CHAR(LG) SEMINAR (2025-10-24)

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Contents

1. Representations of p-adic reductive groups

Let F be a non-archimedean local field with residual characteristic p. Let $\mathbf{G}$ be a connected reductive group over F. We write $G := \mathbf{G}(F)$, which is a locally profinite topological group (with respect to the topology coming from that of F).

In the following, we always take the coefficient field of any representation to be $\mathbb{C}$ (or algebraically closed field of characteristic 0).

Recall that (explained by Cheng-Chiang last week) we say that a representation (π, V) of G is *smooth* if the stabilizer of any $v \in V$ in G is an open subgroup of G. In other words, any $v \in V$ lies in V^K for some open compact subgroup $K \subset G$.[1]

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[1]This condition is equivalent to that the action $G \times V \to V$ is continuous with respect to the discrete topology on V. Hence, in particular, the notion of smoothness does not depend on the topology of the coefficient field.

We are interested in

$$\mathrm{Rep}(G) := (\text{category of smooth representations of } G),$$
$$\mathrm{Irr}(G) := \{\text{irreducible smooth representations of } G\}/\sim.$$

Why? The possible explanations I can easily come up with are, for example,

- $\mathrm{Rep}(G)$ should be parallel to $\mathrm{Rep}(\mathbb{G}(\mathbb{R}))$ for a real reductive group $\mathbb{G}(\mathbb{R})$ by so-called *Harish-Chandra's Lefschetz principle* (although p-adic and real reductive groups look totally different!).
- Irreducible smooth representations of G arise as local components of *automorphic representations*, which is a modern generalization of the classical notion of modular forms.
- $\Pi(G)$ is expected to be related to *Galois representations* (so-called the *local Langlands correspondence*).
- $\mathrm{Rep}(G)$ should be something "generalizing"[2] $\mathrm{Rep}(\mathbb{G}(\mathbb{F}_q))$ for a finite group of Lie type $\mathbb{G}(\mathbb{F}_q)$,
- and so on...

(Actually, all of these are deeply related to each other. So maybe I should say I can come up with just one reason.)

Today I'm going try to give a rough overview of what kinds of topics have been investigated in this are (but I'll not say anything about the local Langlands correspondence).

2. Hecke algebra and types

2.1. **Hecke algebra.** (reference: [BH06])

We first considier the space of "test functions":

$$C_c^\infty(G) := \{f\colon G \to \mathbb{C} \mid f \text{ is locally constant and compactly supported}\}.$$

By fixing a Haar measure dg on G, we can introduce a $\mathbb{C}$-algebra structure on $C_c^\infty(G)$. To be more precise, for any $f_1, f_2 \in C_c^\infty(G)$, we define $f_1 * f_2 \in C_c^\infty(G)$ by

$$\int_G f_1(gh) \cdot f_2(h^{-1})\, dg.$$

This operation $*$ is called the *convolution product* for test functions. We can check that $C_c^\infty(G)$ is an associative $\mathbb{C}$-algebra with respect to $*$. But please be careful that $C_c^\infty(G)$ is non-commutative and even non-unital in general. We call this algebra the *Hecke algebra of* G and often denote by $\mathcal{H}(G)$.

Suppose that (π, V) is a smooth representation of G. Then we may regard (π, V) as an $\mathcal{H}(G)$-module in the following manner. For any $f \in \mathcal{H}(G)$, we define an operator "$\pi(f)$" on V by

$$v \mapsto \pi(f)(v) := \int_G f(g)\pi(g)v\, dg.$$

This gives a well-defined action of $\mathcal{H}(G)$ on V. Moreover, the smoothness of (π, V) implies that $\mathcal{H}(G) \cdot V = V$; note that this does not necessarily hold for general $\mathcal{H}(G)$-modules because $\mathcal{H}(G)$ may not have the unit element! In general, we say that an $\mathcal{H}(G)$-module V is *smooth* if $\mathcal{H}(G) \cdot V = V$.

[2] I don't know the meaning of this.

The importance of the Hecke algebra is encapsulated in the following fact (but it is not extremely difficult to prove this fact; see, e.g., [BH06, §4.2]):

Fact 2.1. *The procedure above gives a categorical equivalence between*

$$\mathrm{Rep}(G) \xrightarrow{\cong} \mathrm{Mod}(\mathcal{H}(G)),$$

where the latter denotes the category of smooth $\mathcal{H}(G)$-modules.

2.2. **Types.** (reference: [BK98])

By Fact 2.1, in order to study $\mathrm{Rep}(G)$, it is enough to "understand $\mathcal{H}(G)$ completely". But of course this change of viewpoint does not immediately make the problem so easier because the Hecke algebra $\mathcal{H}(G)$ is too huge to analyze.

The key idea is to note the *idempotents* of $\mathcal{H}(G)$. For example, let K be an open compact subgroup of G. Then we can easily check that the normalized characteristic function of K

$$e_K := \frac{1}{dg(K)} \cdot \mathbb{1}_K \in \mathcal{H}(G)$$

is an idempotent, i.e., $e_K * e_K = e_K$. This implies that the subset

$$\mathcal{H}(G,K) := e_K * \mathcal{H}(G) * e_K \subset \mathcal{H}(G)$$

is a subalgebra with unit e_K. (Another description of $\mathcal{H}(G,K)$ is: $\mathcal{H}(G,K) = \{f \in \mathcal{H}(G) \mid f \text{ is bi-}K\text{-invariant}\}$.)

