xi+\eta} \tag{2}$$

Thus by (iii), $\mathcal{L}_\xi$ is a Kummer local system.

$\varphi : T \to T'$ induces $\hat{\varphi} : \hat{X}(T') \to \hat{X}(T)$ by postcomposition.

☆ **Lemma 1.3.** *For $\xi \in \hat{X}(T')$ we have $\varphi^* \mathcal{L}_{\xi,T'} = \mathcal{L}_{\hat{\varphi}(\xi),T}$*

Theorem 1

$\xi \mapsto \mathcal{L}_\xi$ defines an isomorphism

$$\hat{X}(T) \xrightarrow{\sim} \{\text{Kummer local systems on } T\}/_\sim =: \mathrm{KLS}(T)$$

Proof. We just need to show surjectivity. (1) Because $(\cdot)^m$ is Galois, constituents of $(\cdot)^m_*(\underline{E}) \longleftrightarrow$ representations of $\mathrm{Gal}(\cdot)^m$, which is abelian and thus

$$(\cdot)^m_*(\underline{E}) = \bigoplus_{\xi \in \hat{X},\ m\xi = 0} \mathcal{L}_\xi \tag{3}$$

(2) Given $\mathcal{L}$ a Kummer local system, $\mathcal{L}$ is defined by a one dimensional representation $\rho : \pi_1(T, e) \to E^\times$ s.t. $\mathrm{im}\,\rho$ lies in the group of $m-$th roots of unity, for some m prime to p. By the Galois correspondence, this gives us a Galois cover $\pi : M \to T$ s.t. $\mathrm{Gal}(M/T) = \mathrm{im}\,\rho$ and $\mathcal{L}$ will be a constituent of $\pi_*(\underline{E})$ exactly as above. But $\mathrm{im}\,\rho$ is a subgroup of $E^\times$ and thus cyclic of degree dividing m.

(2) T is normal, and Z is also normal, being finite over T and thus $k(M)/k(T)$ is Galois with Galois group $\mathrm{im}\,\rho$. By Kummer theory(for fields), it follows that $k(M) = k(T)(a^{1/m})$ for some $a \in k(M)$. Consider the map $k(M) \to k(T)$ given by the $m-$th power map. π Galois $\implies$ it's unramified and so we can extend the birational map $T \dashrightarrow M$ to an actual map? As $k(T) \to k(M)$ is just inclusion it follows that

$$\begin{tikzcd} T \arrow[r] \arrow[rr, bend left, "(\cdot)^m"] & M \arrow[r, "\pi"] & T \end{tikzcd}$$

and thus all constituents of $\pi_*(\underline{E})$ will be constituents of $(\cdot)^m_*(\underline{E})$ so by Eq. (3) we are done. ∎

Lemma 1.4. *If $\xi \neq 0$, then $H^*(T, \mathcal{L}_\xi) = H^*_c(T, \mathcal{L}_\xi) = 0$.*

Proof. Taking cohomology of Eq. (3) we have that

$$H^*(T, \underline{E}) = H^*(T, \underline{E}) \oplus \bigoplus_{\xi \neq 0,\ m\xi = 0} H^*(T, \mathcal{L}_\xi)$$

∎

Lemma 1.5. $\mathbb{D}(\mathcal{L}_\xi) \cong \mathcal{L}_{-\xi}[2 \dim T]$.

Proof. Consider the analogous situation over $\mathbb{C}$, and X a topological $\mathbb{C}$ manifold. Then for a local system $\mathbb{D}(\mathcal{L}) = \mathcal{L}^\vee[\dim X]$. ∎

1.1. Finite Ground Fields

Now assume $\Bbbk = \overline{\mathbb{F}_q}$, $F =$(absolute)Frobenius. $F \curvearrowright \widehat{X}$ and KLS(T) by $F^\times$.

