§5. Character sheaves

2024.10.$\overset{15}{8}$ (火)
@ CS seminar

- k : alg. cl. field
- G : conn. red. gp $/k$
- $T \subset G$: a max. torus $\rightsquigarrow W,\ X := X^*(T)$
- $E := \overline{\mathbb{Q}}_\ell$ or $\mathbb{C}$.

<u>Recall</u> (Cailan)

$\left|\begin{array}{l} \cdot\ \xi \in \hat{X} := X \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}/X \ \circlearrowleft W. \\ \cdot\ w \in W'_\xi := \mathrm{Stab}_W(\xi). \end{array}\right.$

$\rightsquigarrow A_{\xi,w}$: an irr. perv. shf. on G, B-eq. ($\overset{\text{conj.}}{\circlearrowleft} B$)

$$\left(\begin{array}{c} \mathcal{L}_\xi : \text{Kummer} \xrightarrow{\text{pullback}} \mathcal{L}_{\xi,\dot{w}} \xrightarrow{\text{perv.}} A_{\xi,\dot{w}} \\ T \twoheadleftarrow U_w \times T \times U \xrightarrow{\sim} G_w := BwB \underset{\text{loc. cl.}}{\subset} G. \\ t \leftarrow\!\shortmid (u,t,u') \mapsto u\dot{w}tu' \end{array}\right)$$

- $B \curvearrowright G \times G$; $b\cdot(g,h) := (gb^{-1}, bhb^{-1})$

(G-eq.) $E[\dim G] \boxtimes A_{\xi,w} \xleftarrow[\text{[}-\dim B\text{]}]{\text{p.b.}} {}^\exists \widetilde{A}_{\xi,w}$: an irr. perv. shf., G-eq.

$$\begin{tikzcd} G \times G \arrow[r, "\text{quot.}"] \arrow[drr] & G \times_B G \arrow[r, "\sim"] \arrow[d, "\gamma"] \arrow[d, phantom, "\circlearrowleft\quad\circlearrowleft"] & G \times G/B \arrow[dl, "\mathrm{pr}_1"'] \\ & & G \end{tikzcd}$$

$G \curvearrowright$ $x\cdot(g,h) = (xg,h)$; $(g,h) \mapsto ghg^{-1}$; $G \overset{\text{conj.}}{\curvearrowright}$; $(g,h) \mapsto (ghg^{-1}, g)$, proj.

$\rightsquigarrow C_{\xi,w} := \underset{G\text{-eq. proper.}}{\gamma_*} \widetilde{A}_{\xi,w}$ ... G-eq. semisimple complex ~~perv. shf~~ on G.

(by decomposition thm of Deligne–Gabber BBDG)

i.e. $C_{\xi,w} \simeq \bigoplus_i \underbrace{{}^pH^i}_{} [-i]\ C_{\xi,w}$.

(s-s. obj. $\in \mathcal{M}G$ $(:= \mathrm{Perv}(G))$ (G-eq.).)

<u>Def.</u> "A character shf on G" is an irr. constituent of ${}^pH^i\ C_{\xi,w}$ for some $\xi \in \hat{X}$ & $w \in W'_\xi$ (→ necessarily G-eq.)

More general version

- $V_r := \underbrace{(G \times^B \cdots \times^B G)}_{r} \circlearrowleft b\cdot(g_1,\dots,g_r) = (bg_1, g_2, \dots, g_{r-1}, g_r b^{-1})$.

- $Y_r := G \times_B V_r := (G \times V_r)/B \quad \circlearrowleft b\cdot(g,v) = (gb^{-1}, b\cdot v)$.

- $\mathbb{s} := (w_1, \dots, w_r) \in W^r \quad (\rightsquigarrow w := w_1 \cdots w_r)$.

- Let $\xi \in \hat{X}$ s.t. $w \in W'_\xi$.

$\rightsquigarrow A_{\xi,\mathbb{s}} := A_{w_2\cdots w_r\xi, w_1} \boxtimes A_{w_3\cdots w_r \xi, w_2} \boxtimes \cdots \boxtimes A_{\xi, w_r}$

$\cdots$ an irr. perv. shf. on G^r,

pull-back of a B-eq. irr. perv. shf. on $G \times^B \cdots \times^B G$.

(up to $[-(r-1)\dim B]$)

(∵ use Cailan: $A_{\xi,w}$ has wt $w\xi$ for left B-trans act.
$-\xi$ for right.)

