On characters of p-adic reductive groups

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Let F be a non-archimedean local field, with residue field $k \cong \mathbb{F}_q$. For example $F = k((t))$. Let $\mathbb{G}$ be a connected reductive group over F. Let $G = \mathbb{G}(F)$. This is a totally disconnected locally compact Hausdorff topological group. We also write $\mathfrak{g} = (\operatorname{Lie}\mathbb{G})(F)$, $\mathfrak{g}^* = (\operatorname{Lie}\mathbb{G})^*(F)$.

The goal of this note is to discuss the phenomenon that

The Fourier transform of a local character of G is supported on the nilpotent cone.

Specifically, we are referring to the Harish-Chandra–Howe local character expansion. Originally, the proof of Howe and Harish-Chandra in the 70's "works over F." In the 90's and 00's, Waldspurger, DeBacker and Adler-Korman gave a new proof and sometimes stronger results by "working over k." We discuss this latter approach. The new proof needs more hypotheses, and because of that we assume $p = \operatorname{char}(k) > 6 \cdot \operatorname{rank}_{\bar{F}} \mathbb{G}$ today.

1. Moy-Prasad theory

Let $\mathbb{S} \subset \mathbb{G}$ be a maximal split torus. In this section, for simplicity of exhibition we assume that $\mathbb{S} \subset \mathbb{G}/_{\mathbb{Z}} \hookrightarrow \mathrm{GL}_n/_{\mathbb{Z}}$ are split reductive over $\mathbb{Z}$, and that $\mathbb{S}$ is mapped into the diagonal. We have a valuation map $\mathrm{val} : F \to \mathbb{Z} \sqcup +\infty$. Let $\mathcal{A} := X_*(\mathbb{S}) \otimes \mathbb{R}$. Every entry in the $n \times n$ square gives a character $\alpha \in X^*(\mathbb{S})$. For any $x \in \mathcal{A}$ we define

$$\text{For } r \in \mathbb{R}, \quad \mathfrak{g}_{x,r} = \{X \in \mathfrak{g} \mid \text{the entry of } X \text{ at } \alpha \text{ has valuation} \geq r + \langle x, \alpha \rangle\}$$

$$\text{For } r \in \mathbb{R}, \quad \mathfrak{g}^*_{x,r} = \{X \in \mathfrak{g}^* \mid \text{the entry of } X \text{ at } \alpha \text{ has valuation} \geq r + \langle x, \alpha \rangle\}$$

$$\text{For } r \in \mathbb{R}_{>0}, \quad G_{x,r} = \{g \in G \mid \text{the entry of } g - \mathrm{id} \text{ at } \alpha \text{ has valuation} \geq r + \langle x, \alpha \rangle\}$$

Note that $\mathfrak{g}_{x,r}$ and $\mathfrak{g}^*_{x,r}$ are lattices, and $G_{x,r}$ are open compact subgroups. In fact $G_{x,n}$ ($n \in \mathbb{Z}_{>0}$) forms a topological basis for any $x \in \mathcal{A}$. Write $G_{x,r+} := G_{x,r+\epsilon}$ for very small $\epsilon > 0$. It can be shown that $G_{x,r+} \lhd G_{x,r}$ is normal. We assume there is a "logarithm"[1] map $\log : \bigcup_x G_{x,r} \to \bigcup_x \mathfrak{g}_{x,r}$ satisfying [DeB02, Hypothesis 3.2.1]. Then

$$G_{x,r}/G_{x,r+} \xrightarrow[\sim]{\log} \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+} \cong \operatorname{Hom}_k(\mathfrak{g}^*_{x,-r}/\mathfrak{g}^*_{x,(-r)+}, k).$$

Let C be any field of characteristic 0, e.g. $C = \overline{\mathbb{Q}_\ell}$; all our functions and representations will be C-coefficient. Suppose we have a non-trivial homomorphism $\psi : (k, +) \to C^\times$. Consider any $\Gamma \in \mathfrak{g}^*_{x,-r}/\mathfrak{g}^*_{x,(-r)+}$ and

$$\begin{array}{rcl} \psi_\Gamma : G_{x,r} \twoheadrightarrow G_{x,r}/G_{x,r+} \xrightarrow[\sim]{\log} \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+} \cong & \operatorname{Hom}_k(\mathfrak{g}^*_{x,-r}/\mathfrak{g}^*_{x,(-r)+}, k) & \to \quad C^\times \\ & \phi & \mapsto \quad \psi(\phi(\Gamma)) \end{array}$$

[1] I think under our assumption on p, Kazhdan-Varshavsky quasi-logarithm [BKV16, Appendix C] always exists and serves our purpose, but there are non-trivial things to verify and that's not done in the literature. The readers are encouraged to pretend it is some approximation to classical logarithm on matrices.