Proposition 2

$$\mathrm{KLS}(T)^F \cong \mathrm{Hom}_{\mathrm{Grp}}(T^F, E^*)$$

Proof. By Lemma 1.3 and Theorem 1 we have $\mathrm{KLS}(T)^F \cong (\hat{X})^F$. As $F(\alpha) = q\alpha$, it follows that any $(\hat{X})^F$ must be generated by fixed elements of the form $\dfrac{a}{m} \otimes \alpha$. But then

$$\frac{a}{m} \otimes q\alpha = \frac{a}{m} \otimes \alpha \iff \frac{(q-1)a}{m} \in \mathbb{Z} \iff \frac{a}{m} \in \frac{1}{q-1}\mathbb{Z}$$

and thus $(\hat{X})^F$ is generated by $\left\{\dfrac{\alpha}{q-1}\right\}_{\alpha \in X} \cong X/(F-1)X$. Fix an isomorphism $X \cong \mathbb{Z}^n$. Then $X/(F-1)X = \mathbb{Z}^n/(q-1)\mathbb{Z}^n = (\mathbb{Z}/(q-1)\mathbb{Z})^n$. On the other hand as E is algebraically closed of characteristic 0,

$$\mathrm{Hom}_{\mathrm{Grp}}(T^F, E^*) = \mathrm{Hom}_{\mathrm{Grp}}(\mathbb{G}_m(\mathbb{F}_q)^n, E^*) = (\mathbb{Z}/(q-1)\mathbb{Z})^n$$

■

2. Algebraic Preliminaries

Now T is a maximal torus in G, a connected, reductive, affine algebraic group over $\Bbbk$.

2.1. Stabilizer Subgroups

Let $\xi = m^{-1} \otimes \alpha \in \widehat{X}$, $Q =$root lattice

Definition 2.1.

$$W_\xi = \{w \in W \mid w\alpha - \alpha \in mQ\} \subseteq W'_\xi = \{w \in W \mid w(\xi) = \xi\} = \{w \in W \mid w\alpha - \alpha \in mX\}$$

Lemma 2.2. *There is a homomorphism* $W'_\xi \big/ W_\xi \to \mathrm{Hom}(Z(G)/Z(G)^\circ, E^*)$.

3. Perv(G)

3.1. Weights of Torus actions

Let Y be an algebraic variety with a left $T-$action $a : T \times Y \to Y$.

Definition 3.1. *A perverse sheaf K has weight $\mathcal{L}$ (or ξ) if* $a^*K[\dim T] \cong \mathcal{L}[\dim T] \boxtimes K$ $(\ldots \mathcal{L}_\xi[\dim T] \ldots)$.

Remark. If $\mathcal{L} = \underline{E}$ then this is the same as saying that K is $G-$equivariant.

Let G be a connected affine algebraic group and $\varphi : G \to T$ and suppose Y has a $G-$action

Definition 3.2. *A perverse sheaf K has weight $\mathcal{L}$ (relative to a and φ) if* $a^*K[\dim G] \cong \varphi^*\mathcal{L}[\dim G] \boxtimes K$ $(\ldots \mathcal{L}_\xi[\dim G] \ldots)$.

☆**Lemma 3.3.** *Assume that U is a locally closed, smooth, irreducible $G-$stable subvariety of Y. Let $\mathcal{L}$ be a local system on U s.t. $\mathcal{L}[\dim U]$ on U has weight $\xi \in \hat{X}$. Then the perverse extension $I(U, \mathcal{L})$ has weight ξ.*

Proof. As $\Bbbk = \overline{\Bbbk}$, it's perfect and thus G is smooth and thus a is smooth so a^* commutes with perverse extension. ■

3.2. $\mathbf{A}_{\xi,\dot{\mathbf{w}}}$

For $w \in W$, let $\dot{w} \in N_G T$ be a lift. For $\alpha \in R$, let U_α be corresponding root subgroup.

Definition 3.4. *Given $w \in W$, let $U_w = \left\langle U_\alpha | \, \alpha \in R^+, -w^{-1}(\alpha) \in R^+ \right\rangle$*

Let $G_w = B\dot{w}B$. G_w is smooth and the map $U_w \times T \times U \to G_w$ sending $(u, t, u') \mapsto u\dot{w}tu'$ is an isomorphism of varieties. Thus we can define $pr_w : G_w \to T$ by $pr_w(u\dot{w}tu') = t$.