$E[\dim G] \boxtimes A_{\xi,\mathbb{s}} \xleftarrow[\text{\& shift}]{\text{pull-back}} \exists\, \tilde{A}_{\xi,\mathbb{s}}$: an irr. perv. shf. G-eq.

$$\begin{tikzcd}
G \times G^r \arrow[r, two heads] & G \times V_r \arrow[r, two heads] \arrow[d, "\text{prod. on } V_r"'] \arrow[dr, phantom, "\circlearrowright"] & Y_r := G \times_B V_r \arrow[d, "\pi_r"] \\
& G \times G \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & G \times_B G \arrow[dl, "\gamma"] \\
& G &
\end{tikzcd}
\qquad \gamma_r := \gamma \circ \pi_r$$

$(g, g_1, \dots, g_r) \mapsto g g_1 \cdots g_r g^{-1}$

$(g,h) \mapsto ghg^{-1}$

$\rightsquigarrow C_{\xi,\mathbb{s}} := (\gamma \circ \pi_r)_* \tilde{A}_{\xi,\mathbb{s}} \quad \cdots$ a G-eq. s-s. ~~perv.~~ complex

- $\hat{G} := \{\text{char. sheaves on } G\}/\sim$

$\cup$

$\hat{G}(\xi) := \{\text{char. sheaves in } C_{\xi,w} \mid w \in W'_\xi\}. \quad (\xi \in \hat{X})$.

$\hat{G}(\xi, \mathbb{s}) := \{\text{char. sheaves in } C_{\xi,\mathbb{s}}\} \quad \left(\xi \in \hat{X},\ \mathbb{s} = (w_1, \dots, w_r) \in W^r \text{ s.t. } w_1 \cdots w_r \in W'_\xi\right)$.

Lem. $\widehat{G}(\xi) = \bigcup_{\substack{\mathbb{S}=(s_1,\dots,s_r)\in W^r \\ s_i \in S\cup\{e\} \\ s_1\cdots s_r \in W_\xi}} \widehat{G}(\xi, \mathbb{S})$. $\quad$ ($S \subset W$: simple reflections).

pf. Let $A \in \widehat{G}(\xi)$, hence A "$\subset$" $C_{\xi,w}$ for ${}^\exists w \in W_\xi$.

バッサリ削っても良い

"can be shown on the Hecke alg. side"

$$\left(\begin{array}{l}\text{precise meaning}: C_{\xi,w} = \bigoplus_i {}^pH^i \cancel{}[-i]. \quad C_{\xi,w}. \\ \qquad A \subset {}^pH^i C_{\xi,w} \quad {}^\exists w, {}^\exists i\end{array}\right)$$

Let $w = s_1 \cdots s_r$ be a reduced expression.

$\to \mathbb{S} := (s_1, \dots, s_r)$

Recall (Cailan). $\qquad$ Cailan: $\mathcal{H}_{O,W}$.

$O := W.\xi \rightsquigarrow K_O := \bigoplus_{\substack{\eta\in O \\ w \in W}} \mathbb{Z}[t^{\pm 1}]\cdot e_{\eta,w}$. Hecke alg.

h: (s-s. cpx. of the form $\bigoplus_{\substack{\eta\in O \\ w\in W}} A_{\eta,w}[n_{\eta,w}]$) $\longmapsto \sum_{\substack{\eta\in O \\ w\in W}} t^{-n_{\eta,w}} \cdot \underbrace{C_{\eta,w}}_{\text{KL basis.}}$

$\in$ Groth. gp. of the geometric Hecke cat.

$\cdots$ $\mathbb{Z}[t^{\pm 1}]$-alg. isom.

(In particular, $h(A_{y\eta,x} * A_{\eta,y}) = C_{y\eta,x} \cdot C_{\eta,y}$.)

(Thm 3.3.4) <u>Formula (5)</u> after Thm.

Let $x = sy > y$ $(s \in S)$.