Definition 1. *We say* Γ *and* ψ_Γ *are* **nilpotent**[2] *if* $\Gamma + \mathfrak{g}^*_{x,(-r)+}$ *contains a nilpotent element in* $\mathfrak{g}^*$.

Definition 2. *A smooth representation* π *is a vector space* V *over* C *and homomorphism* $\pi : G \to \mathrm{Aut}(V)$ *such that for any* $v \in V$ *we have* $\mathrm{Stab}_G(v)$ *open.*

Theorem 3. *(Moy-Prasad* [MP94, Theorem 5.2]*) Let* π *be an irreducible smooth representation. Then* $\exists!\, r(\pi) \in \mathbb{Q}_{\geq 0}$ *(typically called the* **depth***) such that*

(i) *For* $0 < r < r(\pi)$*, we have* $\mathrm{Hom}_{G_{x,r}}(\psi_\Gamma, \pi) = 0$ *for any* $x \in \mathcal{A}$ *and* $\Gamma \in \mathfrak{g}^*_{x,-r}/\mathfrak{g}^*_{x,(-r)+}$.
(ii) *For* $0 < r = r(\pi)$*, we have* $\mathrm{Hom}_{G_{x,r}}(\psi_\Gamma, \pi) = 0$ *when* Γ *is nilpotent.*
(iii) *For* $r > r(\pi)$*, we have* $\mathrm{Hom}_{G_{x,r}}(\psi_\Gamma, \pi) = 0$ *when* Γ *is non-nilpotent.*

Sketch. The number $r(\pi)$ is defined as the infimum of r for which there exists $y \in \mathcal{A}$ such that the fixed subspace $\pi^{G_{y,r}} \neq 0$. We will only sketch (iii). Let $r > r(\pi)$. Fix $0 \neq v \in \pi^{G_{y,r}}$. Suppose $\mathrm{Hom}_{G_{x,r}}(\psi_\Gamma, \pi) \neq 0$. Since $G_{x,r}$ is compact we also have $\mathrm{Hom}_{G_{x,r}}(\pi, \psi_\Gamma) \neq 0$. We have to prove that Γ is nilpotent. Let $0 \neq \alpha \in \mathrm{Hom}_{G_{x,r}}(\pi, \psi_\Gamma)$. Since π is irreducible, there exists $g \in G$ such that $\alpha(g \cdot v) \neq 0$. Then

$$\forall h \in G_{y,r} \text{ and } {}^g h := ghg^{-1},\ \ {}^g h \cdot \alpha(g\cdot v) = \alpha({}^g h \cdot g \cdot v) = \alpha(g \cdot h \cdot v) = \alpha(g \cdot v)$$
$$\implies \forall h \in G_{y,r} \cap (G_{x,r})^g,\ \ \psi_\Gamma(\alpha(g\cdot v)) = \psi_\Gamma({}^g h \cdot \alpha(g\cdot v)) = \psi(\langle \log({}^g h), \Gamma\rangle)\cdot\psi_\Gamma(\alpha(g\cdot v))$$
$$\implies \psi(\langle \log({}^g h), \Gamma\rangle) = 0,\ \forall {}^g h \in {}^g(G_{y,r}) \cap G_{x,r}$$

Since $\log({}^g(G_{y,r}) \cap G_{x,r}) = {}^g(\mathfrak{g}_{y,r}) \cap \mathfrak{g}_{x,r}$, the above implies

$$\implies \Gamma \in \mathrm{im}(\mathfrak{g}^*_{x,-r} \cap {}^g(\mathfrak{g}^*_{y,(-r)+}) \to \mathfrak{g}^*_{x,-r}/\mathfrak{g}^*_{x,(-r)+})$$

Coming from $\mathfrak{g}^*_{y,(-r)+}$, one can represent Γ by a matrix whose eigenvalues have valuations $> -r$. From that it is possible (after non-trivial Bruhtat-Tits theory) to show that Γ is nilpotent in Definition 1. $\square$

2. Invariant distributions

Definition 4. *The space of* **test functions** *on* G *is*

$$C_c^\infty(G) := \{f : G \to C \mid f \text{ is locally constant and } \mathrm{supp}(f) \text{ is compact}\}.$$

Definition 5. *An (G-)invariant distribution is a linear map* $\Theta : C_c^\infty(G) \to C$ *such that* $\Theta({}^g f) = \Theta(f)$ *for any* $f \in C_c^\infty(G)$ *and* $g \in G$*. Here* $({}^g f)(h) := f(g^{-1}hg)$.

Test functions and (G-)invariant distributions on $\mathfrak{g}$ and $\mathfrak{g}^*$ are defined similarly.