For a smooth representation (π, V) of G (hence regareded also as an $\mathcal{H}(G)$-module), we consider the action of the subring $\mathcal{H}(G,K)$ of $\mathcal{H}(G)$. The point is that $\pi(e_K)$ gives the projector $V \to V^K$, hence the action of $\mathcal{H}(G,K)$ on V factors through the projector $V \to V^K$. If we let $\mathrm{Rep}(G,K)$ be the subcategory

$$\mathrm{Rep}(G,K) := ((\pi, V) \in \mathrm{Rep}(G) \text{ such that } V^K \text{ generates } V)$$

of $\mathrm{Rep}(G)$, then we get a functor

$$\mathrm{Rep}(G,K) \to \mathrm{Mod}(\mathcal{H}(G,K))\colon (\pi, V) \mapsto V^K,$$

where the action of $\mathcal{H}(G,K)$ on V^K is the one induced by that of $\mathcal{H}(G)$ on V.

This procedure can be generalized as follows. We fix an open compact subgroup K of G and its smooth representation ρ (which must be automatically finite-dimensional). We define a test function $e_\rho \in \mathcal{H}(G)$ by

$$e_\rho(g) := \frac{\dim \rho}{dg(K)} \cdot \begin{cases} \Theta_\rho(g^{-1}) & \text{if } g \in K, \\ 0 & \text{othrewise,} \end{cases}$$

where $\Theta_\rho\colon K \to \mathbb{C}$ denotes the trace character of ρ. Then e_ρ is an idempotent of $\mathcal{H}(G)$, hence gives rises to a subalgebra with unit e_ρ

$$\mathcal{H}(G,\rho) := e_\rho * \mathcal{H}(G) * e_\rho \subset \mathcal{H}(G).$$

Similarly to before, we consider the following subcategory of $\mathrm{Rep}(G)$:

$$\mathrm{Rep}(G,\rho) := ((\pi, V) \in \mathrm{Rep}(G) \text{ such that } V^\rho \text{ generates } V),$$

where V^ρ denotes the ρ-isotypic part of $(\pi, V)|_K$. Then we get a functor

$$\mathrm{Rep}(G,\rho) \to \mathrm{Mod}(\mathcal{H}(G,\rho))\colon (\pi, V) \mapsto V^\rho.$$

If this functor gives a categorical equivalence, we can analyze the subcategory $\mathrm{Rep}(G,\rho)$ of $\mathrm{Rep}(G)$ through the module category for a $\mathbb{C}$-algebra $\mathcal{H}(G,\rho)$, which is still non-commutative in general, but much smaller than $\mathcal{H}(G)$ (and has the unit!).

In fact, this functor is not always an equivalence. But this observation naturally leads us to the following definition.

Definition 2.2. We say that (K, ρ) is a *type* if the functor

$$\mathrm{Rep}(G, \rho) \to \mathrm{Mod}(\mathcal{H}(G, \rho))\colon (\pi, V) \mapsto V^{\rho}.$$

gives an equivalence of categories.

Example 2.3. Let $I \subset G$ be an *Iwahori subgroup* of G. For example, when $G = \mathrm{GL}_n(\mathbb{Q}_p)$, you can take I to be the subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ consisting of matrices whose modulo-p-reductions are upper-triangular. In this case, $(I, \mathbb{1})$ is known to be a type. In other words, $\mathrm{Rep}(G, I)$ is equivalent to $\mathrm{Mod}(\mathcal{H}(G, I))$ by the above association. The $\mathbb{C}$-algebra $\mathcal{H}(G, I)$ is sometimes (or often) called the *Iwahori–Hecke algebra*.[3]

Remark 2.4. There are several equivalent characterizations/definitions of the algebra $\mathcal{H}(G, \rho)$. For example, $\mathcal{H}(G, \rho)$ is isomorphic to the endomorphism algebra $\mathrm{End}_G(\text{c-}\mathrm{Ind}_K^G \rho)$ (see [BK98, Section 2] for details).

3. Supercuspidal representations

3.1. Induction and compact induction. (reference: [BH06])

Recall that, when a representation of a subgroup is given, we can construct a representation of a bigger group using it; this is called the *induction* of representations. In the context of representation theory of p-adic groups, we typically consider two kinds of inductions; the *(smooth) induction* and the *compact induction*. Let us briefly review them here.

For any closed subgroup $H \subset G$ and its smooth representation (ρ, W), we define the *(smooth) induction* $(\mathrm{Ind}_H^G \rho, \mathrm{Ind}_H^G W)$, which is a smooth representation of G, by

$$\mathrm{Ind}_H^G W := \{f\colon G \to W \mid \exists K \text{ s.t. } f(hgk) = \sigma(h)(f(g)) \text{ for any } h \in H,\, k \in K\}$$

(K is some open compact subgroup of G depending on f), where G acts on $\mathrm{Ind}_H^G W$ via right-translation of W-valued functions. This gives a functor

$$\mathrm{Ind}_H^G\colon \mathrm{Rep}(H) \to \mathrm{Rep}(G).$$

When H is open, we define the *compact induction* $(\text{c-}\mathrm{Ind}_H^G \rho, \text{c-}\mathrm{Ind}_H^G W)$ in the same way, but additionally requiring that the support of $f\colon G \to W$ is compact-modulo-H. This also gives a functor

$$\text{c-}\mathrm{Ind}_H^G\colon \mathrm{Rep}(H) \to \mathrm{Rep}(G).$$

On the other hand, we can also define the restriction functor in the obvious manner:

$$\mathrm{Res}_H^G\colon \mathrm{Rep}(G) \to \mathrm{Rep}(H).$$

In fact, these are adjunctions to each other

$$\text{c-}\mathrm{Ind}_H^G \dashv \mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G,$$

both of which are referred to as the *Frobenius reciprocity*.

[3] There is a more general and abstract notion of the "Iwahori–Hecke algebra". The Iwahori–Hecke algebra in our context is a special case of more general "Iwahori–Hecke algebras".

3.2. **Parabolic induction and supercuspidal representations.** (reference: [BH06])

Definition 3.1. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ with a Levi subgroup $\mathbf{M}$ (defined over F). Let π_M be a smooth representation of M. We define the *parabolic induction* of π_M to G by

$$i_P^G(\pi_M) := \mathrm{Ind}_P^G(\pi_M \otimes \delta_P),$$

where π_M is regarded as a representation of P through the natural quotient (by the unipotent radical) map $P \twoheadrightarrow M$ and δ_P denotes the "modulus character" of P.

