Reminder of setup

- G : connected reductive group / $k = \bar{k}$

(B, T) Borel pair

$R = R(G,T) \supseteq R^+ \supseteq \Delta$
(positive roots) . (base)

$W = W(G,T) = N_G(T)/T \supseteq S = \{s_\alpha \,;\, \alpha \in \Delta\}$

$X = X^*(T)$ $\qquad \hat{X} = X \otimes \mathbb{Z}_{(p)} / X$

$Y = X_*(T)$ $\qquad \hat{X} \times Y \to \mathbb{Z}_{(p)}/\mathbb{Z}$.

$\xi \in \hat{X}$, $R_\xi = \{\alpha \in R \,;\, \langle \xi, \check{\alpha} \rangle \in \mathbb{Z}\}$

$R_\xi^+ = R_\xi \cap R^+$ $\qquad W_\xi = W(R_\xi) \subseteq W_\xi' = \mathrm{Stab}_W(\xi)$.

Prop. $\xi \in \hat{X}$, $w \in W_\xi \cap W_\xi'$ $\Rightarrow$ $\operatorname{ind}^P_G C^L_{\xi,w} \cong C^G_{\xi,w}$.

$$
\begin{tikzcd}
L \times_{B_L} \bar{L}_w \arrow[d, "r^L"'] & G \times (P \times^B \bar{G}_w) \arrow[l, "\alpha_1"'] \arrow[r, "\beta_1"] \arrow[d, "id \times r^G"] & G \times^B \bar{G}_w \arrow[d, "\varepsilon"] \arrow[dr, "r^G"] & \\
L & G \times P \arrow[l, "\alpha"'] \arrow[r, "\beta"] & G \times^P P \arrow[r] & G
\end{tikzcd}
$$

Since $\pi_P : P \to L$ $\rightsquigarrow$ $\bar{G}_w \to \bar{L}_w$ is a fibration with fibre $\cong U$
(via Bruhat decomp)

$\Rightarrow$ $\pi_P^* A^L_{\xi,w}[\dim U] \cong A^G_{\xi,w}$.

$$\alpha_1^* \tilde{A}^L_{\xi,w}[\dim U] = \beta_1^* \tilde{A}^G_{\xi,w}$$

$C^G_{\xi,w} = r^G_* \tilde{A}^G_{\xi,w} = \delta_*(\varepsilon_* \tilde{A}^G_{\xi,w})$ $\quad$ $\beta^* \varepsilon_*(\tilde{A}^G_{\xi,w}) = (id \times r_G)_* (\beta_1^* \tilde{A}^G_{\xi,w})$

$= (id \times r_G)_* \alpha_1^* (\tilde{A}^L_{\xi,w})[\dim U] = \alpha^* C^L_{\xi,w}$

$\Rightarrow$ $C^G_{\xi,w} \cong \operatorname{ind}_P^G C^L_{\xi,w}$

$\not{t}$

- Hecke algebra : given $\mathcal{O} \in \hat{X}/W$. $K_{\mathcal{O}}$: alg. / $\mathbb{Z}[t^{\pm 1}]$ with basis $e_{\xi, w}$, $\xi \in \mathcal{O}$, $w \in W$.

a/ $e_{\xi,x} e_{\eta,y} = 0$ if $\xi \neq y\eta$

b/ $e_{y\eta, s} e_{\eta, y} = \begin{cases} e_{\eta, sy} & \text{if } sy > y \text{ or } s \notin W_{y\eta} \\ (t^2 - 1) e_{\eta y} + t^2 e_{\eta, sy} & \text{if } sy < y \text{ and } s \in W_{y\eta} \end{cases}$

c/ $\sum_{\xi \in \mathcal{O}} e_{\xi, e} = 1$

bar (= KL) involution : $e_{\xi,x} \mapsto \overline{e_{\xi,x}} \in e_{x\xi} K$ $\quad \xi \in \mathcal{O}, x \in W$.

s.t. $e_{x\xi, x^{-1}} \cdot \overline{e_{\xi,x}} = e_{\xi,e}$

$t \mapsto t^{-1}$

Parabolic subgroups & induced modules.