Definition 3.5. *Given $(\xi, w) \in \widehat{X} \times W$ let $G_{\xi,\dot{w}}(\mathcal{L}_{\xi,\dot{w}}$ in notes$) = pr_w^*(\mathcal{L}_\xi)$. Define $A_{\xi,\dot{w}} = I(G_w, G_{\xi,\dot{w}}) \in$ $\mathrm{Perv}(G)$.*

Lemma 3.6. *(i) $A_{\xi,\dot{w}}$ has weight $w(\xi)$ for left $B-$action and weight $-\xi$ for right $B-$action.*

(ii) If $w\xi = \xi$ then $A_{\xi,\dot{w}}$ is equivariant for the conjugation action of B

(iii) $\mathbb{D}(A_{\xi,w}) = A_{-\xi,w}$

Proof. (i) (1) It suffices to prove $G_{\xi,\dot{w}}$ has the appropriate weights by Lemma 3.3.

(2) Consider the case $w = e$. Then $G_e = B$, $\varphi = pr_w = \pi : B \to T \implies G_{\xi,\dot{w}} = \pi^*(\mathcal{L}_\xi)$ and $a = m$ for the left action. π is a group homomorphism so the following diagram commutes

$$\begin{tikzcd} B \times B \arrow[r, "m"] \arrow[d, "\pi\times\pi"'] & B \arrow[r, "\pi"] & T \\ T \times T \arrow[urr, "m"'] & & \end{tikzcd}$$

Thus

$$m^*(\pi^*(\mathcal{L}_\xi)) = (\pi \times \pi)^*(m^*(\mathcal{L}_\xi)) \overset{AS\ prop}{=\!=} (\pi \times \pi)^*(\mathcal{L}_\xi \boxtimes \mathcal{L}_\xi) = \pi^*(\mathcal{L}_\xi) \boxtimes \pi^*(\mathcal{L}_\xi)$$

as desired. The right action is given by $(b_1, b_2) \mapsto b_1 b_2^{-1}$. Let $\iota : T \to T$ be the inversion map $t \mapsto t^{-1}$. Then m becomes $m \circ (\mathrm{id} \times \iota)$ and $\iota^*(\mathcal{L}_\xi) = \mathcal{L}_{-\xi} \implies G_{\xi,\dot{w}}$ has weight $-\xi$ for right $B-$action.

(3) Now let w be general. Each U_α is an eigenspace for the adjoint action of T and thus for $u \in U_\alpha$, $ut = tu'$ for some $u' \in U_\alpha$. For the right action, given $(g, b) = (u\dot{w}tu', u(b)\pi(b)) \in G_w \times B$, we have

$$(u\dot{w}tu', b) \overset{a}{\mapsto} u\dot{w}tu'\pi(b)^{-1}u(b)^{-1} = u\dot{w}t\pi(b)^{-1}u''u(b)^{-1} \xmapsto{pr_w} = t\pi(b) = pr_w(g)\pi(b)^{-1}$$

Thus $pr_w \circ a = m \circ (\mathrm{id} \times \iota) \circ (pr_w \times \pi) \implies G_{\xi,\dot{w}}$ has weight $-\xi$ for right $B-$action. The right action is similar except we pick up $w(\xi)$ when commuting past $\dot{w}$. ■

Proposition 3.7. *(i) $\mathcal{H}^\bullet(A_{\xi,\dot{w}})|_{G_x} \otimes G_{-\xi,\dot{w}}$ is a constant sheaf on G_x.*

(ii) If $\mathcal{H}^\bullet(A_{\xi,\dot{w}})|_{G_x} \neq 0$ then $x(\xi) = w(\xi)$ and $x \leq w$.

