We fix k a field. Everything is over k. All stacks are ∞-stacks. Our goal is to define relative dimension, equi-dimensionality and semi-small-ness for morphisms between infinite-dimensional spaces.

1. Finite type situation

Definition 1. *Let X be a stack over k. Let $[X]$ be the set of points on X (equivalence classes(?) of morphisms from $\mathrm{Spec}(K)$, K/k any extension). We'll sometimes abbreviate $[X]$ to X if there is no danger of confusion.*

Definition 2. *Let X be an algebraic space of finite type. Denote by $\dim_x(X)$ the maximum of the dimensions of the irreducible components of X containing x.*

Definition 3. *Let $f : X \to Y$ be a morphism of algebraic spaces of finite type. We define $\underline{\dim}_f : X \to \mathbb{Z}$ by $\underline{\dim}_f(x) := \dim_x(X) - \dim_{f(x)}(Y)$. We say*

*(1) f is **weakly equi-dimensional** if $\underline{\dim}_f$ is locally constant.*

*(2) f is **equi-dimensional** if f is weakly equi-dimensional, and $\underline{\dim}_f(x) = \dim_x(f^{-1}(f(x)))$ for every $x \in X$.*

*(3) f is universally open equi-dimensional (**uoed**) if f is universally open and weakly equi-dimensional.*

Lemma 4. *Let $f : X \to Y$ be an open morphism of algebraic spaces of finite type. Then $\underline{\dim}_f(x) = \dim_x(f^{-1}(f(x)))$. In particular,* uoed *morphisms of algebraic spaces of finite type are equi-dimensional.*

Corollary 5. *Suppose we have a Cartesian square of algebraic spaces of finite type:*

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow{\scriptstyle\psi} & \ulcorner & \downarrow{\scriptstyle\phi} \\ X & \xrightarrow{f} & Y \end{array}$$

such that either f or ϕ is universally open (for example, when either is flat). Then $\underline{\dim}_g = \psi^*\underline{\dim}_f$.

Proof. Let $x' \in X'$ and $x = \psi(x') \in X$. We have to show that $\underline{\dim}_g(x') = \underline{\dim}_f(x)$. Let $y' = g(x')$ and $y = f(x) = \phi(y')$. We have to show that

$$\dim_{x'}(X') + \dim_y(Y) = \dim_x(X) + \dim_{y'}(Y'). \tag{1}$$

Suppose ϕ is universally open; in particular ψ is also open. Lemma 4 gives $\dim_{y'}(Y') - \dim_y(Y) = \dim_{y'}(\phi^{-1}(\phi(y')))$, and likewise $\dim_{x'}(X') - \dim_x(X) = \dim_{x'}(\psi^{-1}(\psi(x')))$. Since $\psi^{-1}(\psi(x')) \cong \phi^{-1}(\phi(y'))$, we have equality (1). The case when f is universally open is identical. □

Corollary 6. *The class of* uoed *morphism (for algebraic spaces of finite type) is closed under compositions and base change.*

2. lfpr morphisms

Definition 7. *Recall that*

*(1) A ring homomorphism $\rho : A \to B$ is **of finite presentation** (fp) if ρ is isomorphic to $A \to A[x_1, ..., x_n](f_1, ..., f_n)$ for some $f_1, ..., f_n \in A[x_1, ..., x_n]$.*

(2) A morphism $f : X \to Y$ of schemes is **locally fp** *if it is locally (on X) of the above form.*

(3) A morphism $f : X \to Y$ of algebraic spaces is **locally fp** *if we have a commutative diagram*

$$\begin{CD} X' @>{f'}>> Y' \\ @VVV @VVV \\ X @>{f}>> Y \end{CD}$$

where the vertical maps are étale covers and $f' : X' \to Y'$ is a locally fp morphism of schemes.