Definition 3.2. We say that an irreducible smooth representation (π, V) of G is *supercuspidal* if there does not exist any pair $(\mathbf{M}, \pi_M)$ of a proper Levi subgroup $\mathbf{M} \subsetneq \mathbf{G}$ and an irreducible smooth representation π_M of M such that π is realized as a subquotient of $i_P^G(\pi_M)$.

Fact 3.3. *For any irreducible smooth representation (π, V) of G, there uniquely (up to G-conjugation) exists a pair $(\mathbf{M}, \pi_M)$ of a Levi subgroup $\mathbf{M} \subset \mathbf{G}$ and an irreducible supercuspidal representation π_M of M such that π is realized as a subquotient of $i_P^G(\pi_M)$.*

Definition 3.4. We call the pair $(\mathbf{M}, \pi_M)$ associated to π as in the previous fact the *cuspidal support* of π. Also, we call such a pair *cuspidal pair* of G.

We may naively understand that supercuspidal representations are "building blocks" of $\mathrm{Rep}(G)$.

The following lemma is useful to construct supercuspidal representations:

Lemma 3.5. *Let K be an open compact-mod-center subgroup of G. Let ρ be an irreducible smooth representation of K. If the compact induction* $\text{c-Ind}_K^G \rho$ *is of finite length, then its all irreducible constituents are supercuspidal.*

The converse is also expected to be true, but it has been a long-standing "folklore conjecture":

Conjecture 3.6. *Any irreducible supercuspidal representation of G is realized in the compact induction of an irreducible smooth representation of an open compact-mod-center subgroup.*

4. Bernstein decomposition

4.1. **Bernstein decomposition.** (reference: [Ber84, Roc09])

We consider an equivalence relation on the set of cuspidal pairs of G as follows:

> we say cuspidal pairs $(\mathbf{M}, \pi_M)$ and $(\mathbf{M}', \pi_{M'})$ are *inertially equivalent* if there exists an unramified character $\chi\colon M \to \mathbb{C}^\times$ such that $(\mathbf{M}, \pi_M \otimes \chi)$ and $(\mathbf{M}', \pi_{M'})$ are G-conjugate.

Here, loosely speaking, an "unramified character" means that χ is trivial on "$\mathbf{M}(\mathcal{O}_F)$".

Let us write $\mathfrak{C}(G)$ for the set of inertially-equivalence classes of cuspidal pairs of G. Then the uniqueness fact on the cuspidal supports especially implies the following decomposition of $\mathrm{Irr}(G)$:

$$\mathrm{Irr}(G) = \bigsqcup_{\mathfrak{s} \in \mathfrak{C}(G)} \mathrm{Irr}_{\mathfrak{s}}(G),$$

where $\mathrm{Irr}_{\mathfrak{s}}(G)$ denotes the set of isomorphism classes of irreducible smooth representations of G whose cuspidal supports lie in $\mathfrak{s}$.

In fact, this decomposition can be upgraded to a decomposition of $\mathrm{Rep}(G)$. Let $\mathrm{Rep}_{\mathfrak{s}}(G)$ be the subcategory of $\mathrm{Rep}(G)$ consisting of smooth representations whose irreducible subquotients are all in $\mathrm{Irr}_{\mathfrak{s}}(G)$.

Theorem 4.1 (Bernstein decomposition)**.** *We have a decomposition*

$$\mathrm{Rep}(G) \cong \prod_{\mathfrak{s}\in\mathfrak{C}(G)} \mathrm{Rep}_{\mathfrak{s}}(G).$$

This decomposition is in fact the "finest" one (I'll explain this later in a bit more detail). To understand $\mathrm{Rep}(G)$, it is enough to investigate each block $\mathrm{Rep}_{\mathfrak{s}}(G)$.

4.2. **Depth decomposition.** Please recall that Cheng-Chiang introduced an invariant for irreducible smooth representations of p-adic reductive groups called the *depth*. In fact, we also can also decompose $\mathrm{Rep}(G)$ according to the depth as follows:

Theorem 4.2 (depth decomposition)**.** *We have a decomposition*

$$\mathrm{Rep}(G) \cong \prod_{r\in\mathbb{Q}_{\geq 0}} \mathrm{Rep}_r(G),$$

where $\mathrm{Rep}_r(G)$ *denotes the subcategory of* $\mathrm{Rep}(G)$ *consisting of smooth representations whose irreducible subquotients all have depth* r*.*

But this decomposition is coarser than the Bernstein decomposition in the sense that each block $\mathrm{Rep}_r(G)$ further decomposes as the product of several blocks $\mathrm{Rep}_{\mathfrak{s}}(G)$.

4.3. **Types for Bernstein blocks.** Recall that, by definition, a pair (K,ρ) is called a type when the functor

$$\mathrm{Rep}(G,\rho) \cong \mathrm{Mod}(\mathcal{H}(G,\rho))\colon V \mapsto V^{\rho}$$

is an equivalence of categories. In fact, types are closely related to the Bernstein decomposition as follows:

Fact 4.3 ([BK98, Theorem 4.3])**.** *Let* K *be an open compact subgroup of* G *and* ρ *an irreducible smooth representation of* K*. The pair* (K,ρ) *is a type if and only if*

$$\mathrm{Rep}(G,\rho) \cong \prod_{\mathfrak{s}\in\mathfrak{S}} \mathrm{Rep}_{\mathfrak{s}}(G)$$

for a finite subset $\mathfrak{S}\subset\mathfrak{C}(G)$*.*

The finite subset $\mathfrak{S}$ as in this fact is uniquely determined by a given type (K,ρ); let us say that (K,ρ) is an $\mathfrak{S}$*-type* in this situation. When $\mathfrak{S}$ is a singleton $\{\mathfrak{s}\}$, we simply say (K,ρ) is an $\mathfrak{s}$*-type*.