Proof. (i) Notice that if we restrict to the left U_x action on G_w, then $pr_w(a_{U_x}(u, g)) = pr_w(\pi_2(u, g))$. Thus $G_{\xi,\dot{w}} = pr_w^*(\mathcal{L}_{\xi,w})$ is automatically U_x equivariant. By Lemma 3.3,

$$A_{\xi,\dot{w}} \text{ is } U_x \text{ equivariant} \implies S = \mathcal{H}^\bullet(A_{\xi,\dot{w}})|_{G_x} \text{ is } U_x \text{ equivariant}$$

as equivariant sheaves form an abelian category. $A_{\xi,\dot{w}}$ is perverse and so S is also constructible. Since $G_x = U_x \times B$, the U_x equivariance of S and constructibility implies that S is actually a local system. $A_{\xi,\dot{w}}$ has weight $-\xi$ for right $B-$action and thus $S \otimes G_{-\xi,\dot{w}}$ is a $U_x \times B-$ equivariant local system on $U_x \times B$ and therefore constant. ■

Let $G \overset{B}{\times} G$ be the quotient of $B \curvearrowright G \times G$ via $b(g,h) = (gb^{-1}, bh)$. For V, Z left $B-$stable sets, $V \overset{B}{\times} Z \to G$ is proper if V, Z are closed and thus $\pi : G \overset{B}{\times} G \to G$ is proper. Now recall the following equivariance trick

Lemma 3.8. *Let $f : X \to Y$ be a principal $G-$bundle. Then $K \in \mathrm{Perv}(X)$ is $G-$equivariant $\iff$ $\exists L \in \mathrm{Perv}(Y)$ s.t. $K = f^*(L[\dim G])$.*

Given $\xi \in \hat{X}$, $x, y \in W$, by Lemma 3.6 we see that $A_{\xi,\dot{x}} \boxtimes A_{y^{-1}(\xi),\dot{y}}$ is a $B-$equivariant irreducible perverse sheaf on $G \times G$. Applying the above lemma to $G \times G \xrightarrow{p} G \overset{B}{\times} G$, we obtain an irreducible perverse sheaf $A_{\xi,\dot{x},\dot{y}}$ on $G \overset{B}{\times} G$.

Definition 3.9.

$$A_{\xi,\dot{x}} * A_{y^{-1}(\xi),\dot{y}} = \pi_*(A_{\xi,\dot{x},\dot{y}}) \in \mathrm{Semi}(G)$$

3.3. Convolution Formulas

Lemma 3.10. *(a) Let X be a smooth irreducible variety of dimension d and let $D_1, \dots, D_r$ be smooth divisors*[3] *with normal crossings in X. Suppose $\mathcal{L}$ is a $1-$dim local system on $X \setminus \cup_{i=1}^r D_i$ which factors through a finite quotient of $\pi_1(X \setminus \cup_{i=1}^r D_i)$ of order prime to $\mathrm{char}\,\Bbbk$. Then $H^i(I(X \setminus \cup_{i=1}^r D_i, \mathcal{L})) = 0$ if $i \neq -\dim X$.*

(b) If J is the set of $i \in [1,r]$ s.t. the local monodromy of $\mathcal{L}$ around D_i is non-trivial and $U = X \setminus \cup_{i \in J} D_i$ then $\mathcal{L}$ can be extended to a local system $\overline{\mathcal{L}}$ on U. Moreover, $I(U, \mathcal{L}) = \iota_0(\overline{\mathcal{L}}[d])$.

Theorem 3
Let i_0 be extension by 0.

(i) If $s \in W_\xi$, $G_{\xi,s}$ extends to a local system $\overline{G_{\xi,s}}$ on $\overline{G_s}$ and $A_{\xi,s} = \iota_0(\overline{G_{\xi,s}}[\dim G_s])$. If $s \notin W_\xi$, $A_{\xi,s} = \iota_0(G_{\xi,s}[\dim G_s])$.

(ii) If $s \in W_\xi$, $A_{\xi,s} * A_{\xi,s} = A_{\xi,s}[1] \oplus A_{\xi,s}[-1]$. If $s \notin W_\xi$, $A_{s(\xi),s} * A_{\xi,s} = A_{\xi,e}$.