- Let $I \subseteq S \subseteq W$. $W_I = \langle I \rangle$. $P := P_I = \bigcup_{w \in W_I} \overset{G_w}{\overset{\shortparallel}{\overbrace{BwB}}} \subseteq G$

$$\begin{array}{ccc} & P & \\ & \cup\!\!| & \\ T \subseteq & L & \end{array} \qquad P \underset{\pi}{\twoheadrightarrow} L$$

$B_I = B \cap L$ $\quad R_I = R(L,T)$ $\quad R_I^+ = R^+ \cap R_I$. $\quad \Delta_I = \Delta \cap R_I$

Minimal left coset representatives: ${}^I W \subseteq W \longrightarrow W_I \backslash W$, ${}^I W \xrightarrow{\sim} W_I \backslash W$.

characterised by $v \in {}^I W \Leftrightarrow \pi(vBv^{-1} \cap P) = B_I$.

For $v \in {}^I W$, put $O_v = PvB = \bigcup_{w \in W_I} G_{wv}$.

- Let $O \in \hat{X}/W$. $K = K_O$ $\quad O = \coprod_i O_i$, $\quad O_i \in \hat{X}/W_I$.

$K_i = K_{O_i} = \langle e_{\xi, w} \rangle_{\xi \in O_i, w \in W_I}$ $\quad K_I = \bigoplus_i K_i$ idempotent subalg.

Lem. K is a free left K_I-module with basis $\{u_v := \sum_{\xi \in G} e_{\xi, v} \;;\; v \in {}^I W\}$

Pf: $(G \times W_I) \times {}^I W \xrightarrow{\sim} G \times W \quad \rightsquigarrow \quad K_I \otimes \mathbb{Z}\,{}^I W \xrightarrow{\sim} K$

$(\xi, w, v) \mapsto (\xi, wv) \qquad\qquad e_{\xi, w} \otimes v \mapsto e_{\xi, w} \cdot u_v = e_{\xi, wv}$

(because $l(wv) = l(w) + l(v)$).

#.

· trace map: define $tr_I : K/[K,K] \to K_I/[K_I, K_I]$ via

$$u \in K, \rightsquigarrow \underset{\text{(right mult.)}}{\rho(u)} \in End_{K_I}(K) \rightsquigarrow tr_I(u) \in K_I/[K_I, K_I]$$

Lem. $tr_I \bar{u} = \overline{tr_I u}$.

Pf $\bar{K_I} = K_I$, $\{\bar{u_v} \;;\; v \in {}^I W\}$ is also a basis. #

• Parabolic restriction of character sheaves.

Tate twist

$i : P \to G$, $\pi : P \to L$. $\mathrm{res} = \mathrm{res}^G_P = \pi_! i^* (\dim U) : D_G(G) \to D_L(L)$.

Fix $\xi \in \hat{X}$, $w \in W_\xi'$. $\rightsquigarrow A_{\xi,w} \in D\left(\frac{U\backslash G/U}{T}\right)$

$$\downarrow$$

$C^G_{\xi,w} := C_{\xi,w} \in D_G(G)$ semisimple cpx.

Thm. i/ $\mathrm{res}\, C^G_{\xi,w} \cong \bigoplus_{\substack{\eta \in \mathcal{O} \\ x \in (W_L)'_\eta}} C^L_{\eta,x} \otimes (\text{multiplicity})$

ii/

$$\begin{tikzcd}
K_G/[K_G,K_G] \arrow[r, "tr_I"] \arrow[d, "\tau^G"] \arrow[dr, phantom, "\circlearrowright"] & K_L/[K_L,K_L] \arrow[d, "\tau^L"] \\
K_0(D_G(G)) \arrow[r, "\mathrm{res}"] & K_0(D_L(L))
\end{tikzcd}$$

i.e. $\tau^L tr_I [A^G_{\xi,w}] = [\mathrm{res}\, C^G_{\xi,w}]$

Coro. $A \in \hat{G}$. Then

i/ A is adm ii/ $P \subseteq G$ parab $\Rightarrow res_P^G A = \bigoplus$ character shves.