If (K,ρ) is an $\mathfrak{s}$-type, we get

$$\mathrm{Rep}_{\mathfrak{s}}(G) \cong \mathrm{Mod}(\mathcal{H}(G,\rho)).$$

Thus the natural question we come up with here is:

can we always find an $\mathfrak{s}$-type for each $\mathfrak{s}\in\mathfrak{C}(G)$?

If this is true, then, combined with the Bernstein decomposition, the study of $\mathrm{Rep}(G)$ ultimately comes down to studying the structure of Hecke algebras $\mathcal{H}(G,\rho)$ of an $\mathfrak{s}$-type (K,ρ) for each $\mathfrak{s}\in\mathfrak{C}(G)$.

However, the answer has not been known at present. For example, let us consider an inertially-equivalence class $\mathfrak{s} = [(\mathbf{G},\pi)]$ whose Levi part is $\mathbf{G}$ itself. If we can find an $\mathfrak{s}$-type (K, ρ), then it especially means that $\pi \in \mathrm{Rep}_{\mathfrak{s}}(G) = \mathrm{Rep}(G, \rho)$, hence, in particular, $\pi^\rho \neq 0$. In other words, we have $\mathrm{Hom}_K(\rho, \pi) \neq 0$. By Frobenius reciprocity, this is equivalent to $\mathrm{Hom}_G(\text{c-Ind}_K^G \rho, \pi) \neq 0$. This means that π is realized in the compact induction of (K, ρ).[4]

So, much remains unknown, but what we want to do is in some sense clear.

(1) For each $\mathfrak{s}$, we want to construct an $\mathfrak{s}$-type (K, ρ).
(2) For each type (K, ρ), we want to determine the structure of $\mathcal{H}(G, \rho)$.

Concerning (1), Moy–Prasad ([MP94, MP96]) constructed $\mathfrak{s}$-types for depth-zero inertially equivalence classes $\mathfrak{s} \in \mathfrak{C}(G)$. Jiu-Kang Yu ([Yu01]) constructed $\mathfrak{s}$-types for "a large class" of inertial equivalence classes of the form $\mathfrak{s} = (\mathbf{G}, \pi)$, i.e., the Levi part of $\mathfrak{s}$ is $\mathbf{G}$. The meaning of "a large class" here is that Yu's construction exhausts $\mathfrak{s}$-types for all such $\mathfrak{s}$ when p is sufficiently large (depending on $\mathbf{G}$). Note that this especially means that Conjecture 3.6 is true when p is sufficiently large. This "exhaustion" result was firstly established by Ju-Lee Kim in [Kim07], then Fintzen provided another proof [Fin21] which improves the bound on p much better. Fintzen's assumption on p is only that p does not divide the order of the Weyl group of $\mathbf{G}$ (e.g., when $\mathbf{G} = \mathrm{GL}_n$, this means that $p \nmid n$). Also, Yu's construction has been generalized to $\mathfrak{s}$ whose Levi part is not necessarily $\mathbf{G}$ itself by Kim–Yu [KY17].

Note that, depending on groups, sometimes even better bounds are known; for example, for $\mathbf{G} = \mathrm{GL}_n$, Bushnell–Kutzko [BK93] constructed all $\mathfrak{s}$-types without any assumption on p.

Regarding (2), there appears to have been rapid progress in recent years, which I do not comprehend at all, honestly speaking. The important references are [Oha24, AFMO24b, AFMO24a]. My shallow understanding on what has been happening is like this:

- Yu's construction of an $\mathfrak{s}$-type (say (K, ρ)) involves a depth-zero supercuspidal representaion of its "twisted" Levi subgroups G^0 as its input data.
- Recently people realized that Hecke algebras $\mathcal{H}(G, \rho)$ for such types (K, ρ) is isomorphic to that for (K^0, ρ_0), which is a type for the depth-zero part.
- Therefore, the problem reduces to understanding Hecke algebras for depth-zero types.

Actually, Cheng-Chiang and I invited Kazuma Ohara to Taipei; currently it's planned that he will stay here from 2025-11-13 to 2025-11-19. We can ask him!!

5. Explicit construction of supercuspidal types

5.1. **Depth-zero case: Moy–Prasad types.** Recall that Cheng-Chiang introduced the notion of an *apartment* $\mathcal{A}$ of G; each point $\mathbf{x} \in \mathcal{A}$ gives rise to an open compact subgroup $G_{x,0} \subset G$ called the *parahoric subgroup* and a descending sequence of open compact subgroups $\{G_{\mathbf{x},r}\}_{r \in \mathbb{R}_{\geq 0}}$ called the *Moy–Prasad filtration*. The apartment $\mathcal{A}$ is defined by fixing an maximal split torus of $\mathbf{G}$. Roughly speaking, we can construct a poly-simplicial set $\mathcal{B}(G)$ equipped with an action of G by varying the choice of an maximal split torus and gluing together all the apartments of G. The set $\mathcal{B}(G)$ is called the *Bruhat–Tits building* of G. So, again, each point $\mathbf{x} \in \mathcal{B}(G)$ gives rise to a parahoric subgroup $G_{\mathbf{x},0}$ and its Moy–Prasad filtration.

[4] Precisely speaking, we have to be careful about the center (to apply the Frobenius reciprocity, we first have to extend ρ to a compact-mod-center subgroup using the central character of π).

