Character Sheaves: Preliminaries 2

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October 1st, 2024

1. Generalized Hecke Algebras

Let $\mathcal{O}$ be a $W-$orbit in $\hat{X} = \hat{X}(T) = \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} X \Big/ 1 \otimes_{\mathbb{Z}} X$. Denote by $K_{\mathcal{O}}$ the free $\mathbb{Z}[t, t^{-1}]$ module with "standard" basis $\{e_{\xi,w}\}_{\xi \in \mathcal{O}, w \in W}$.

Theorem 1.1. *$\mathcal{H}_{\mathcal{O},W}$ is the unique $\mathbb{Z}[t, t^{-1}]$ algebra on $K_{\mathcal{O}}$ s.t. for $\xi, \eta \in \mathcal{O}$, $x, y \in W$, $s \in S$*

(a) $e_{\xi,x} e_{\eta,y} = 0$ *if* $\xi \neq y\eta$

(b) $1 = \sum_{\xi \in \mathcal{O}} e_{\xi,e}$

(c) $$e_{y(\eta),s} e_{\eta,y} = \begin{cases} e_{\eta,sy} & \text{if } sy > y \text{ or } s \notin W_{y(\eta)} \\ (t^2 - 1) e_{\eta,y} + t^2 e_{\eta,sy} & \text{if } sy < y \text{ and } s \in W_{y(\eta)} \end{cases}$$

(d) $$e_{s(\xi),x} e_{\xi,s} = \begin{cases} e_{\xi,xs} & \text{if } xs > x \text{ or } s \notin W_{\xi} \\ (t^2 - 1) e_{\xi,x} + t^2 e_{\xi,xs} & \text{if } xs < x \text{ and } s \in W_{\xi} \end{cases}$$

(e) $e_{y(\eta),x} e_{\eta,y} = e_{\eta,xy}$ *if* $\ell(xy) = \ell(x) + \ell(y)$

Remark. For $\mathcal{O} = \{0\}$, we recover the usual Hecke algebra (by setting $e_{0,w} = t^{\ell(w)} \delta_s$).

Lemma 1.2 (Bar Automorphism)**.** *There is a unique automorphism $\overline{(\cdot)} : \mathcal{H}_{\mathcal{O},W} \to \mathcal{H}_{\mathcal{O},W}$ sending $t \mapsto t^{-1}$ and*

$$e_{x(\xi),x^{-1}} \overline{e_{\xi,x}} = e_{\xi,e}$$

Remark. This is not an involution in general, and for $\xi = \{0\}$ we get usual bar.

Lemma 1.3. *Let W^{ξ}_{min} is the set of minimal length coset representatives of W_{ξ}. Then every element $w \in W$ has a unique decomposition*

$$w = w^* w_1 \text{ where } w^* \in W^{\xi}_{min} \text{ and } w_1 \in W_{\xi}$$

Definition 1.4. *Given $w \in W$, $\xi \in \hat{X}$, set $\ell_{\xi}(w) = \ell(w_1)$.*

Proposition 1.5 (KL basis)**.** *Given $e_{\xi,x}$ there is a unique element $C_{\xi,x}$ in $\mathcal{H}_{\mathcal{O},W}$ s.t.*

(i) $\overline{C_{\xi,x}} = C_{\xi,x}$

(ii) $$C_{\xi,x} = t^{-\ell_{\xi}(x)} \sum_{\substack{y \in xW_{\xi} \\ y \leq_{\xi} x}} P_{\xi,y,x}(t^2) e_{\xi,y} \text{ where } P_{\xi,y,x} \in \mathbb{Z}[t], P_{\xi,x,x} = 1, \ \deg P_{\xi,y,x} \leq \frac{1}{2}(\ell_{\xi}(x) - \ell_{\xi}(y) - 1).$$

☆ **Example 1.**

$$C_{\xi,e} = e_{\xi,e} \qquad C_{\xi,s} = \begin{cases} e_{\xi,s} & \text{if } s \notin W_\xi \\ t^{-1}(e_{\xi,s} + e_{\xi,e}) & \text{if } s \in W_\xi \end{cases}$$