Then $C_{\xi,x} = \begin{cases} C_{y\xi,s}\cdot C_{\xi,y} & \text{if } y^{-1}sy \notin W_\xi \\ C_{y\xi,s}\cdot C_{\xi,y} - \sum\limits_{\substack{z\in yW_\xi \\ z <_\xi y,\ sz<z}} \mu_\xi(z,y)\cdot C_{\xi,z} & \text{if } y^{-1}sy \in W_\xi \end{cases}$

$\mu_\xi(z,y)$: coeff. of $t^{l_\xi(y)-l_\xi(z)-1}$ in $P_{\xi,zy}(t)$

$\rightsquigarrow C_{s_2\cdots s_r\xi, s_1} \cdot C_{s_3\cdots s_r\xi, s_2} \cdots C_{\xi, s_r}$ contains $t^{0}\cdot C_{\xi,w}$ with nonzero coeff.

Hence $A_{\xi,w}$ is contained in $A_{s_2\cdots s_r\xi,s_1} * \cdots$.

$\rightsquigarrow C_{\xi,w} := \gamma_* \widetilde{A}_{\xi,w}$ "$\subset$" $C_{\xi,\mathbb{S}} := (\gamma\circ\pi(r))_* A_{\xi,\mathbb{S}}$.

(proper base change).

Conversely, let $\mathbf{s} := (s_1,\dots,s_r)$ s.t. $s_i \in S\cup\{e\}$, $w := s_1\cdots s_r \in W'_\xi$.

Again through K_0, we can check:

$\forall$ irr. constituent of $C_{\xi,\mathbf{s}}$ is $C_{\xi,w}$ for $\exists\, w\in W'_\xi$ $\square$.

<u>Lem.</u> Let $\mathbf{s} := (w_1,\dots,w_r) \in W^r$ s.t. $w_1\cdots w_r \in W'_\xi$.

Put $\mathbf{s}' := (w_r, w_1,\dots,w_{r-1})$. ($\to w_r w_1\cdots w_{r-1} \in W'_{w_r\xi}$.)

Then $C_{\xi,\mathbf{s}} \simeq C_{w_r\xi,\mathbf{s}'}$.

<u>prf.</u> Put $\varphi : Y_r \xrightarrow{\sim} Y_r\ ;\ (g,g_1,\dots,g_r) \mapsto (g g_r^{-1}, g_r, g_1,\dots,g_{r-1})$

Recall: $Y_r := \underbrace{G\times_B}_{(g,v)\sim(gb^{-1},\,b.v)} \underbrace{(G\times^B\cdots\times^B G)}_{(g_1,\dots,g_i,g_{i+1},\dots,g_r)\sim(g_1,\dots,g_ib,\,b^{-1}g_{i+1},\dots,g_r)} =: V_r$

$E\boxtimes A_{\xi,\mathbf{s}} \rightsquigarrow \widetilde{A}_{\xi,\mathbf{s}}$

$G\times G_r \to G\times_B V_r$

$$\begin{tikzcd}
(g,g_1,\dots,g_r)\in Y_r \arrow[r, "\varphi"', "\sim"] \arrow[d, mapsto] & Y_r \arrow[dl, "r_r"] \\
g\cdot g_1\cdots g_r\cdot g^{-1} \in G &
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{A}_{\xi,\mathbf{s}} \arrow[r, "\varphi"] \arrow[d, "r_*"] & \widetilde{A}_{w_r\xi,\mathbf{s}'} \arrow[dl, "r_*"] \\
C_{\xi,\mathbf{s}} = C_{w_r\xi,\mathbf{s}'} &
\end{tikzcd}$$

with $r_r : Y_r \to G$ the vertical map $\circlearrowright$, and $\circlearrowright$ in the right square. $\square$

$$\left(A_{\xi,\mathbf{s}} = A_{w_2\cdots w_r\xi,\,w_1} \boxtimes \cdots \boxtimes A_{w_r\xi,\,w_{r-1}} \boxtimes A_{\xi,\,w_r} \right)$$

<u>Lem.</u> Let $\xi, \eta \in \widehat{X}$ s.t. $\eta \notin W \cdot \xi$.
Then, for any $x \in W_\xi'$, $y \in W_\eta'$, $H^i_c(G, \mathcal{C}_{\xi,x} \otimes \mathcal{C}_{-\eta,y}) = 0$ $\forall i$.

pf. $Y_w := G \times_B G_w \subset \overline{Y}_w = G \times_B \overline{G_w} \xrightarrow[:= \gamma|_{\overline{Y}_w}]{\gamma_w} G$.