Proof. (i) For $G_s \xrightarrow{f} \overline{G_s} \xrightarrow{g} G$, we have

$$A_{\xi,s} = (g \circ f)_{!*}(G_{\xi,s}) = g_{!*}(f_{!*}(G_{\xi,s}))$$

$\overline{G_s} = G_e \sqcup G_s = P_s$ is closed and so $g_{!*} = \iota_0$. $G_e = B$ is a smooth divisor in $\overline{G_s}$ which is smooth and so

$$f_{!*}(G_{\xi,s}) = \begin{cases} \overline{G_{\xi,s}}[\dim G_s] & \text{if } s \in W_\xi \\ \iota_0(G_{\xi,s}[\dim G_s]) & \text{if } s \notin W_\xi \end{cases}$$

by applying part (b) of the lemma above.

(ii) $\underline{\text{For } s \in W_\xi}$ Since $A_{\xi,s} * A_{\xi,s} = \pi_*(A_{\xi,s,s}) \in \mathrm{Semi}(G)$ it suffices to compute the table of stalks to compute the isomorphism class. (1) We claim that $A := A_{\xi,s,s} = \iota_0(\overline{G_{\xi,s,s}}[\dim G_s + 1])$ where $\overline{G_{\xi,s,s}}$ is a local system on $\overline{G_s} \overset{B}{\times} \overline{G_s}$. By definition,

$$A_{\xi,s} \boxtimes A_{\xi,s} = p^*(A[\dim B])$$

[3] As X is smooth, Weil=Cartier divisors.

By part (i) the LHS is a local system in cohomological degree $[2\dim G_s]$ and by the equivalence of categories $\mathrm{Sh}_B(G\times G)\cong \mathrm{Sh}(G\overset{B}{\times} G)$, A must be a local system supported on $\overline{G_s}\overset{B}{\times}\overline{G_s}$ in cohomological degree $[2\dim G_s-\dim B]=[\dim G_s+1]$.

(2) Any fiber of $\pi:\overline{G_s}\overset{B}{\times}\overline{G_s}\to\overline{G_s}$ is isomorphic to $\overline{G_s}/B=P_s/B=\mathbb{P}^1_{\Bbbk}$. $\overline{G_{\xi,s,s}}|_{fiber}$ is then a local system (being the pullback of a local system). But $\mathbb{P}^1(\mathbb{C})\overset{homeo}{\simeq}S^2$ and thus $\pi_1(\mathbb{P}^1(\mathbb{C}))=0$ and the same will be true for $\pi_1^{\acute{e}t}(\mathbb{P}^1_{\overline{\mathbb{F}_p}})$ and thus $\overline{G_{\xi,s,s}}|_{fiber}\cong\underline{E}$. Thus for $g\in\overline{G_s}$

$$H^i(\pi_*(A))_g\overset{PBC}{=\!=}H^i(\pi^{-1}(g),\overline{G_{\xi,s,s}}[\dim G_s+1]|_{fiber})=H^{i+\dim G_s+1}(\mathbb{P}^1_{\Bbbk},\underline{E})$$

Thus the table of stalks for $\pi_*(A)$ is of the form on the left while TOS for $A_{\xi,s}=\iota_0(G_{\xi,s}[\dim G_s])$ is on the right

	$-\dim G_s-1$	$\dim G_s+1$
G_s	$\Bbbk$	$\Bbbk$
G_e	$\Bbbk$	$\Bbbk$

	$-\dim G_s$
G_s	$\Bbbk$
G_e	$\Bbbk$

and we conclude $\pi_*(A)=A_{\xi,s}[1]\oplus A_{\xi,s}[-1]$.