In the following, we explain a construction of types for depth-zero supercuspidal representations of G. Let us take any vertex $\mathbf{x} \in \mathcal{B}(G)$; then it gives a maximal parahoric subgroup $G_{\mathbf{x},0}$. We consider the quotient $G_{\mathbf{x},0}/G_{\mathbf{x},0+}$. In fact, this quotient group can be identified with the group $\mathbb{G}(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational points of a connected reductive group $\mathbb{G}$ over $\mathbb{F}_q$.[5]

We take an irreducible cuspidal representation ρ of $\mathbb{G}(\mathbb{F}_q)$.[6] By pulling back it along $G_{\mathbf{x},0} \twoheadrightarrow G_{\mathbf{x},0}/G_{\mathbf{x},0+} \cong \mathbb{G}(\mathbb{F}_q)$, we get a representation of $G_{\mathbf{x},0}$.

Theorem 5.1 ([MP94, MP96])**.** *The pair $(G_{\mathbf{x},0}, \rho)$ is an $\mathfrak{s}$-type for $\mathfrak{s} = [(G, \pi)]$ whose π is a depth-zero supercuspidal representation. Furthermore, any such $\mathfrak{s} \in \mathfrak{C}(G)$ has an $\mathfrak{s}$-type of this form.*

The relationship between ρ and π in the above theorem is more explicated as follows. Let Z be the center of G. We take any extension of ρ to a representation $\tilde{\rho}$ of $ZG_{\mathbf{x},0}$. Then its compact induction $\text{c-Ind}_{ZG_{\mathbf{x},0}}^{G} \tilde{\rho}$ decomposes into a finite direct sum of irreducible supercuspidal representations of G. Any irreducible supercuspidal representation of G obtained in this way has the same inertial equivalence class (i.e., unique up to unramified character twisting); π can be taken to be any of them.

5.2. **Positive-depth case: Adler's toral types.** We next explain Adler's construction [Adl98] of types for positive-depth supercuspidal representations of G. In the following, we assume that p is odd and does not divide the order of the Weyl group of $\mathbf{G}$.[7]

We first take an elliptic maximal torus $\mathbf{S}$ of $\mathbf{G}$ which is tamely ramified.[8] Then we may associate to it a point $\mathbf{x} \in \mathcal{B}(G)$ (we can identify $\mathcal{B}(S)$ with a subset of $\mathcal{B}(G)$, which consists just of a single point). Accordingly, we get a parahoric subgroup $G_{\mathbf{x},0}$ and its Moy–Prasad filtration $\{G_{\mathbf{x},r}\}_{r \in \mathbb{R}_{\geq 0}}$.

We next take a character $\theta\colon S \to \mathbb{C}^\times$ of depth $r > 0$, so θ is trivial on S_{r+} but non-trivial on S_r. We further assume that θ is *G-generic of depth r*; the meaning is as follows. By Cheng-Chiang's manner, we can associate to it a dual Lie algebra element $X^* \in \mathfrak{s}^*_{-r}$. Namely, by fixing a non-trivial additive character $\psi_{\mathbb{F}_q} : \mathbb{F}_q \to \mathbb{C}^\times$ of the residue field $\mathbb{F}_q$, we have

$$x\theta(\exp(X)) = \psi_{\mathbb{F}_q}(\langle X^*, X\rangle)$$

for any $X \in \mathfrak{s}_r$. For each coroot $\alpha^\vee$ of $\mathbf{S}$ in $\mathbf{G}$, we put $H_{\alpha^\vee} \in \mathfrak{s}$ to be $d\alpha^\vee(1)$. Then we say that θ is G-generic of depth r if X^* satisfies

$$\mathrm{val}_F(\langle X^*, H_{\alpha^\vee}\rangle) = -r$$

for any coroot $\alpha^\vee$.

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[5]A little bit more precisely speaking, there exists a smooth integral model $\mathcal{G}_{\mathbf{x}}$ of $\mathbf{G}$ over the ring of integers $\mathcal{O}_F$ such that $\mathcal{G}_{\mathbf{x}}(\mathcal{O}_F) = G_{\mathbf{x},0}$. Then we can consider the reductive quotient of the special fiber of $\mathcal{G}_{\mathbf{x}}$, which is a connected reductive group $\mathbb{G}$ over the residue field $\mathbb{F}_q$ of F. Then, we in fact have $\mathbb{G}(\mathbb{F}_q) \cong G_{\mathbf{x},0}/G_{\mathbf{x},0+}$.

[6]For representations of $\mathbb{G}(\mathbb{F}_q)$, we define the notion of cuspidality in the same way, i.e., define them to be the ones not realized in any proper parabolic induction.

[7]The latter assumption is actually not necessary in Adler's construction, but here we assume it for simplicity.

[8]Recall that a maximal torus $\mathbf{S}$ of $\mathbf{G}$ is called *elliptic* if its quotient by the center of $\mathbf{G}$ is anisotropic, i.e., has split rank 0.

We put $s := \frac{r}{2}$ and consider the following groups:

$$K := S_0 \cdot G_{\mathbf{x},s} \quad \supset \quad J := (S,G)_{\mathbf{x},(r,s)} \quad \supset \quad J_+ := (S,G)_{\mathbf{x},(r,s+)}.$$

We can extend $\theta|_{S_r}$ to a character J_+ in a canonical way (i.e., "trivial outside S_r"); let us write $\hat{\theta}$ for it. We consider the commutator product

$$[-,-]\colon J \times J \to J.$$

The image is in fact contained in J_+. If we compose it with

$$\hat{\theta}\colon J_+ \twoheadrightarrow J_+/\operatorname{Ker}(\hat{\theta}) \cong \mu_p(\mathbb{C}^\times) \subset \mathbb{C}^\times,$$

then the resulting pairing factors through $(J/J_+) \times (J/J_+)$. Let us put $V := J/J_+$. Then, by fixing an identification $\mu_p(\mathbb{C}^\times) \cong \mathbb{F}_p$, we get an $\mathbb{F}_p$-bilinear form

$$\langle -,- \rangle\colon V \times V \to \mathbb{F}_p,$$

which is in fact symplectic (i.e., alternating and non-degenerate).