Pf. Induction on $\dim G$.

Suppose $A \in {}^pH^i(C^G_{\mathfrak{z},w})$ $[C_{\mathfrak{z},w}] = \sum_i f_i(t)[A_i]$.

$f_i \in \mathbb{N}[t^{\pm 1}]$. $f_i(t) = f_i(t^{-1})$ by relative hard Lefschetz.

$res\, C^G_{\mathfrak{z},w} = \bigoplus_{\eta,x} C^L_{\eta,x} \otimes (\text{mult.})$ is adm and ind_P^G is t-exact on $C^L_{\eta,x}$.

$\Rightarrow res\, C^G_{\mathfrak{z},w} \in {}^pD^{\leq 0}$. Braden $\Rightarrow res\, C^G_{\mathfrak{z},w} \in {}^pD^{\geq 0}$.

$\Rightarrow res^G C_{\mathfrak{z},w} = \bigoplus$ character sheaves (ii).

i/. Trivial if A cusp. Otherwise $\exists$ char. C s.t. $\mathrm{Hom}(res_P^G A, C) \neq 0$

$\quad \wr\!\wr$

$\mathrm{Hom}(A, ind_Q^G C)$.

$\Rightarrow A$ adm. ∎

Thm. is equivalent to:

Thm' $\forall\ \underline{s} = (s_1 \sim s_r) \in S^r$

i/ $\operatorname{res} C^G_{\xi,\underline{s}} \simeq \bigoplus_{\substack{x \in W'_{I,\eta} \\ \eta \in \mathfrak{G}}} C^L_{\eta,x} \otimes (\text{multiplicity})$

ii/ $\tau^L \operatorname{tr}_I [\underbrace{A^G_{s_2 \cdots s_r \xi, s_1} * A^G_{s_3 \cdots s_r \xi, s_2} * \cdots * A^G_{\xi, s_r}}_{= A^G_{\xi,\underline{s}}}]$

$= [\operatorname{res} C^G_{\xi,\underline{s}}]$

Outline:

$$
\begin{tikzcd}[column sep=small]
 & & & \coprod_{\underline{v}} Z_{\underline{v}} \arrow[r, "p", "\text{affine bundle}"'] \arrow[d, phantom, "\text{stratif} \;\; ="] & \coprod_{\underline{s}'} L \times^{B_L} (\bar{L}_{s_1'} \times^{B_L} \cdots \times^{B_L} \bar{L}_{s_r'}) = \bar{Y}^L_{\underline{s}'} \arrow[dr] & \\
 & & \bar{Y}^G_{\underline{s}} \arrow[r, phantom, "\supset"] & \bar{Y}^P_{\underline{s}} \arrow[dl] \arrow[dr] & & L \\
\bar{G}_{s_1} \times^B \cdots \times^B \bar{G}_{s_r} \arrow[r, hook] \arrow[d] & G \times^{B,Ad} (\bar{G}_{s_1} \times^B \cdots \times^B \bar{G}_{s_r}) \arrow[d] \arrow[drr] & & & P \arrow[ur] \arrow[dl] & \\
G \arrow[r, hook] & G \times^{B,Ad} G \arrow[rr] & & G & &
\end{tikzcd}
$$

$$\overset{\omega}{A}_{\xi,\underline{s}} \longrightarrow \tilde{A}_{\xi,\underline{s}} \longrightarrow C_{\xi,\underline{s}}$$