For $s\notin W_\xi$, $A_{s(\xi),s}*A_{\xi,s}=\pi_*(A_{s(\xi),s,s})$. Because $G_s\overset{B}{\times}G_w$ is smooth, $(A_{\xi,s,w})|_{G_s\overset{B}{\times}G_w}=G_{\xi,s,w}[\dim G_s+1]$ for some local system $G_{\xi,s,w}$ as $A_{\xi,s,w}$ is perverse. We now need a generalization of Springer's Lemma

Lemma 3.11. *Let $q:G_s\overset{B}{\times}G_w\to G$ be the restriction of π. If $sw<w$, $G_s\overset{B}{\times}G_w\xrightarrow{q}G_w\sqcup G_{sw}$ is an isomorphism.*

(a) If $g\in G_{sw}$ then $q^{-1}(g)\overset{homeo}{\simeq}\Bbbk$ and $G_{\xi,s,w}|_{q^{-1}(g)}\cong\underline{E}$.

(b) If $g\in G_w$ then $q^{-1}(g)\overset{homeo}{\simeq}\Bbbk^*$ and $G_{\xi,s,w}|_{q^{-1}(g)}\cong\mathcal{L}$ where $\mathcal{L}$ is a nontrivial local system.

For $U=G_s\overset{B}{\times}G_s$, consider the fiber diagram

$$\begin{tikzcd} q^{-1}(g)=\pi^{-1}(g)\cap U \arrow[r, "h_2"] \arrow[d, "j_2"] & U \arrow[d, "j"] \\ \pi^{-1}(g) \arrow[r, "h"] & \overline{G_s}\overset{B}{\times}\overline{G_s} \end{tikzcd}$$

By part (i), $A=A_{s(\xi),s,s}=j_!(G_{s(\xi),s,s}[\dim G_s+1])$. Let $a_{pt}:\pi^{-1}(g)\to pt$. Then

$$\begin{aligned} H^i(\pi_*(A))_g &\overset{PBC}{=\!=}H^i(\pi^{-1}(g),A|_{fiber})=(a_{pt})_*h^*(j_!(G_{s(\xi),s,s}[\dim G_s+1]) \\ &=(a_{pt})_*(j_2)_!h_2^*(G_{s(\xi),s,s}[\dim G_s+1])\overset{a_{pt}\ proper}{=\!=}(a_{pt})_!(j_2)_!h_2^*(G_{s(\xi),s,s}[\dim G_s+1]) \\ &=H_c^{i+\dim G_s+1}(q^{-1}(g),G_{s(\xi),s,s}|_{q^{-1}(g)}) \end{aligned}$$

By Lemma 3.11 there are two cases: (a) $g\in G_e$, then we obtain

$$H_c^{i+\dim G_s+1}(\Bbbk,\underline{E})=\begin{cases}\Bbbk & \text{if } i=-\dim G_s+1=\dim G_e(\Bbbk``="\mathbb{C}=\mathbb{R}^2)\\ 0 & \text{otherwise}\end{cases}$$

$(b) g\in G_s$, then we obtain

$$H_c^{i+\dim G_s+1}(\Bbbk^*,\mathcal{L})\overset{PV duality}{=\!=}H^{-(i+\dim G_s+1)}(\Bbbk^*,\mathbb{D}(\mathcal{L}))^\vee\overset{\Bbbk^*\ orient\ manifold}{=\!=}H^{-(i+\dim G_s+1)}(\Bbbk^*,\mathcal{L}^\vee[1])^\vee$$

We claim the RHS above is always 0. Recall that for local systems $\mathcal{L}_T = (M, T)$ on $\mathbb{C}^\times$

$$H^0(\mathbb{C}^\times, \mathcal{L}_T) = M^T, \qquad H^1(\mathbb{C}^\times, \mathcal{L}_T) = M_T, \qquad H^i(\mathbb{C}^\times, \mathcal{L}_T) = 0 \ i \neq 0, 1$$

Because $\mathcal{L}^\vee$ is a nontrivial 1−dimensional local system, $M^T = M_T = 0$ and thus the claim. Putting (a) and (b) together, the TOS for $\pi_*(A)$ is

	$-\dim G_e$
G_s	0
G_e	$\mathbb{k}$

Part (i) tells us that $A_{\xi,e} = i_0(G_{\xi,e}[\dim G_e])$ so the above is also the TOS for $A_{\xi,e}$ and thus $\pi_*(A) = A_{\xi,e}$. ■