Here, let us recall the general theory of Heisenberg groups. We put a group structure on the product set $H(V) := V \times \mathbb{F}_p$ by

$$(v,z) \cdot (v',z') := \bigl(v+v', z+z'+\tfrac{1}{2}\langle v,v'\rangle\bigr).$$

This group is called the *Heisenberg group* associated to the symplectic space $(V, \langle -,- \rangle)$, which is non-abelian and has $\{0\} \times \mathbb{F}_p$ as its center. The important fact about Heisenberg groups is the following:

Theorem 5.2 (Stone–von Neumann theorem)**.** *For any non-trivial character $\psi\colon \mathbb{F}_p \to \mathbb{C}^\times$, there uniquely exists (up to isomorphism) an irreducible representation (τ_ψ, W) of $H(V)$ with central character ψ.*

Note that $H(V)$ has a natural action of the symplectic group $\mathrm{Sp}_{\mathbb{F}_p}(V)$ by $g \cdot (v,z) := (gv, z)$ for any $g \in \mathrm{Sp}_{\mathbb{F}_p}(V)$ and $(v,z) \in H(V)$. For an irreducible representation (τ_ψ, W) of $H(V)$ with central character ψ, let us consider its pull-back (τ_ψ^g, W) along $g \in \mathrm{Sp}_{\mathbb{F}_p}(V)$, i.e., $\tau_\psi^g(h) := \tau_\psi(g \cdot h)$. Since (τ_ψ^g, W) again has central character ψ, the uniqueness part of the Stone–von Neumann theorem implies that τ_ψ^g is isomorphic to τ_ψ. Let us fix an isomorphism $\phi_g\colon (\tau_\psi, W) \to (\tau_\psi^g, W)$ for each $g \in \mathrm{Sp}_{\mathbb{F}_p}(V)$. Note that, since such ϕ_g is unique up to scalar multiplication by the irreducibility of τ_ψ, the assignment $g \mapsto \phi_g$ gives a homomorphism $\mathrm{Sp}_{\mathbb{F}_p}(V) \to \mathrm{PGL}(W)$. This also means that we obtain a projective representation of the semi-direct product group

$$\mathrm{Sp}_{\mathbb{F}_p}(V) \ltimes H(V) \to \mathrm{PGL}(W)\colon (g,h) \mapsto \phi_g \circ \tau_\psi(h).$$

The question here is: can we lift this projective representation to a genuine representation? Another way to say this is: can we find a set of isomorphisms $\{\phi_g\}_{g \in \mathrm{Sp}_{\mathbb{F}_p}(V)}$ such that the $\phi_{gg'} = \phi_g \circ \phi_{g'}$ holds for any $g, g' \in \mathrm{Sp}_{\mathbb{F}_p}(V)$? In fact, it is known that such a lift indeed exists uniquely if $p > 3$. The resulting representation of $\mathrm{Sp}_{\mathbb{F}_p}(V) \ltimes H(V)$ is called the *Heisenberg–Weil representation* associated to ψ, for which we write (ω_ψ, W).

Let us go back to Adler's construction. The point of the construction is that the conjugation action of S_0 on G preserves J and J_+ and induces a symplectic action on $V = J/J_+$. Hence we get a homomorphism

$$S_0 \ltimes J \twoheadrightarrow S_0 \ltimes J/\operatorname{Ker}(\hat{\theta}) = S_0 \ltimes H(V) \to \mathrm{Sp}_{\mathbb{F}_p}(V) \ltimes H(V) \xrightarrow{\omega_\psi} \mathrm{GL}(W).$$

We write $\tilde{\omega}_{\psi}$ for this representation of $S_0 \ltimes J$. On the other hand, we pull-back $\theta|_{S_0}$ to $S_0 \ltimes J$.

Fact 5.3. *The tensor product representation*

$$\rho := (\theta|_{S_0}) \otimes \tilde{\omega}_{\psi}$$

of $S_0 \ltimes J$ factors through the product homomorphism $S_0 \ltimes J \twoheadrightarrow S_0 \cdot J = K$, hence defines a representation of K.

Theorem 5.4. *The pair (K, ρ) is an $\mathfrak{s}$-type for a Bernstein block $[(G, \pi)]$ whose π is a supercuspidal representation of depth r.*

5.3. **Positive-depth case: Yu's tame types.** I finally sketch the outline of Yu's construction of types for more general positive-depth supercuspidal representations of G [Yu01]. The input data for Yu's construction is a tuple $\Psi = (\vec{\mathbf{G}}, \vec{\phi}, \vec{r}, \mathbf{x}, \rho_0')$ called a *Yu-datum.* An oversimplified explanation on each ingredient is as follows:

- $\vec{\mathbf{G}}$ is a sequence ($\mathbf{G}^0 \subsetneq \mathbf{G}^1 \subsetneq \cdots \subsetneq \mathbf{G}^d = \mathbf{G}$) of tame twisted Levi subgroups, which means that each $\mathbf{G}^i$ is an F-rational subgroup of $\mathbf{G}$ which becomes a Levi subgroup of $\mathbf{G}$ after taking the base change along a tamely ramified extension of F;
- $\mathbf{x}$ is a vertex of the Bruhat–Tits building $\mathcal{B}(\mathbf{G}^0, F)$ of $\mathbf{G}^0$;
- $\vec{r}$ is a sequence $(0 \leq r_0 < \cdots < r_{d-1} \leq r_d)$ of real numbers;
- $\vec{\phi}$ is a sequence $(\phi_0, \ldots, \phi_d)$ of characters ϕ_i of G^i of depth r_i satisfying a "genericity condition", which is defined in a similar manner to the genericity condition used in Adler's construction;
- ρ_0 is an irreducible cuspidal representation of $G^0_{\mathbf{x},0}$.