$\downarrow pr_1$ G/B $\qquad$ $A_{\xi,w}$ is supp'd here.

$$\begin{tikzcd}
\overline{Y}_{x,y} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{Y}_x \times \overline{Y}_y \arrow[d, "{(\gamma_x, \gamma_y)}", "\text{proper.}"'] \\
G \arrow[r, hook, "\Delta"] & G \times G
\end{tikzcd}$$

$\underset{p.b.c.}{\rightsquigarrow}$ enough to show $H^i_c(\overline{Y}_{x,y}, \widehat{A}_{\xi,x} \otimes \widehat{A}_{-\eta,y}) = 0$.

$\mathcal{C}_{\xi,x} \otimes \mathcal{C}_{-\eta,y} \xleftarrow[\Delta^*]{} \mathcal{C}_{\xi,x} \boxtimes \mathcal{C}_{-\eta,y}$

$\overline{Y}_{x,y} = \coprod_{\substack{u,v \in W \\ u \leq x \\ v \leq y}} \underbrace{Y_{u,v}}_{\text{loc. cl.}}$ $\rightsquigarrow$ enough to show $H^i_c(Y_{u,v}, \widehat{A}_{\xi,x} \otimes \widehat{A}_{-\eta,y}) = 0$.

<u>Claim</u>. $(\widehat{A}_{\xi,x} \otimes \widehat{A}_{-\eta,y})|_{Y_{u,v}} \neq 0 \Rightarrow u\xi = x\xi$ & $v\eta = y\eta$.
Moreover, in this case,
$$(\widehat{A}_{\xi,x} \boxtimes \widehat{A}_{-\eta,y})|_{Y_{u,v}} \simeq (\widehat{A}_{\xi,u} \otimes \widehat{A}_{-\eta,v})|_{Y_{u,v}}.$$

$\rightsquigarrow$ enough to show $H^i_c(Y_{u,v}, \widehat{A}_{\xi,u} \boxtimes \widehat{A}_{-\eta,v}) = 0$.

Let $\alpha : Y_{u,v} \hookrightarrow Y_u \times Y_v \xrightarrow{pr_1 \times pr_1} G/B \times G/B$.
$\rightsquigarrow$ enough to show $R\alpha_!(\widehat{A}_{\xi,u} \boxtimes \widehat{A}_{-\eta,v}) = 0$.
(can be checked stalk-wise).

$\widetilde{\mathcal{L}}_{\xi,u}$ G-eq

$$\begin{tikzcd}
\alpha^{-1}(a) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y_{u,v} \arrow[d, "\alpha\ G\text{-}e"] \\
a \arrow[r] & G/B \times G/B
\end{tikzcd}$$

By p.b. thm, enough to show $H^i_c(\alpha^{-1}(a), \widehat{\mathcal{L}}_{\xi,u} \boxtimes \widehat{\mathcal{L}}_{-\eta,v}) = 0$.
$\simeq \widehat{A}_{\xi,u}|_{Y_u}$

G-equivariance $\Rightarrow$ a may be any rep. of G-orbits.

Take $a := (B, w(B)) \in G/B \times G/B$. $(w \in W)$ $\rightsquigarrow$ exhausts. by Bruhat decom

$\rightsquigarrow \alpha^{-1}(a) \simeq \{(g,h) \in G_u \times G_v \mid g = \dot{w} h \dot{w}^{-1}\} \xrightarrow[pr.]{\sim} G_u \simeq U_u \times T \times U$

$\widehat{\mathcal{L}}_{\xi,u} \boxtimes \mathcal{L}_{-\eta,v} \longleftrightarrow \underbrace{}_{\text{const}} \otimes \mathcal{L}_{\xi - w^{-1}(\eta)} \otimes \underbrace{}_{\text{const}}$

(pull back of to here)

$\rightsquigarrow$ Claim vanishing.

<u>Thm.</u> Let $\xi, \eta \in \hat{X}$.

(i) $\eta \in W\xi \Rightarrow \hat{G}(\xi) = \hat{G}(\eta)$

(ii) $\eta \notin W\xi \Rightarrow \hat{G}(\xi) \cap \hat{G}(\eta) = \phi$.