Z can be stratified $\bar{Y}^P_{\underline{s}} = \coprod_{\underline{v} \in ({}^I W)^r} Z_{\underline{v}}$

$$Z_{\underline{v}} = \{ [g_0 : (g_1 \dots g_r)] \in \bar{Y}^G_{\underline{s}} \mid g_0 \cdots g_i \in P \dot{v}_i B,\ g_0 g_1 \cdots g_r g_0^{-1} \in P \}$$

We shall study thus the sheaf $\mathcal{L}^G_{\xi,s}$ on G_s and $\mathcal{L}^G_{\xi,e}$ on G_e restricted to $\dot{v}_1^{-1} P \dot{v}_2$ for various $v_1, v_2 \in {}^I W$.

compare them with $\mathcal{L}^L_{\xi,s}$ and $\mathcal{L}^L_{\xi,e}$ and twists by v_1 or v_2.

- Now, fix $v_1, v_2 \in {}^I W$ $s \in S$ and put

$$Z = \bar{G}_s \cap \dot{v}_1^{-1} P \dot{v}_2 \qquad Z_s = G_s \cap Z \qquad Z_e = B \cap Z .$$

$$\begin{tikzcd} g \arrow[d, mapsto] \in & Z = Z_e \amalg Z_s \arrow[d, "\varphi"] \\ \pi(\dot{v}_1 g \dot{v}_2^{-1}) & L \end{tikzcd}$$

- im φ is union of B_I-double cosets: $v \in {}^I W$ is characterised by

$$v^{-1} R_I^+ \subseteq R^+ \iff v R^+ \cap R_I^- = \emptyset \iff \pi(\dot{v} B \dot{v}^{-1} \cap P) = B_I$$

Thus $B_I \varphi(Z_e) B_I = \pi(\dot{v}_1 B \dot{v}_1^{-1} \cap P) \pi(\underbrace{\dot{v}_1 Z_e \dot{v}_2^{-1}}_{\dot{v}_1 B \dot{v}_2^{-1} \cap P}) \pi(\dot{v}_2 B \dot{v}_2^{-1} \cap P)$

$\subseteq \pi(\dot{v}_1 Z_e \dot{v}_2^{-1}) = \varphi(Z_e)$

Similarly, $B_I \varphi(Z_s) B_I \subseteq \varphi(Z_s)$ $\qquad \varphi_e := \varphi|_{Z_e} \quad \varphi_s := \varphi|_{Z_s}$

Lem. $Z \neq \emptyset \Rightarrow v_1 = v_2$ or $v_1 s = v_2$. and one of the following holds:

Case a/ $v_1 s \notin {}^I W$. Then $v_1 = v_2 < v_1 s$ and $\sigma = v_1 s v_2^{-1} \in I$.

$$\varphi Z_s = L_\sigma, \ \varphi Z_e = L_e \text{ and } \varphi_s^* \mathcal{L}^L_{\xi,\sigma} = \mathcal{L}^G_{v_1^{-1}\xi, s}\big|_{Z_s}$$

Case b/ $v_1 s \in {}^I W$, $\varphi Z = L_e \ (= B_I)$.

bi/ $v_1 = v_2 < v_1 s$. Then $Z = Z_e$ and $\varphi_e^* \mathcal{L}^L_{\xi,e} \simeq \mathcal{L}^G_{v_1^{-1}\xi,e}\big|_{Z_e}$

bii/ $v_1 = v_2 > v_1 s$ Then $Z = Z_e \cup Z_s$ and $\varphi_s^* \mathcal{L}^L_{\xi,e} \simeq \mathcal{L}^G_{v_1^{-1}\xi,s}\big|_{Z_s}$ if $s \in W_{v_1^{-1}\xi}$

biii/ $v_1 s = v_2$. Then $Z = Z_s$ and $\varphi_s^* \mathcal{L}^L_{\xi,e} \simeq \mathcal{L}^G_{v_2^{-1}\xi,s}\big|_{Z_s}$

Pf. $g \in \dot v_1 Z \dot v_2^{-1} \Rightarrow g \dot v_2 \in \dot v_1 \bar G_s$. Let $x \in W_I$ s.t. $g \in G_x$.