(4) A morphism $f : \mathcal{X} \to \mathcal{Y}$ of stacks is **locally fp-representable** (**lfpr**) *if for every morphism $Y \to \mathcal{Y}$ from an affine scheme Y, we have that $\mathcal{X} \times_{\mathcal{Y}} Y$ is an algebraic space and $\mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is a locally fp morphism of algebraic spaces.*

Definition 8. *Let $f : X \to Y$ be a locally* fp *morphism of 0-placid schemes. Consider a Cartesian square*

$$\begin{CD} X @>{f}>> Y \\ @V{p_X}VV @VV{p_Y}V \\ X_\alpha @>{f_\alpha}>> Y_\alpha \end{CD}$$

where X_α and Y_α are affine schemes of finite types, and that p_X and p_Y are strongly pro-smooth. Then we define $\underline{\dim}_f := p_X^* \underline{\dim}_{f_\alpha}$.

Lemma 9. *Definition 8 is well-defined.*

Proof. Any two such diagrams are dominated by a third one, so we have to prove that when we have a Cartesian square of affine schemes of finite types:

$$\begin{CD} X_\gamma @>{f_\gamma}>> Y_\gamma \\ @V{p_X}VV @VV{p_Y}V \\ X_\alpha @>{f_\alpha}>> Y_\alpha \end{CD}$$

where p_X and p_Y are smooth, we have that $\underline{\dim}_{f_\gamma} = p_X^*(\underline{\dim}_{f_\alpha})$. This follows from Corollary 5. □

Definition 10. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a* lfpr *morphism of placid stacks. We define $\underline{\dim}_f : [\mathcal{X}] \to \mathbb{Z}$ to be the function such that $\psi^*(\underline{\dim}_f) = \underline{\dim}_g$ for every commutative square*

$$\begin{CD} X @>{g}>> Y \\ @V{\psi}VV @VV{\phi}V \\ \mathcal{X} @>{f}>> \mathcal{Y} \end{CD}$$

where X and Y are 0-placid affine schemes, ϕ is (n-)smooth, and that $X \to \mathcal{X} \times_{\mathcal{Y}} Y$ is étale.

Lemma 11. *Definition 10 is well-defined.*

3. General morphisms

Definition 12. *(i) Let $f : X \to Y$ be a morphism of 0-placid schemes. We say f is* **weakly equi-dimensional** *(resp.* **equi-dimensional***, resp.* **universally open***, resp.* **uoed***) if, for every strongly pro-smooth $Y \to Y'$ where Y' is an affine scheme of finite type, there exists a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f'} & Y' \end{array}$$

where $X \to X'$ is also strongly pro-smooth, X' is also an affine scheme of finite type, and f' is weakly equi-dimensional (resp. equi-dimensional, resp. universally open, resp. uoed*).*

(ii) Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of placid stacks, not necessarily lfpr*. We say f is* **weakly equi-dimensional** *(resp.* **equi-dimensional***, resp.* **pro-universally open***, resp.* **uoed***) if, for any 0-placid schemes X and Y and (n-)smooth morphisms $Y \to \mathcal{Y}$ and $X \to \mathcal{X} \times_{\mathcal{Y}} Y$, we have that $X \to \mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is weakly equi-dimensional (resp. equi-dimensional, resp. universally open, resp.* uoed*) as in (i).*

Lemma 13. *If $f : X \to Y$ is morphism of 0-placid schemes, then the definitions in (i) and (ii) are equivalent. Also, if $f : \mathcal{X} \to \mathcal{Y}$ is an* lfpr *morphism, then the definitions are equivalent to the naive ones as in Definition 3.*

Example 14. Let G be a connected reductive group and $\mathfrak{c} := \mathfrak{g}//G$. We'll see (in §3 of the paper) that $L^+(\mathfrak{g}) \to L^+(\mathfrak{c})$ is flat, and thus universally open. It is not lfpr unless G is a torus. Moreover, we will have fp-locally closed $\mathfrak{c}_{w,r} \subset L^+(\mathfrak{c})$ and its pre-image $\mathfrak{g}_{w,r} \subset L^+(\mathfrak{g})$ such that $\mathfrak{g}_{w,r} \to \mathfrak{c}_{w,r}$ is uoed.