<u>prf.</u> (i) For any $w \in W'_\xi$ & $s \in S \setminus W'_\xi$,

$$\underset{\substack{\| \text{ Cailan} \\ C_{\xi,w}}}{C_{\xi,(s,s,w)}} = \underbrace{C_{s\xi,(s,w,s)}}_{\text{any irr. const lies in } \hat{G}(s\xi)} \quad \text{by Lem.}$$

(compute on the Hecke alg side)

$$\rightsquigarrow \hat{G}(\xi) \subset \hat{G}(s\xi).$$

On the other hand, $\supset$ is obviously true for $s \in W'_\xi$ ($\Rightarrow s\xi = \xi$).

$$\therefore \hat{G}(\xi) \subset \hat{G}(w\xi) \text{ for } {}^\forall w \in W.$$

$$\rightsquigarrow \hat{G}(\xi) = \hat{G}(w\xi). \quad //$$

(ii) $K \in \hat{G}(\xi)$, $K' \in \hat{G}(\eta)$.

$$\rightsquigarrow K \not\cong K' \Leftrightarrow H_c^0(G, K \otimes \underbrace{DK'}_{\in \hat{G}(-\eta)}) = 0 \quad \text{--- holds for any perv. shf.}$$

$$\Uparrow$$

$$\text{Lem: } H_c^*(G, C_{\xi,x} \otimes C_{-\eta,y}) = 0 \qquad \square$$

§6. Cuspidal perv. sheaves

· $G \supset P = LU$: parabolic

$$\rightsquigarrow \mathrm{res}^G_P : \mathcal{D}G \longrightarrow \mathcal{D}L$$
$$K \longmapsto (\pi_P)_! \circ i^*(K)$$

$$\begin{tikzcd} L & P \arrow[l, "\pi_P"'] \arrow[r, hook, "i"] & G \end{tikzcd}$$

"parabolic restriction".

<u>Note</u> (function-sheaf dictionary).

When $k = \overline{\mathbb{F}_q}$ & G is def'd / $\mathbb{F}_q$,

for any $K \in \mathcal{D}^b G$ s.t. $\exists \psi : \mathrm{Fr}_G^* K \xrightarrow{\sim} K$,

we can define a function

$$\chi^G_{K,\psi} : G(\mathbb{F}_q) \longrightarrow E \quad \text{by} \quad g \longmapsto \sum_i (-1)^i \cdot \mathrm{Tr}(\psi_g \mid (\mathcal{H}^i K)_g).$$

In fact, $\chi^L_{\mathrm{res}^G_P K(\dim U), \psi} = |U(\mathbb{F}_q)|^{-1} \cdot \sum_{u \in U(\mathbb{F}_q)} \chi^G_{K,\psi}(l \cdot u)$ for $\forall l \in L(\mathbb{F}_q)$.

"Tate twist" (ψ)

... nothing but the parab. rest. ("Jacquet functor")

in rep. theory of fin. gps of Lie type.

$\perp$

<u>Def.</u> A perv. shf $K \in \mathcal{M}G$ is "cuspidal" (resp. "strongly cusp.") if

(a) K has a weight for $Z(G)^\circ$-trans. action.

(b) K is G-equiv. (conjugation).

(c) For any $P = LU \subset G$ proper parabolic, $\mathrm{res}^G_P K \in \mathcal{D}L^{<0}$ (w.r.t. perv. t-str.)

(resp. $\mathrm{res}^G_P K = 0$).

$\perp$

$\left(\Longleftrightarrow {}^pH^i(\mathrm{res}^G_P K) = 0 \text{ for } i \geq 0 \right).$

$\left(\begin{array}{l} \text{by def. of perv. t-str,} \\ \dim \operatorname{supp} H^i(\mathrm{res}^G_P K) < -i \quad \forall i \end{array} \right).$

Stratification of G

Def. For semisimple G, we call a conj. class $C \subset G$ "isolated" if $\forall g \in C$, $Z_G(g_s)$ & G have the same semisimple rk.

(g_s: s-s. part of the Jordan decomp.)

$\left(\Leftrightarrow Z_G(g_s) \text{ is not contained in any proper Levi of } G \right)$.

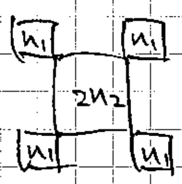
e.g. ◎ $G = SL_n$.

$Z(G)$ ⋯ only isolated semisimple elements.

◎ $G = Sp_{2n} := \left\{ g \in GL_{2n} \;\middle|\; {}^t g \cdot \begin{pmatrix} 0 & & -1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} \cdot g = \begin{pmatrix} 0 & & -1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} \right\}$.

$\cup$

$Sp_{2n_1} \times Sp_{2n_2}$ $(n = n_1 + n_2)$.