To any such datum, Yu associated a type of G. Roughly speaking, his idea is to combine Moy–Prasad's and Adler's constructions in an inductive manner. Recall that $(G_{\mathbf{x},0}, \rho_0)$ gives a depth-zero type of G^0 by Moy–Prasad's theorem. By imitating Adler's construction (use $(\mathbf{G}^0, \mathbf{G}^1)$ instead of $(\mathbf{S}, \mathbf{G})$), we can construct a type (K^1, ρ_1) of G^1, where $K^1 = G^0_{\mathbf{x},0} G^1_{\mathbf{x},\frac{r_0}{2}}$. By doing this again, we can furthermore construct a type (K^2, ρ_2) of G^2, where $K^2 = G^0_{\mathbf{x},0} G^1_{\mathbf{x},\frac{r_0}{2}} G^2_{\mathbf{x},\frac{r_1}{2}}$. Repeating this procedure, we finally get a type (K^d, ρ_d) of $G^d = G$, where $K^d = G^0_{\mathbf{x},0} G^1_{\mathbf{x},\frac{r_0}{2}} \cdots G^d_{\mathbf{x},\frac{r_{d-1}}{2}}$.

Appendix A. Complementary remarks

A.1. **Bernstein center.**

Definition A.1. We call the categorical center $\mathfrak{z}(\mathrm{Rep}(G))$ of $\mathrm{Rep}(G)$ the *Bernstein center* of G. More precisely, $\mathfrak{z}(G)$ is defined to be the endomorphism ring of the identity functor $\mathrm{id}_{\mathrm{Rep}(G)} \colon \mathrm{Rep}(G) \to \mathrm{Rep}(G)$. We write $\mathfrak{z}(G) := \mathfrak{z}(\mathrm{Rep}(G))$ in short.

In more concrete terms, an element $z \in \mathfrak{z}(G)$ is a family of endomorphisms $\{\pi(z) \in \mathrm{End}_G(V)\}_{(\pi,V)\in\mathrm{Rep}(G)}$ such that, for any morphism $f \colon (\pi, V) \to (\pi', V')$ in $\mathrm{Rep}(G)$, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ {\scriptstyle \pi(z)}\downarrow & & \downarrow{\scriptstyle \pi'(z)} \\ V & \xrightarrow{f} & V'. \end{array}$$

The Bernstein center is a $\mathbb{C}$-algebra, which acts on any smooth representation of G. Note that, in general, if we have any product decomposition $\mathrm{Rep}(G) \cong \prod_i \mathrm{Rep}(G)_i$, then we accordingly have a decomposition of the Bernstein center $\mathfrak{z}(G) \cong \prod_i \mathfrak{z}(\mathrm{Rep}(G)_i)$. In particular, the Bernstein decomposition (Theorem 4.1)

$$\mathrm{Rep}(G) \cong \prod_{\mathfrak{s}\in\mathfrak{C}(G)} \mathrm{Rep}_{\mathfrak{s}}(G)$$

induces a decomposition

$$\mathfrak{z}(G) \cong \prod_{\mathfrak{s}\in\mathfrak{C}(G)} \mathfrak{z}_{\mathfrak{s}}$$

where we write $\mathfrak{z}_{\mathfrak{s}} := \mathfrak{z}(\mathrm{Rep}_{\mathfrak{s}}(G))$. In fact, the center $\mathfrak{z}_{\mathfrak{s}}$ of each block is an integral domain. More precisely, $\mathfrak{z}(G)$ can be thought of as the coordinate ring $\mathbb{C}[\Omega(G)]$ of a reduced affine scheme $\Omega(G)$ over $\mathbb{C}$, which is called the *Bernstein variety*. The connected components of the Bernstein variety are labeled by $\mathfrak{C}(G)$; say $\Omega(G) = \bigsqcup_{\mathfrak{s}\in\mathfrak{C}(G)} \Omega_{\mathfrak{s}}(G)$. Then each $\Omega_{\mathfrak{s}}(G)$ is irreducible and $\mathfrak{z}_{\mathfrak{s}}$ is regarded as its coordinate ring $\mathbb{C}[\Omega_{\mathfrak{s}}(G)]$. Especially, $\mathfrak{z}_{\mathfrak{s}}$ has no nontrivial idempotent, which means that we cannot have a finer decomposition of $\mathrm{Rep}(G)$ than the Bernstein decomposition.

A.2. **Bernstein center and distribution characters.** (reference: [Hai14, Var24])
Recall (from Cheng-Chiang's talk) that, for any irreducible smooth representation $(\pi, V) \in \mathrm{Irr}(G)$, we can define its *distribution character*[9][10] Θ_π by

$$\Theta_\pi \colon C_c^\infty(G) \to \mathbb{C}; \quad f \mapsto \mathrm{tr}(\pi(f) \mid V).$$

Recall that, in representation theory of finite groups, the notion of the *trace character* is extremely important. For any finite-dimensional representation (σ, W) of a finite group Γ, we define its trace character $\Theta_\sigma \colon \Gamma \to \mathbb{C}$ by $\Theta_\sigma(g) := \mathrm{tr}(\sigma(g) \mid W)$. Then, for example, any two irreducible representations are isomorphic to each other if and only if their trace characters coincide. So basically *characters tell all* in representation theory of finite groups. On the other hand, we can also define the "distribution character" for a representation π of Γ exactly in the same manner as above. In fact, in the finite group case, the distribution character and the trace character are essentially equivalent; they determine each other completely.