Example 6. It is a non-trivial fact that if π is an irreducible smooth representation, then

$$\pi(f) : v \mapsto \int_G f(g)\pi(g)v\, dg$$

is a well-defined finite sum for any v and $\pi(f)$ has finite-dimensional image. If we choose a Haar measure on G (always possible) such that $|H| \in \mathbb{Q}_{>0}$ for any open compact subgroups $H \subset G$, then

$$f \mapsto \Theta_\pi(f) := \mathrm{Tr}(\pi(f)) \in C.$$

[2] They are called "degenerate" by Moy and Prasad.

is an invariant distribution, called the **character** of π.

Before we proceed, we have to define Fourier transforms. Denote again by $\psi : \mathcal{O}_F \twoheadrightarrow \mathcal{O}_F/\mathfrak{m}_F = k \xrightarrow{\psi} C^\times$. Assume we can extend ψ to $\tilde{\psi} : F \to C^\times$ as a homomorphism (always possible if either $\mathrm{char}(F) = p$, or $\mathrm{char}(F) = 0$ and $C^\times$ has all p-power roots of unity). Fix one such $\tilde{\psi}$. Fix on $\mathfrak{g}$ a Haar measure so that every lattice has measure in $\mathbb{Q}_{>0}$. For $f \in C_c^\infty(\mathfrak{g})$ we define the Fourier transform $\hat{f} \in C_c^\infty(\mathfrak{g}^*)$ by

$$\hat{f}(X) := \int_{Y \in \mathfrak{g}} f(Y)\tilde{\psi}(\langle X, Y\rangle)dY.$$

This is always a finite sum valued in C.

Let Θ be an invariant distribution on G and $r \in \mathbb{R}_{>0}$. We say Θ **has depth** $< r$ iff the following property holds: For any $x \in \mathcal{A}$, $s \geq r$, $\Gamma \in \mathfrak{g}^*_{x,-s}/\mathfrak{g}^*_{x,(-s)+}$ non-nilpotent, and smooth $\rho \in \mathrm{Irr}(G_{x,r})$ satisfying $\mathrm{Hom}_{G_{x,s}}(\psi_\Gamma, \rho) \neq 0$, we have

$$\Theta(\theta_\rho) = 0$$

where $\theta_\rho \in C_c^\infty(G_{x,r})$ is the character of ρ viewed as a test function.

Theorem 7. *(DeBacker, essentially* [DeB02, Theorem 3.5.2][3]*) Suppose Θ has depth $< r$. Then*

$$\text{“}\mathrm{supp}(\log^*(\Theta|_{G_r}))\ \textit{is contained in the nilpotent cone.”}$$

This means, there exists an invariant distribution J_Θ on $\mathfrak{g}^*$ such that

(1) J_Θ is supported on the nilpotent cone, i.e. if $f \in C_c^\infty(\mathfrak{g}^*)$ is such that $\mathrm{supp}(f)$ *contains no nilpotent elements in $\mathfrak{g}^*$, then $J_\Theta(f) = 0$.*

(2) $\Theta(f) = J_\pi(\widehat{f \circ \log})$ for any $x \in \mathcal{A}$ and $f \in C_c^\infty(G_{x,r})$.

Corollary 8. *(Harish-Chandra–Howe local character expansion) Let π be an irreducible smooth representation and $r > r(\pi)$. Then the character Θ_π has depth $< r$ and therefore*

$$\mathrm{supp}(\log^*(\Theta_\pi|_{G_r}))\ \textit{is contained in the nilpotent cone.}$$

Remark 9. Theorem 7 and Corollary 8 should be viewed as statements about the invariant distributions near $\mathrm{id} \in G$. There are generalization for the invariant distributions near any semisimple $\gamma \in G$ by Adler-Korman [AK07, Theorem 12.1 and Corollary 12.9].

References

[AK07] Jeffrey D. Adler and Jonathan Korman, *The local character expansion near a tame, semisimple element*, Amer. J. Math. **129** (2007), no. 2, 381–403. MR 2306039 (2008a:22020)

[BKV16] Roman Bezrukavnikov, David Kazhdan, and Yakov Varshavsky, *On the depth r Bernstein projector*, Selecta Math. (N.S.) **22** (2016), no. 4, 2271–2311. MR 3573958

[DeB02] Stephen DeBacker, *Homogeneity results for invariant distributions of a reductive p-adic group*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 3, 391–422. MR 1914003 (2003i:22019)

[3]Technically, the proof is written for $C = \mathbb{C}$ and indirectly makes use of the calculus notion of absolute convergence. There are additional tricks to generalize to any characteristic zero field.

[MP94] Allen Moy and Gopal Prasad, *Unrefined minimal K-types for p-adic groups*, Invent. Math. **116** (1994), no. 1-3, 393–408. MR 1253198 (95f:22023)