Suppose $s \notin W_\xi$. By above, we have $C_{\xi,s} = e_{\xi,s}$ and $C_{s(\xi),s} = e_{s(\xi),s}$ as $W_{s(\xi)} = sW_\xi s^{-1}$. We then compute

$$C_{s(\xi),s}C_{\xi,s} = e_{s(\xi),s}e_{\xi,s} = e_{\xi,e} = C_{\xi,e}$$

Likewise we have that

$$C_{\xi,x}C_{\xi,s} = (t+t^{-1})C_{\xi,s} \ \text{ if } s \in W_\xi \text{ and } xs < x$$

Proposition 1.6. *(a) $\operatorname{span}_{\mathbb{Z}[t^{\pm 1}]}\{e_{\xi,w}\}_{w\in W_\xi}$ forms a subalgebra isomorphic to $\mathcal{H}(W_\xi)$.*

(b) $\operatorname{span}_{\mathbb{Z}[t^{\pm 1}]}\{e_{\xi,w}\}_{w\in W'_\xi}$ forms a subalgebra. As a $\mathbb{Z}[t^{\pm 1}]-$module it is isomorphic to $\mathcal{H}(W_\xi)\otimes\mathbb{Z}[W'_\xi \cap W^\xi_{min}]$.

Corollary 1.7. *Let $x, y \in W$ s.t. $y \in xW_\xi$. Write $x = x^*x_1$, $y = y^*y_1$ where $x^*, y^* \in W^\xi_{min}$ and $x_1, y_1 \in W_\xi$. Then*

$$C_{\xi,x} = e_{\xi,x^*}C_{\xi,x_1} \qquad \text{and } P_{\xi,y,x} = P_{\xi,y_1,x_1} = P_{y_1,x_1}$$

where P_{y_1,x_1} is the KL poly for $\mathcal{H}(W_\xi)$.

2. Motivation for $A_{\xi,w}$

Theorem 2.1. *Let π be an irreducible representation of $G(\mathbb{F}_q)$ in characteristic 0. Then either*

(i) $\exists!$ Levi subgroup L contained in some proper parabolic P, where $P = L \ltimes U$ and a unique cuspidal representation ρ of $L(\mathbb{F}_q)$ s.t.

$$\operatorname{Hom}_{G(\mathbb{F}_q)}\left(\pi, \operatorname{Ind}_{P(\mathbb{F}_q)}^{G(\mathbb{F}_q)}\rho\right) \neq 0$$

(ii) π is cuspidal

Thus, to understand representations of $G(\mathbb{F}_q)$ we need to understand (i) irreducible constituents of $\operatorname{Ind}_{P(\mathbb{F}_q)}^{G(\mathbb{F}_q)}\rho$. In the special case $P = B$, then $L = T$, we want to understand $\operatorname{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}\rho$ where χ is a character of $T(\mathbb{F}_q)$ (these are the principal series representations of $G(\mathbb{F}_q)$). But recall

$$\operatorname{KLS}(T)^F \cong \operatorname{Hom}_{\mathrm{Grp}}(T^F, E^*)$$

Furthermore given $\mathcal{L}_\xi$ a Kummer local system on T, if $w \in W_\xi$, recall that

$$A_{\xi,\dot{w}} \text{ is equivariant for } B^{ad}$$

In particular, $A_{\xi,\dot{e}} = I(B, G_{\xi,e})$ is B^{ad} equivariant for any ξ. If $\mathcal{L}_\xi \in \operatorname{KLS}(T)^F$, then $A_{\xi,\dot{e}}$ is our geometric incarnation of the $B(\mathbb{F}_q)$ representation χ. In talks to come, we will see how to realize $\operatorname{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}$ geometrically.

Remark. At the other extreme, notice that $A_{0,w} = I(G_w, \underline{E}) =: \mathbb{IC}_w$. This has weight $w(0) = 0$ on the left and 0 on the right and thus $\mathbb{IC}_w \in D^b_{B\times B}(G) \cong D^b_B(G/B)$ sending $\mathbb{IC}_w$ to the usual $\mathrm{IC}_w \in D^b_B(G/B)$. Hence, we should think of $A_{\xi,w}$ as generalized IC complexes on flag varieties that now live on G.