$\subset$ $2n$.

$\parallel$

$Z_G(\mathrm{diag}(\underbrace{1, \dots, 1}_{n_1}, \underbrace{-1, \dots, -1}_{2n_2}, \underbrace{1, \dots, 1}_{n_1}))$.

$\hookrightarrow$ isolated s-s. element.

(**Note**: char $k \neq 2$

$\rightsquigarrow$ $s \in Sp_{2n}$ or SO_n is iso. s-s.

$\Leftrightarrow$ ± 1 are only eigenvals of s.)

Fact $\#\{ C \subset G \mid \text{iso. conj. classes} \} < \infty$.

(s-s.)

Def. For conn. red. G, we call the inverse image of an iso. conj. class of $G/Z(G)^\circ$ an "isolated class".

$\hookrightarrow$ Note: not a conj. class!

- G : conn. red. / k
- (L, Σ) : $\left(\begin{array}{l} - L \subset G : \text{Levi.} \\ - \Sigma \subset L : \text{iso. class.} \end{array} \right.$

$\rightsquigarrow \Sigma_{reg} := \{ x \in \Sigma \mid Z_G(x_s)^\circ \subset L \}$.

$$\begin{cases} \cdot\ \Sigma_{reg} \subset \Sigma : \text{non-empty op.} \\ \cdot\ x \in \Sigma_{reg} \Rightarrow L \text{ is the smallest Levi s.t. } Z_G(x_s)^\circ \subset L \end{cases}$$

$\cdot\ Y_{(L,\Sigma)} := \bigcup_{g \in G} g \cdot \Sigma_{reg} \cdot g^{-1} \subset G.$

$\rightsquigarrow$

$$\begin{tikzcd} G \times \Sigma_{reg} \arrow[r, two heads] \arrow[dr] & G \times_L \Sigma_{reg} \arrow[d, "\gamma"] \\ & Y_{(L,\Sigma)} \end{tikzcd} \qquad l \cdot (g,x) := (g l^{-1}, l x l^{-1})$$

$(g,x) \mapsto g x g^{-1}$

γ ... Gal. cov. w/ Gal. gp. $W(L,\Sigma) := \underset{\{n \in G \mid nLn^{-1} = L,\ n\Sigma n^{-1} = \Sigma\}}{\underset{\parallel}{N(L,\Sigma)}}/L.$

CSO (CSII).

Fact. $G = \bigcup_{(L,\Sigma)} \underbrace{Y_{(L,\Sigma)}}_{\substack{G\text{-inv. loc. cl. sm. irr. subvar.} \\ \text{of } \dim G - \dim L + \dim \Sigma.}}$

Note. $\{(L,\Sigma) \mid L \subset G : \text{Levi},\ \Sigma \subset L : \text{iso}\}/G\text{-conj}$ is finite

$\rightsquigarrow$ get a stratification $G = \coprod_{\substack{i \in I \\ \text{fin. set}}} Y_{(L_i,\Sigma_i)}$.

$\left(L = T \subset G : \text{max. torus},\ \Sigma = T \rightsquigarrow Y_{(T,T)} = G_{reg} \underset{op}{\subset} G \right).$

$\left(L = G \rightsquigarrow Y_{(G,\Sigma)} = \underset{\overline{\Sigma}.}{\underset{\parallel}{\Sigma_{reg}}} \right).$

Closure of $Y_{(L,\Sigma)}$.

$\cdot\ P = \underset{\overline{\Sigma} \cdot U}{\underset{\cup}{L \cdot U}} \subset G$: parabolic.

$\rightsquigarrow$

$$\begin{tikzcd} G \times \Sigma_{reg} \arrow[r, hook] \arrow[d, phantom, "\cup"] \arrow[dr, phantom, "\circlearrowleft"] & G \times \overline{\Sigma} \cdot U \arrow[d, phantom, "\cup"] \\ G \times_L \Sigma_{reg} \arrow[r, hook, "\alpha", "\text{open}"'] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\square"] & G \times_P \overline{\Sigma} \cdot U \arrow[d, "\delta"] \\ Y_{(L,\Sigma)} \arrow[r, hook] & \overline{Y_{(L,\Sigma)}} \end{tikzcd}$$

$Y_{(L',\Sigma')} \subset \overline{Y_{(L,\Sigma)}} \iff \exists x \in \overline{\Sigma} \cdot U$ is conj. to an element of Σ'_{reg}.