Corollary 15. uoed *morphisms are equi-dimensional.*

Lemma 16. *Suppose we have a Cartesian square of placid stacks*

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow{\scriptstyle\psi} & \ulcorner & \downarrow{\scriptstyle\phi} \\ X & \xrightarrow{f} & Y \end{array}$$

such that f is lfpr *and ϕ is pro-universally open. Then $\underline{\dim}_g = \psi^* \underline{\dim}_f$.*

Definition 17. *A morphism $f : \mathcal{X} \to \mathcal{Y}$ of stacks is an fp-open/fp-closed/fp-locally closed embedding if, for every morphism $Y \to \mathcal{Y}$ from an affine scheme Y, we have that $\mathcal{X} \times_{\mathcal{Y}} Y$ is a scheme and $\mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is fp-open/fp-closed/fp-locally closed. If $\mathcal{X}$ and $\mathcal{Y}$ are placid, then we say f has codimension d if $\underline{\dim}_f = -d$.*

Definition 18. *Let $\mathcal{X}$ be a stack, and let $\{X_\alpha\}_{\alpha \in \mathcal{I}}$ be a collection of fp-locally closed reduced substacks of $\mathcal{X}$. We say $\{X_\alpha\}$ is a*

(1) **finite constructible stratification** *of* $\mathcal{X}$*, if there is an increasing sequence* $\emptyset = \mathcal{X}_0 \subset \mathcal{X}_1 \subset ... \subset \mathcal{X}_n = \mathcal{X}$ *and a bijection*

$$\begin{array}{ccc} \{1,...,n\} & \longleftrightarrow & \mathcal{I} \\ i & \mapsto & \alpha_i \end{array}$$

such that $\mathcal{X}_{\alpha_i} \cong (\mathcal{X}_i \backslash \mathcal{X}_{i-1})_{\mathrm{red}}$.

(2) *A* **bounded constructible stratification** *of* $\mathcal{X}$ *if we have a filtered colimit*

$$\mathcal{X} = \operatorname{colim} \mathcal{X}_U$$

by fp-open substacks such that

(a) *For each* $\alpha \in \mathcal{I}$*, either* $\mathcal{X}_\alpha \subset \mathcal{X}_U$ *or* $\mathcal{X}_\alpha \cap \mathcal{X}_U = \emptyset$*. Write* $I_U := \{\alpha \in \mathcal{I} \mid \mathcal{X}_\alpha \subset \mathcal{X}_U\}$.

(b) $\{X_\alpha\}_{I_U}$ *forms a finite constructible stratification.*

(3) *A* **constructible stratification** *of* $\mathcal{X}$ *if we have a filtered colimit*

$$\mathcal{X} = \operatorname{colim} \mathcal{X}_\lambda$$

by fp-closed substacks such that

(a) *For each* $\alpha \in \mathcal{I}$*, either* $\mathcal{X}_\alpha \subset \mathcal{X}_\lambda$ *or* $\mathcal{X}_\alpha \cap \mathcal{X}_\lambda = \emptyset$.

(b) $\{X_\alpha\}_{I_\lambda}$ *forms a bounded constructible stratification.*

Definition 19. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a morphism of placid stacks. Let* $\{\mathcal{Y}_\alpha\}$ *be a constructible stratification of* Y*. Let* $\mathcal{X}_\alpha = f^{-1}(\mathcal{Y}_\alpha)_{\mathrm{red}}$*, and* $f_\alpha : \mathcal{X}_\alpha \to \mathcal{Y}_\alpha$ *the restriction. Suppose for each* α *we have*

(1) X_α *is of codimension* b_α *for some* b_α*,*

(2) f_α *is* lfpr *and* $\underline{\dim}_{f_\alpha} \equiv \delta_\alpha$ *for some* δ_α.

Then we say f *is* **semi-small** *if* $\delta_\alpha \leq b_\alpha$ *of every* α*. In this case, a stratum* $\mathcal{Y}_\alpha$ *is called* f-**relevant** *if* $\delta_\alpha = b_\alpha$.

Lemma 20. *Let* $f : X \to Y$ *be a morphism of algebraic spaces of finite type, for which any fiber is equi-dimensional. Then it has a finite constructible stratification with respect to which* f *is semi-small iff* f *is semi-small in the classical sense.*

Remark 21. In case it is unclear, the goal is to prove that the affine Springer map is semi-small, and then prove that the semi-small property implies that the !-pushforward of the dualizing sheaf is perverse (though not necessarily semisimple).