$P = \bigcup_{w \in W_I} B \dot w B$. Now, $\dot v_1 \bar G_s \subseteq G_{v_1} \cup G_{v_1 s}$

Since $\underset{\substack{\Psi \\ g \dot v_2}}{G_x \cdot G_{v_2}} = G_{x v_2}$ $(\ell(x v_2) = \ell(x)\ell(v_2)) \Rightarrow x v_2 = v_1$ or $v_1 s$.

- If $v_1 s \notin {}^I W$, then $v_1 s > v_1$

$\sigma := v_1 s v_1^{-1}$: reflection. $\sigma v_1 s = \underset{{}^I W}{\underset{\cap}{v_1}} < \underset{{}^I W}{\underset{\notin}{v_1 s}}$, $l(\sigma v_1 s) = l(v_1 s) - 1$.

$v_1 s = u u'$ red $\underset{u_1 u_2 \cdots u_r}{\underset{\|}{u}} \in W_I$, $\underset{u_1' u_2' \cdots u_{r'}'}{\underset{\|\,\text{red}}{u'}} \in {}^I W$ $\underset{\text{exchange}}{\Rightarrow}$ $\sigma v_1 s = u_1 \underbrace{\cdots \hat{u_r} u_1' \cdots u_{r'}'}_{\text{reduced}}$.

$\Rightarrow u = \sigma \in I$.

Suppose $x \neq e$, then $x v_2 \neq v_1 \Rightarrow x v_2 = v_1 s$

Now $v_1 = \sigma v_1 s = \sigma x v_2$ and $\sigma x \in W_I$ $\Rightarrow \sigma = \sigma^{-1} = x$ and $v_1 = v_2$

Suppose $x = e$, then $x v_2 \neq v_1 s \Rightarrow v_1 = v_2$

so $v_1 = v_2$ in both cases

•• $\varphi(Z_e) = \pi(\dot{v}_1 B \dot{v}_1^{-1} \cap P) = B_I = L_e$

•• $\dot{v}_1 G_s \dot{v}_1^{-1} \cap P \subseteq B \dot{v}_1 \dot{s} B \dot{v}_1^{-1} \cap P = \underset{P}{\underset{\cap}{\underbrace{B \dot{\sigma}}}} \dot{v}_1 B \dot{v}_1^{-1} \cap P = B\dot{\sigma}(\dot{v}_1 B \dot{v}_1^{-1} \cap P)$

$\Rightarrow \varphi(Z_s) \subseteq B_I \dot{\sigma} B_I = L_\sigma$ so $\varphi(Z) \subseteq \overline{L_\sigma}$

• Assume $v_1 s \in {}^I W \Rightarrow \quad x v_2 = \begin{cases} v_1 \in {}^I W \\ v_1 s \in {}^I W \end{cases} \quad \Rightarrow x = e.$

.. if $v_1 = v_2 < v_1 s$, then $\dot v_1 G_s \dot v_1^{-1} \cap P \subseteq B \dot v_1 \dot s B \dot v_1^{-1} \cap P = B \cdot (\dot v_1 \dot s B \dot v_1^{-1} \cap P)$
$(v_1 s \in {}^I W)$

$= B \cdot (\dot v_1 \dot s B \cap P \dot v_1) \dot v_1^{-1} = \emptyset$ since $v_1 s \notin W_I \cdot v_1$

$\Rightarrow Z = Z_e$

.. if $v_1 \neq v_2$, then $v_1 s = v_2$ and $\dot v_1 G_e \dot v_2^{-1} \subseteq B \dot v_1 \dot s B \dot v_1^{-1} \cap P = \emptyset$

$\Rightarrow Z = Z_s$

Comparison of local systems :