Theorems on cusp. perv. sheaves. 2024.10.22 (K).

Recall (L,Σ) : $\begin{cases} - L\subset G : \text{Levi}. \\ - L \supset \Sigma : \text{isolated class.} \supset \Sigma_{reg} \end{cases}$

$\rightsquigarrow Y(L,\Sigma) := \bigcup_{g\in G} g\cdot \Sigma_{reg}\cdot g^{-1} \subset G.$

$\rightsquigarrow G = \bigsqcup_{(L,\Sigma)/\sim} Y(L,\Sigma)$ stratification. ($L=G \Rightarrow Y(G,\Sigma)=\Sigma_{reg}=\Sigma$

Thm. (1) Any irr. cusp. perv. shf K on G is of the form
$K = I(\overline{\Sigma},\mathcal{L})$, where $\exists!\,\Sigma\subset G$: iso, $\exists!\,\mathcal{L}$: irr. loc. sys. on Σ.

prf. Choose
- (L,Σ)
- $V\subset Y(L,\Sigma)$: dense, loc. cl. sm. irr. subvar, G-conj. inv. & $Z(G)^0$-trans. inv.
- $\mathcal{L}$: irr. loc. sys. on V. G-eq. & same $Z(G)^0$-wt as K.

first just choose $(V,\mathcal{L})$. Then replace V w/ $V\cap Y(L,\Sigma)$.

($\rightsquigarrow$ goal: to show $L=G$).

- Let $P=L\cdot U\subset G$: parab.
- Let $g\in V\cap\Sigma_{reg}$ ($\rightsquigarrow$ Fact: $\{ugu^{-1} \mid u\in U\} = g\cdot U$.

$$\begin{tikzcd} & & G\times_L(V\cap\Sigma_{reg}) \arrow[r, phantom, "\subset"] \arrow[d] \arrow[dr, phantom, "\square"] & G\times_L\Sigma_{reg} \arrow[d, "\text{gal.}"] & & \circledast \\ g\cdot U \arrow[r, phantom, "\subset"] & & V \arrow[r, phantom, "\subset"] & Y(L,\Sigma) \arrow[r, phantom, "\subset"] & \overline{Y(L,\Sigma)} \arrow[r, phantom, "\subset"] & G. \end{tikzcd}$$

$\mathcal{L}|_{g.U}[\dim V] \leftsquigarrow \mathcal{L}[\dim V] \xleftarrow{\text{rest.}} K = I(V,\mathcal{L})$. supp'd here

U-eq $\Rightarrow$ const. G-eq.

put $d:=\dim U$. $\rightsquigarrow H_c^{2d}(g.U,\mathcal{L}) \neq 0$ ($U\simeq \mathbb{A}^d$).

$\rightsquigarrow H^i(\mathrm{res}_P^G K)_g \neq 0$. $i=2d-\dim V$. (proper b.c.)

$$\begin{tikzcd} gU \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & P \arrow[r, "i"] \arrow[d, "\pi_P"] & G \\ g \arrow[r] & L & \end{tikzcd}$$

$(\pi_P)_!\, i^* K = \mathrm{res}_P^G K$

$\therefore\ \operatorname{supp} H^i(\operatorname{res}^G_P K) \supset \underbrace{V \cap \Sigma_{\mathrm{reg}}}_{\dim = \dim V - 2d \text{ by } ⊛}$ $\quad i = 2d - \dim V.$

Cuspidality of $K \Rightarrow P$ cannot be proper. $\square$.

Thm. (2) $K = I(\overline{\Sigma}, \mathcal{L})$: as in the above Thm.
Let $P = LU \subsetneq G$ proper parab & $l \in L$.
Put $e := \dim \Sigma - \dim Z(G) - \dim L + \dim Z_L(l)$.
Then $H^i_c(l\cdot U \cap \Sigma, \mathcal{L}) = 0$ for $\forall\, i \geq e$.

prf. Lusztig's dimension estimate ([CS0]):

$$\dim(l\cdot U \cap \Sigma) \leq \tfrac{1}{2} e.$$

$\rightsquigarrow$ we only need to show $H^e_c(l\cdot U \cap \Sigma, \mathcal{L}) = 0$.

