Representation theory of p-adic groups

2025.10.24.
Char(LG) seminar

- F: non-arch. local field
  (fin. ext. of $\mathbb{Q}_p$, $\mathbb{F}_p((t))$)
  $F \supset \mathcal{O}_F \supset \mathfrak{p}_F$, $\mathcal{O}_F/\mathfrak{p}_F \simeq \mathbb{F}_q$. char $p>0$.

- $\mathbb{G}$: a conn. red. gp / F
  $\rightsquigarrow$ $G := \mathbb{G}(F)$ locally profinite group "p-adic reductive gp"

Recall (π, V) : a rep. of G is "smooth" $\underset{\text{def.}}{\iff}$ $\mathrm{Stab}_G(v) \underset{\text{op}}{\subset} G$ $\forall v \in V$.

☆ We are interested in $\mathrm{Rep}(G)$:= cat. of sm. rep'ns of G. rep ⋯ / $\mathbb{C}$.
$\Pi(G) := \{\text{irr. sm. rep'ns of } G\}/\sim$.

Why?
- parallel to $\mathrm{Rep}(\mathbb{G}(\mathbb{R}))$. Harish-Chandra's Lefschetz principle
- arise as local compo. of automorphic rep'ns
- expected to be related to Galois rep'ns. Langlands corresp.
- "contains" $\mathrm{Rep}(\mathbb{G}(\mathbb{F}_q))$.

etc...

§. Supercuspidal rep'ns.

◎ $\mathrm{Ind}_H^G(-) : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$. induction
- $H \subset G$: a closed subgp
- (ρ, W) : a smooth rep. of H.

$\rightsquigarrow$ define $(\mathrm{Ind}_H^G \rho, \mathrm{Ind}_H^G W)$ a sm. rep. of G by

$$\mathrm{Ind}_H^G W := \{ f : G \underset{\text{smooth}}{\to} W \mid f(hg) = \rho(h) f(g) \ \forall h \in H, \forall g \in G \}$$

(i.e. loc. const.)

$\underset{\mathrm{Ind}_H^G \rho}{\circlearrowleft} G$ via right trans.

- $c\text{-}\mathrm{Ind}_H^G(-) : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$ compact induction
  - $H < G$: an open subgp
  - (ρ, W) : a sm. rep. of H

$\rightsquigarrow$ define $(c\mathrm{Ind}_H^G \rho, c\mathrm{Ind}_H^G W)$ by

$$c\mathrm{Ind}_H^G W := \{ f : G \to W \mid \underbrace{\qquad\qquad /\!/ \qquad\qquad}_{\text{smooth \& supp}f\text{ : cpt mod } H} \},$$

Note. $\mathrm{Res}_H^G(-) : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$

$$c\mathrm{Ind}_H^G(-) \dashv \mathrm{Res}_H^G(-) \dashv \mathrm{Ind}_H^G(-)$$ adjunctions

Frobenius reciprocities.

Def $G \supset \mathbb{P} = \mathbb{M} \cdot \mathbb{U}_{\mathbb{P}}$ parabolic w/ Levi $\mathbb{M}$

π_M : a sm. rep. of M.

$\rightsquigarrow$ $i_P^G \pi_M := \mathrm{Ind}_P^G(\pi_M \otimes \delta_P)$. parabolic ind.

regarded as a rep. of P ($\twoheadrightarrow M$).

modulus char $P \twoheadrightarrow M \to \mathbb{C}^\times$.

Def. π : an irr. sm. rep. of G is supercuspidal

$\overset{\text{def.}}{\Longleftrightarrow}$ π cannot be a subquot. of $i_P^G \pi_M$ for any $G \supsetneq \mathbb{P}$ & π_M.

Fact For any irr. sm. rep π of G,

there is a unique (up to G-conj.) pair (M, π_M) of

- a Levi M ($\subset \mathbb{P}$ parab.)
- an irr. s.c. rep. π_M of M

called a cusp. pair.

s.t. π is a subquot of $i_P^G \pi_M$.

called the cusp. supp. of π.

supercusp. repn's are "atoms / building blocks" of Rep(G).

Lem. $\left\{ \begin{array}{l} \cdot K \subset G \text{ an op. cpt-mod-center subgp} \\ \cdot \rho : \text{irr. sm. rep. of } K. \end{array} \right.$
If $c\mathrm{Ind}_K^G \rho$ is of fin. length, then all the irr. constituents are s.c.