Unfortunately, in the context of p-adic groups, irreducible smooth representations are infinite-dimensional unless they are 1-dimensional. So we cannot define their trace characters. However, the above definition of distribution characters still makes sense (this is because the image of $\pi(f)$ is always finite-dimensional as explained by Cheng-Chiang). Moreover, in this setting, we can prove that, for exmaple, any two irreducible smooth representations are isomorphic if and only if their distribution characters coincide. Hence, the basic philosophy is again "characters tell all". We want to understand $\mathrm{Rep}(G)$ through the distribution characters.[11]

Based on this philosophy, let us consider how the projection to each Bernstein block

$$\mathrm{Rep}(G) \to \mathrm{Rep}_{\mathfrak{s}}(G) \colon V \mapsto V_{\mathfrak{s}}$$

[9]By "distribution", we just mean a $\mathbb{C}$-linear functional $C_c^\infty(G) \to \mathbb{C}$.
[10]We often call Θ_π the *Harish-Chandra character of* π.
[11]Note that, to extend the notion of distribution characters to general smooth representations (not necessarily irreducible), we need to assume that the smooth representation is of finite length.

can be described in terms of distribution characters when V is of finite length. To explain it, we introduce another expression of the Bernstein center $\mathfrak{z}(G)$. We let $\mathcal{D}(G)$ be the space of distributions on G:

$$\mathcal{D}(G) := \mathrm{Hom}_{\mathbb{C}}(C_c^{\infty}(G), \mathbb{C}).$$

Recall that $D \in \mathcal{D}(G)$ is said to be *invariant* if $D({}^g f) = D(f)$ for any $f \in C_c^{\infty}(G)$ and $g \in G$, where ${}^g f(x) := f(g^{-1}xg)$. We write $\mathcal{D}^G(G)$ for the subspace of $\mathcal{D}(G)$ of invariant distributions.

For any $D \in \mathcal{D}(G)$ and $f \in C_c^{\infty}(G)$, we define $D * f \in C^{\infty}(G)$ by

$$D * f(g) := D(\rho_g(f^{\vee})),$$

where ρ_g denotes the right-translation by g and $f^{\vee}(g) := f(g^{-1})$, hence $\rho_g(f^{\vee})(x) = f(g^{-1}x^{-1})$. Note that $D * f$ is smooth, but may not be compactly supported. This is why the subscript "c" is not put in "$C^{\infty}(G)$" above.

Definition A.2. We say that $D \in \mathcal{D}(G)$ is *essentially compact* if $D * f \in C_c^{\infty}(G)$ for any $f \in C_c^{\infty}(G)$.

We write $\mathcal{D}_c(G)$ for the subspace of $\mathcal{D}(G)$ consisting of essentially compact distributions. We put

$$\mathcal{D}_c^G(G) := \mathcal{D}^G(G) \cap \mathcal{D}_c(G).$$

Then we can define the convolution product $*$ on $\mathcal{D}_c^G(G)$ as follows. For any $D, D' \in \mathcal{D}_c^G(G)$, we define $D * D' \in \mathcal{D}(G)$ by

$$(D * D')(f) := D((D' * f^{\vee})^{\vee}).$$

We can check that $D * D'$ belongs to $\mathcal{D}_c^G(G)$.

Fact A.3. *The product $*$ makes $\mathcal{D}_c^G(G)$ a commutative $\mathbb{C}$-algebra. Furthermore, there exists an isomorphism*

$$\iota\colon \mathcal{D}_c^G(G) \xrightarrow{\cong} \mathfrak{z}(G)$$

of $\mathbb{C}$-algebras such that, for any smooth representation $(\pi, V) \in \mathrm{Rep}(G)$, we have

$$\pi(z * f) = \pi(\iota(z)) \circ \pi(f)$$

for any $z \in \mathcal{D}_c^G(G)$ and $f \in C_c^{\infty}(G)$.

Now let us go back to our original question. Let $(\pi, V) \in \mathrm{Rep}(G)$ be a smooth representation of finite length, which is projected to $(\pi_{\mathfrak{s}}, V_{\mathfrak{s}}) \in \mathrm{Rep}(G)_{\mathfrak{s}}$. We consider the associated distribution characters $\Theta_{\pi}, \Theta_{\pi_{\mathfrak{s}}} \in \mathcal{D}^G(G)$. Hence, by definition, for any $f \in C_c^{\infty}(G)$, we have

$$\Theta_{\pi}(f) = \mathrm{tr}(\pi(f) \mid V)$$

and

$$\Theta_{\pi_{\mathfrak{s}}}(f) = \mathrm{tr}(\pi_{\mathfrak{s}}(f) \mid V_{\mathfrak{s}}).$$

Let $e_{\mathfrak{s}}$ be the unit element of $\mathfrak{z}_{\mathfrak{s}}$, hence regarded as an idempotent of $\mathfrak{z}(G)$ (we have $\mathfrak{z}_{\mathfrak{s}} = e_{\mathfrak{s}} \cdot \mathfrak{z}(G)$). Let $D_{\mathfrak{s}} := \iota^{-1}(e_{\mathfrak{s}}) \in \mathcal{D}_c^G(G)$. Then, by noting that $\pi(e_{\mathfrak{s}})$ is the projector $V \twoheadrightarrow V_{\mathfrak{s}}$, we see

$$\begin{aligned}\Theta_{\pi_{\mathfrak{s}}}(f) &= \mathrm{tr}(\pi_{\mathfrak{s}}(f) \mid V_{\mathfrak{s}}) = \mathrm{tr}(\pi(f) \circ \pi(e_{\mathfrak{s}}) \mid V) \\ &= \mathrm{tr}(\pi(e_{\mathfrak{s}}) \circ \pi(f) \mid V) = \mathrm{tr}(\pi(D_{\mathfrak{s}} * f) \mid V) = \Theta_{\pi}(D_{\mathfrak{s}} * f).\end{aligned}$$

Here, the second from the last equality follows from the property of ι in Fact A.3.

In summary, in terms of distibution character, the projection $\pi \mapsto \pi_{\mathfrak{s}}$ amounts to pulling back Θ_π along the convolution by $D_{\mathfrak{s}}$:

$$\begin{array}{ccc} C_c^\infty(G) & \xrightarrow{\Theta_\pi} & \mathbb{C} \\ {\scriptstyle D_{\mathfrak{s}}*(-)}\uparrow & \nearrow{\scriptstyle \Theta_{\pi_{\mathfrak{s}}}} & \\ C_c^\infty(G) & & \end{array}$$

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