• Case a) $\begin{pmatrix} v_1 = v_2 \\ \sigma = v_1 s v_1^{-1} \in I \end{pmatrix}$

$$\begin{tikzcd} G_s \arrow[dr, "pr^G"'] & Z_s \arrow[l, hook', "i"'] \arrow[r, "\varphi_s"] \arrow[dr, phantom, "\circlearrowleft"] & L_\sigma \arrow[d, "pr^L"] \\ & T \arrow[r, "v_1"] & T \end{tikzcd}$$

Recall $pr^G : G = U_s \cdot \dot s T U \to T$

$\Rightarrow \varphi_s^* \mathcal{L}^L_{\xi, \sigma} = \mathcal{L}^G_{v_1^{-1}\xi, s}$

bi & biii similarly.

· Case bii/ apply bi/ to $v_1 s$ and $v_2 s$

$\Rightarrow s^{-1} Z s = s^{-1}(\bar{G}_s \cap v_1^{-1} P v_1) s = B \cap s v_1^{-1} P v_1 s$

$\Rightarrow Z = s B s^{-1} \cap v_1^{-1} P v_1$, $\quad Z_e = v_1^{-1}(\underset{\substack{\cap \\ G_{v_1 s} \cup G_{v_1}}}{v_1 \bar{G}_s} \cap \underset{\substack{\cap \\ \bigcup_{w \in {}^I W} G_{w v_1}}}{P v_1}) = v_1^{-1}(v_1 B \cap B v_1) = B \cap v_1^{-1} B v_1$

Let $\alpha \in \Delta$ s.t. $s = s_\alpha$, $X_{-\alpha} \subseteq G$.

$U' := U \cap v_1^{-1} U v_1 \qquad \Rightarrow Z_s = T \cdot (X_{-\alpha} \setminus \{e\}) \cdot U' \subseteq G_s$

We have $\psi : X_{-\alpha} \setminus \{e\} \xrightarrow{\sim} \mathbb{G}_m$ s.t. $pr^G u = \alpha^\vee(\psi u)$ for $u \in X_{-\alpha} \setminus \{e\}$

extends to $\psi : Z_s = T \times (X_{-\alpha} \setminus \{e\}) \cdot U' \to \mathbb{G}_m$.

$$\begin{tikzcd} G_s & Z_s \arrow[l, hook', "i"'] \arrow[r, "\varphi_s"] & L_e \arrow[d, "pr^L"] \\ & T \arrow[from=ul, "pr^G"'] \arrow[r, "v_1 s"] & T \end{tikzcd}$$

Set $\mu : T \times \mathbb{G}_m \to T$
$(t, a) \mapsto t v_1 s(\alpha^\vee(a))$

$\rho = (pr^L \varphi_s, \psi) : Z_s \to T \times \mathbb{G}_m$

Then $v_1 s \circ pr^G \circ i = \mu \circ \rho$

Now $\mathcal{L}^G_{v_1^{-1}\xi, s}\big|_{Z_s} \cong \mathcal{L}^G_{s^{-1}v_1^{-1}\xi, s}\big|_{Z_s} \cong (\mu\circ\rho)^*\mathcal{L}_\xi$ We have $(v_1 s\circ\alpha^\vee)^*\mathcal{L}_\xi = \underset{\text{const.}}{E}$

$(s\in W_{v_1^{-1}\xi})$

$\Rightarrow (\mu\circ\rho)^*\mathcal{L}_\xi \cong (\varphi^\sigma\circ pr_1^*)(\mathcal{L}_\xi)$. $pr_1 : T\times\mathbb{G}_m\to T$. $(pr_1\circ\rho = pr^L\circ\varphi_s)$

$\cong \varphi_s^*\mathcal{L}^L_{\xi,e}$. ∎

<u>Lem.</u> $\varphi : Z\to I_0$ (case a/) and $\varphi : Z_e\to L_e$ (case b/) is a loc. triv fibration by affine spaces.