Conj All s.c. repn's are obtained by cInd

For example, if we choose $\rho = \mathbb{1}$, then

$$\mathrm{End}_G(c\mathrm{Ind}_K^G \mathbb{1}) \simeq \mathrm{Hom}_K(\mathbb{1}, \mathrm{Res}_K^G\, c\mathrm{Ind}_K^G \mathbb{1}).$$

$$\simeq \dim_{\mathbb{C}} \{ f : \underbrace{K \backslash G / K}_{\text{NOT finite almost always.}} \to \mathbb{C} : \text{cpt.supp} \}.$$

$\rightsquigarrow (K, \mathbb{1})$ is not good for getting s.c. repn's.

How to get s.c. repn's $=$ How to find nice $(\overset{\text{cpt-mod-center}}{K}, \overset{\text{fin. dim. rep.}}{\rho})$. call "s.c. types".

Rem. $\mathcal{H}(G, K, \rho) := \mathrm{End}_G(c\mathrm{Ind}_K^G \rho)$ is called the "Hecke algebra" for (K, ρ).
If $c\mathrm{Ind}_K^G \mathbb{1} =: \pi$ is s.c., then

$$\text{"}\pi\text{-part of } \mathrm{Rep}(G)\text{"} \simeq \mathrm{Mod}(\mathcal{H}(G, K, \rho))$$

This identification itself can be generalized to (K, ρ) s.t. $\pi = c\mathrm{Ind}_K^G \mathbb{1}$ is not s.c. (Bushnell-Kutzko's "type")

(e.g. "unramified part of Rep(G)" $\simeq \mathrm{Mod}(\underset{\text{Iwahori-Hecke alg.}}{\mathcal{H}}(G, \underset{\text{Iwahori subgp.}}{I}, \mathbb{1}))$.

formulated in terms of Bernstein decomp. of Rep(G).

§. Depth-zero s.c. reprs.

Recall $x \in \mathcal{A}$. point of apartment of G.

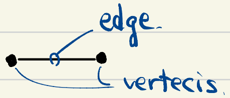
$\rightsquigarrow$ $G_{x,0} \subset G$ parahoric subgp (op. cpt).

$\{G_{x,r}\}_{r \in \mathbb{R}_{\geq 0}}$ Moy-Prasad filtr.

Key $G_{x,0}/G_{x,0+} \simeq \underset{\nwarrow \text{ a conn. red. gp } / \mathbb{F}_q}{\mathbb{G}(\mathbb{F}_q)}$

e.g. $G = GL_2$. $\rightsquigarrow$ $\mathcal{A}$: 1-dim. simplicial set.

$\cup$

C : alcove = 

- x = vertex $\rightsquigarrow$ $G_{x,0}/G_{x,0+} \simeq \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} / \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix} \simeq GL_2(\mathbb{F}_q)$.
- x = edge $\rightsquigarrow$ $G_{x,0}/G_{x,0+} \simeq \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ \mathfrak{p} & \mathcal{O}^\times \end{pmatrix} / \begin{pmatrix} 1+\mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix} \simeq \mathbb{G}_m^2(\mathbb{F}_q)$.

Moy-Prasad's theorem

(1) · $x \in \mathcal{A}$ a vertex. $K := \overset{\text{center}}{Z} \cdot G_{x,0}$

· ρ : an irr. cuspidal rep. of $G_{x,0}/G_{x,0+} \simeq \mathbb{G}(\mathbb{F}_q)$.

$\tilde{\rho}$:= an ext. of ρ to K

$\rightsquigarrow$ $(K, \tilde{\rho})$ is a s.c type s.t. $c\text{-}\mathrm{Ind}_K^G \tilde{\rho}$ is of depth 0

(2) any depth 0 s.c. rep. has a type of this form.

§ Positive-depth s.c. (Adler ('98), Yu ('01))

p : odd & $p \nmid |W_G|$ ← Weyl gp.

- $S \subset G$ an elliptic max. torus, tamely ramified
  - elliptic ← split rank is minimal. S/Z is compact.
  - tamely ramified ← splits / tame ext. of F.