3. Categorification

Theorem 1 (Geometric Hecke Category)
$\mathcal{H}_W^{geo}$ is the monoidal category

$$\mathcal{H}_W^{geo} := \langle \mathrm{IC}_w \,|w \in W\rangle_{\star,[1],\oplus} \subset D_B^b(G/B)$$

i.e. smallest subcategory containing $\{\mathrm{IC}_w\}$ closed under convolution, homological shifts, and direct sums. We then have an isomorphism of algebras

$$K_{\oplus}\left(\mathcal{H}_W^{geo}\right) \cong \mathcal{H}_W$$

given by $h(\mathrm{IC}_w) = C_w$ (the corresponding KL basis element) and extending $\mathbb{Z}[v^{\pm 1}]$ linearly.

Proposition 3.1. *(i) For $x \in W$, $\mathcal{H}^\bullet(A_{\xi,\dot{w}})|_{G_x}$ is locally constant.*

(ii) $\mathcal{H}^\bullet(A_{\xi,\dot{w}})|_{G_x} \otimes G_{-\xi,\dot{w}}$ is a constant sheaf on G_x.

Proof. (i) Notice that if we restrict to the left U_x action on G_w, then $pr_w(a_{U_x}(u,g)) = pr_w(\pi_2(u,g))$. Thus $G_{\xi,\dot{w}} = pr_w^*(\mathcal{L}_{\xi,w})$ is automatically U_x equivariant. Since perverse extension preserves weights,

$$A_{\xi,\dot{w}} \text{ is } U_x \text{ equivariant} \implies S = \mathcal{H}^\bullet(A_{\xi,\dot{w}})|_{G_x} \text{ is } U_x \text{ equivariant}$$

as equivariant sheaves form an abelian category. $A_{\xi,\dot{w}}$ is perverse and so S is also constructible. Since $G_x = U_x \times B$, the U_x equivariance of S and constructibility implies that S is actually a local system. (ii) $A_{\xi,\dot{w}}$ has weight $-\xi$ for right $B-$action and thus $S \otimes G_{-\xi,\dot{w}}$ is a $U_x \times B-$ equivariant local system on $U_x \times B$ and therefore constant. ∎

Definition 3.2. *Let $n_{x,\xi,i} = \mathrm{rk}(\mathcal{H}^i(A_{\xi,\dot{w}})|_{G_x})$. Set*

$$F_{\xi,x,w}(t^2) = t^{\ell(x)+\dim B+\ell_\xi(w)-\ell_\xi(x)} \sum_{i\in\mathbb{Z}} n_{x,\xi,i}t^i$$

Remark. Note that the LHS above is a function of t^2 but the RHS is a function of t. The fact that only even powers of t show up on the RHS is a consequence of the fact that $A_{\xi,\dot{w}}$ is parity.

Theorem 3.3. *$F_{\xi,x,w}$ is the (generalized) KL-polynomial $P_{\xi,x,w}$.*

Theorem 2 (Generalized Geometric Hecke Category)
Given $\mathcal{O}$ a $W-$orbit in $\hat{X}$, $g\mathcal{H}_{\mathcal{O},W}^{geo}$ is the monoidal category

$$g\mathcal{H}_{\mathcal{O},W}^{geo} := \langle A_{\xi,w} \,|\, \xi \in \mathcal{O},\ w \in W\rangle_{\star,[1],\oplus} \subset D_c^b(G)$$

We then have an isomorphism of algebras

$$K_{\oplus}\left(g\mathcal{H}_{\mathcal{O},W}^{geo}\right) \cong \mathcal{H}_{\mathcal{O},W}$$

given by $h(A_{\xi,w}) = C_{\xi,w}$ and extending $\mathbb{Z}[t^{\pm 1}]$ linearly.

Example 2. Compare calculations in Example 1 with our last big theorem from last week

$$\text{If } s \notin W_\xi,\ A_{s(\xi),s} * A_{\xi,s} = A_{\xi,e}, \qquad \text{If } s \in W_\xi,\ A_{\xi,s} * A_{\xi,s} = A_{\xi,s}[1] \oplus A_{\xi,s}[-1]$$