long ex. seq. w.r.t. $l\cdot U = (l\cdot U \cap \Sigma) \sqcup (l\cdot U \cap (\overline{\Sigma} \setminus \Sigma))$:

$$H_c^{e-\dim\Sigma-1}(l\cdot U \cap (\overline{\Sigma}\setminus\Sigma), K) \xrightarrow{\delta} \underbrace{H_c^{e-\dim\Sigma}(l\cdot U \cap \Sigma, K) \ (= H^e_c(l\cdot U \cap \Sigma, \mathcal{L}))}_{\text{want to kill}}$$
$$\longrightarrow H_c^{e-\dim\Sigma}(l\cdot U, K) = 0.$$

(∵) K: cusp. $\Rightarrow \operatorname{res}^G_P K \in DL^{<0}$, i.e. $\dim \operatorname{supp} H^i(\operatorname{res}^G_P K) < -i$.
If $H^{\bullet}_c(l\cdot U, K) \neq 0$, $l \in \operatorname{supp} H^{\bullet}(\operatorname{res}^G_P K)$. $\quad \bullet = e - \dim\Sigma$
K: G-eq. & has wt for $Z(G)^\circ$
$\Rightarrow \underbrace{Z(G)^\circ \cdot (L\cdot l)}_{\dim = \dim L - \dim Z_L(l) + \dim Z(G)} \subset \operatorname{supp}$.
$\therefore\ \dim L - \dim Z_L(l) + \dim Z(G) < -(e - \dim\Sigma)$. ↯.

$\rightsquigarrow$ enough to show: $\delta = 0$.

BBD's reduction:
We may assume $k = \overline{\mathbb{F}}_q$, $G, K, \mathcal{L}$: def'd / $\mathbb{F}_q$.
$\mathcal{L}$: wt 0 (K: pure of wt $\dim \Sigma$).

We may suppose $\dim(l\cdot U \cap \Sigma) = \tfrac{1}{2} e$.

$\rightsquigarrow$ Weil conj.: $H_c^e(\ell\cdot U\cap\Sigma,\mathcal{L})$: pure of wt e.

- $\Sigma'\subset\overline{\Sigma}\setminus\Sigma$: preimage of a conj. class in $G/Z(G)^\circ$.

$\rightsquigarrow$ enough to show: $H_c^{e-\dim\Sigma-1}(\ell\cdot U\cap\Sigma',K)$: wt $<e$.

[spec. seq of hypercoh.
$H_c^p(\ell\cdot U\cap\Sigma',\mathcal{H}^q(K))\Rightarrow H_c^{p+q}(\ell\cdot U\cap\Sigma',K)$.

nonzero only when
- $p\leq 2\dim(\ell\cdot U\cap\Sigma')\leq e-\dim\Sigma+\dim\Sigma$ Lusztig, [CSO].
- $q<-\dim\Sigma'$ ($\underset{\wedge\!\!\!\backslash}{<-\dim\Sigma}$) ($K$: perversity).

$\rightsquigarrow$ When $p+q=e-\dim\Sigma-1$,

only
- $p=2\dim(\ell\cdot U\cap\Sigma')=e-\dim\Sigma-\dim\Sigma'$
- $q=-\dim\Sigma'-1$

is possible.

As Weil conj. $\mathcal{H}^qK$: pure of wt $\dim\Sigma+q$. $\leftarrow$ (def of K: pure)

$\Rightarrow H_c^p(\ell\cdot U\cap\Sigma',\mathcal{H}^qK)$: pure of wt $\underbrace{\dim\Sigma+p+q}_{=e-1}$. $\square$

[(3) Thm. $K=IC(\Sigma,\mathcal{L})$ as before.
For ${}^\forall g\in\Sigma$, $Z_G(g)^\circ/Z(G)^\circ$ is unip.

Outline of pf.
- $S\subset Z_G(g)^\circ$: max. torus.
- $L:=Z_G(S)\subset G$: Levi. ($\to$ to show: $L=G$).
- Suppose $P=\underset{\substack{\cup\\ g}}{L}\cdot U\subsetneq G$ for contradiction.

$V:=g\cdot U\cap\Sigma \overset{\text{trans.}}{\curvearrowleft} U$

$\rightsquigarrow H_c^{2\dim V}(V,\underbrace{\mathcal{L}|_V}_{\text{const.}})\neq 0$.

Thm 2 $\Rightarrow$ $2\dim V<$ "e". But [CSO] implies $\geq$. $\lightning$.