$\rightsquigarrow \exists !\, x \in \mathcal{A}$ is associated.
(S elliptic & tamely-ramified descent)

- $\theta : S \to \mathbb{C}^\times$ char. of depth $r > 0$. $(\theta|_{S_r} \not\equiv \mathbb{1},\ \theta|_{S_{r+}} \equiv \mathbb{1})$.

Assume : "generic" $\overset{\text{def.}}{\Longleftrightarrow}$ If we let $X^* \in \mathfrak{s}^*_{-r}$ $(\mathfrak{s} := \mathrm{Lie}\, S)$ s.t.

$$\theta|_{S_r}(\exp(X)) = \psi_{\mathbb{F}_q}(\langle X, X^* \rangle) \quad {}^\forall X \in \mathfrak{s}_r,$$

$\psi_{\mathbb{F}_q} : \mathbb{F}_q \to \mathbb{C}^\times$ fix.

then $\mathrm{val}_F(\langle \underset{:= d\alpha^\vee(1)}{H_\alpha}, X^* \rangle) = -r$ for ${}^\forall \underset{\text{root}}{\alpha} \in \Phi(G,S)$

$(\alpha^\vee : \mathbb{G}_m \to S$. coroot for $\alpha)$

<u>idea</u>

◎ input : (S, θ)

$\downarrow$ rep. theory of Heisenberg gp.

◎ output : (K, ρ) s.c. type of depth $r > 0$.

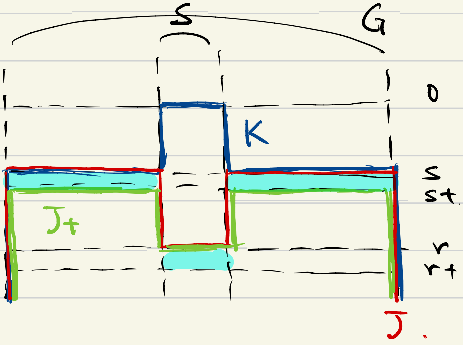
- $s := \frac{r}{2}$.
- $K := S_0 \cdot G_{x,s}$

$\cup$

$J := (S, G)_{x,(r,s)}$

$\cup$

$J_+ := (S, G)_{x,(r,s+)}$.



- $\hat{\theta} : \overset{\mathbb{F}_p\text{-v.s.}}{J_+} \to \mathbb{C}^\times$:= inflation of $\theta|_{S_r}$. $\mathrm{Im}\,\hat{\theta} \subset \mu_p(\mathbb{C})$.

$$\begin{matrix} J_+ \\ \cup \\ \mathrm{Ker}\,\hat{\theta} \supset (S,G)_{x,(r+,s+)}. \end{matrix}$$

- $[-,-] : J \times J \to J$: commutator prod.

$$\begin{tikzcd}
J\times J \arrow[r] \arrow[rd] \arrow[d] & J & \\
 & J_+ \arrow[u, phantom, "\cup"] \arrow[r, "\hat{\theta}"] \arrow[l, phantom, "\circlearrowright"] & \mu_p(\mathbb{C}) \subset \mathbb{C}^\times \arrow[d, phantom, "\wr|\ \leftarrow \text{fix}"] \\
\underbrace{J/J_+}_{\substack{\| \\ V}} \times J/J_+ \arrow[rr, "\substack{\| \\ \langle -,- \rangle}"'] & & \mathbb{F}_p
\end{tikzcd}$$

$\rightsquigarrow$ $(V, \langle -,- \rangle)$: symplectic $\mathbb{F}_p$-v.s.

- $H(V) := V \times \mathbb{F}_p$. "Heisenberg gp".

$$(v,z)\cdot(v',z') = (v+v', v+v'+\tfrac{1}{2}\langle z,z'\rangle)$$

($H(V)$ is non-abelian. $\{0\}\times\mathbb{F}_p$ is center).

> **Stone-von Neumann Thm**
>
> For any $\psi : \mathbb{F}_p \to \mathbb{C}^\times$ nontriv. char,
> $\exists!\,(\tau_\psi, W)$: irr. rep. of $H(V)$ w/ cent. char ψ.

Note $g \in Sp_{\mathbb{F}_p}(V) \curvearrowright H(V)$ by $g\cdot(v,z) := (gv, z)$.

$\rightsquigarrow$ τ_ψ^g : rep. on W given by $\tau_\psi^g(v,z) := \tau_\psi(gv,z)$ is again irr. & has cent. char ψ.

$\therefore$ SvN Thm $\Rightarrow \exists \phi_g : \tau_\psi \simeq \tau_\psi^g$ (unique up to scalar)

<u>Fact</u> can choose $\{\phi_g\}_{g\in \mathrm{Sp}_{\mathbb{F}_p}(V)}$ s.t. $\mathrm{Sp}_{\mathbb{F}_p}(V) \to GL_{\mathbb{C}}(W)$ is hom.

$$g \longmapsto \phi_g$$

$\rightsquigarrow$ get

$$\begin{tikzcd} \mathrm{Sp}_{\mathbb{F}_p}(V)\ltimes H(V) \arrow[rd, "\omega_\psi(g,h) := \phi_g\circ\tau_\psi(h)"] & \circlearrowright \\ H(V) \arrow[u, phantom, "\cup"] \arrow[r, "\tau_\psi"'] & GL_{\mathbb{C}}(W) \end{tikzcd}$$

ω_ψ is called the "Heisenberg-Weil rep".

$$\begin{tikzcd} S_0\ltimes J \arrow[loop above, "\text{conj.}"] \arrow[r, two heads] \arrow[rrr, bend left, "\widetilde{\omega}_\psi"] & S_0\ltimes J/\mathrm{Ker}(\hat{\theta}) = S_0\ltimes H(V) \arrow[r] & \mathrm{Sp}_{\mathbb{F}_p}(V)\ltimes H(V) \arrow[r, "\omega_\psi"] & GL_{\mathbb{C}}(W). \end{tikzcd}$$

$$S_0\ltimes J \twoheadrightarrow S_0 \xrightarrow[\theta|_{S_0}]{} \mathbb{C}^\times$$

<u>Fact</u> $\rho := \widetilde{\omega}_\psi \otimes (\theta|_{S_0})$ factors through $S_0\ltimes J \twoheadrightarrow S_0\cdot J = K$.

<u>Thm</u> (K,ρ) is a s.c. type.

<u>Yu's generalization</u>

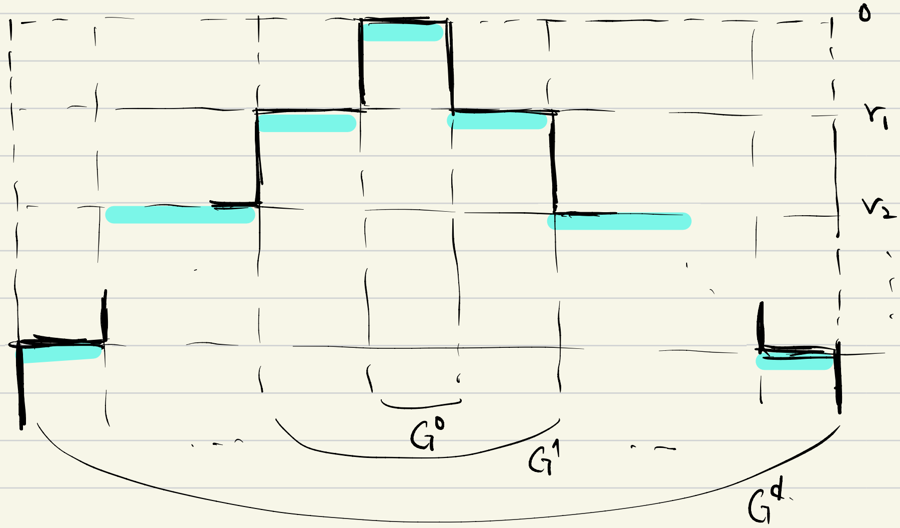
◎ input : $(\vec{G}, \vec{\Phi}, \vec{r}, x, \rho_0)$

$\rightsquigarrow$

◎ output : (K, ρ).

- $\vec{G} = (G^0 < G^1 < \cdots < G^d = G)$
- $\vec{\Phi} = (\phi^0, \ldots, \phi^d)$ $\quad \phi^i : G^i \to \mathbb{C}^\times$ "generic" depth r.
- $\vec{r} = (r_0 \leq \cdots \leq r_d)$
- $x \in \mathcal{A}(G^0)$ vertex
- ρ_0 : irr. s.c. of $G^0_{x,0}/G^0_{x,0+}$

<u>idea</u> extending ρ_i to ρ_{i+1} inductively
via Heisenberg-Weil rep.



Yu's pyramid.