0.1. **Arcs.** We work over $k = \bar{k}$. Let X be a variety and consider the functor

$$L_n^+X : A \mapsto X(A[z]/z^{n+1}).$$

This is representable by a scheme and the truncations

$$L_m^+X \to L_n^+X \ \ , (m \geqslant n)$$

are affine. Set $L^+X : A \mapsto X(A[[z]])$ and observe that we have $L^+X \simeq \lim_n L_n^+X$. There is a $\mathbf{G}_m$ action on all of this, which scales the coordinate z on the formal disc.

Lemma 0.1. *All L_n^+X are smooth if X is, and so are all truncation morphisms. If $U \to X$ is etale ten the natural morphism*

$$L_n^+U \to L_n^+X \times_X U$$

is an equivalence.

Proof. Use infinitesimal lifting criterion for smoothness/ etaleness. ☐

Example. Let $X = \mathbf{A}^d$, then $L_n^+\mathbf{A}^d$ is an affine space of dimension $d(n+1)$. If $\mathbf{A}^d$ has coordinates $x^1, ..., x^d$ then $L_n^+\mathbf{A}^d$ has coordinates x_j^i with $i \in \{1, ..., d\}$ and $j \in \{0, ..., n\}$. The functor of points definition gives a universal morphism

$$L_n^+\mathbf{A}^d \times \mathrm{spec}(k[z]/z^{1+n}) \to \mathbf{A}^d.$$

This amounts to a d-tuple of functions on $L_+^n\mathbf{A}^d \times \mathrm{spec}(k[z]/z^{1+n})$. These functions are given by

$$x^i(z) = \sum_j x_j^i z^j.$$

Then the functor of points definiton makes clear that we have isomorphisms

$$L_n^+(X \times_Y Z) \simeq L_n^+X \times_{L_n^+Y} L_n^+Y$$

and we can use this to get presentations of L_n^+X for an arbitrary affine X, by writing $X = \mathbf{A}^n \times_{\mathbf{A}^m} \{0\}$.

0.2. **Placidity.** So let us note the following - L^+X is infinite type but is quite reasonable in the sense that it is naturally the limit of finite type things along affine morphisms. If X is smooth then L^+X is easily seen to be formally smooth, but something much stronger is true - it is actually the limit of smooth things along smooth (and affine) maps.

Remark. I think if we remember the $\mathbf{G}_m$-equivariance then it is absolutely canonically such a limit - we can work etale locally and reduce to the case of an affine space. Now $\mathcal{O}(L_n^+\mathbf{A}^d) \subset \mathcal{O}(L^+\mathbf{A}^d)$ can be canonically recovered as the sub-algebra generated by vectors of $\mathbf{G}_m$-weight at most n.

We axiomatize this as follows:

Definition 0.2.
- *A scheme is called* strongly pro-smooth *(*sps *henceforth) if it admits a presentation $X \simeq \lim_a X_a$ where the X_a are all smooth, and the transition maps $X_a \to X_b$ are smooth affine.*
- *A morphism $f : X \to Y$ is called strongly pro-smooth if it admits a presentation $f \simeq \lim_a (f_a : X_a \to Y)$ where all f_a are smooth and transition maps are smooth affine.*
- *A scheme X is said to admit a placid presentation if it can be presented as $\lim_a X_a$ where X_a are finite type and transition morphisms are smooth affine.*

Definition 0.3. *An ind-scheme is called ind-placid if it can be presented as the filtered colimit along closed fp-embeddings.*

Remark. The point seems to be that placid, sps etc are reasonably similar to finite, smooth etc. We will try to sketch some of this below.

Example.
- If X is sps then if $X = \lim_a X_a$ is a presentation, each of the maps $X \to X_a$ is sps. Indeed $X = \lim_{a'>a} X_{a'}$ and the maps $X_{a'} \to X_a$ are smooth affine.
- If X is smooth then L^+X is sps.
- If $X \to Y$ is smooth then and X is smooth then $L^+X \to Y$ is sps.
- If X is smooth and Z is singular then $L^+X \times Z$ is placid but not sps (as it is not formally smooth).

Remark. sps morphisms are closed under compositions and preserved by pull-backs. Indeed let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be a morphism. Write $g = \lim_b g_b$ and $f = \lim_a f_a$ so $Y_b \to Z$ and $X_a \to Z$ are smooth. Then write gf as the limit of $f_a^{-1}(Y_b) \to Y_b \to Z$ over pairs (a, b). Pullbacks easy.

0.3. **Some independence of presentation results.** We actually give proofs here so that there is $\neq 0$ content in this lecture.

Lemma 0.4. *Let $X = \lim_a X_a$ placid and $Y = \lim_b Y_b$ placid and consider a flat map $g : X \to Y$. Then it can be written as a limit of maps $g_{a,b} : X_a \to Y_b$ which are flat (here a is sufficiently large depending on b.) Moreover if g is sps then we can take $g_{a,b}$ smooth.*

Proof. $g : X \to Y_b$ factors through X_a for a large enough by finite presentation. Let $g_{a,b}$ be resulting map and consider the flat locus $X'_a \subset X_a$ of this map. We claim that that $p_a : X_a \to X$ factors through X'_a. If so then by finite presentation this morphism factors $X \to X_{a'} \to X'_a$ and this is good enough for flatness. Let us now see the claim: let $x_a = p_a(x)$ be in the image, and set $g(x) = y$. Now $X \to X_a$ is strongly pro-smooth and so $\mathcal{O}_{X,x}$ is fflat $\mathcal{O}_{X_a,x_a}$ algebra. It is also a flat $\mathcal{O}_{Y,y}$ algebra as $X \to Y$ is flat. Now if

$$A \to B \to C$$

is diagram with $A \to C$ flat and $B \to C$ fflat then $A \to B$ is flat. Smooth version skipped, arguments are similar. □

Let X be placid, with two placid presentations (X_a) and (X_b). Then applying the above to the identity functor of X we conclude that we can factor $X \to X_b$ through large enough a with $g_{a,b} : X_a \to X_b$ smooth. This is a sort of independence...

Recall that if $\pi : X \to Y$ is a smooth cover and X is smooth, then so too is Y.

Lemma 0.5. *If $f : X \to Y$ is sps of placid schemes, then if X is sps then so is Y.*

Proof. f is fp so there is some a and a smooth covering $f_a : Z_a \to Y_a$ so that $f = f_a \times_{Y_a} Y \to Y$. Now $X = \lim_{a'>a} Z_a \times_{Y_a} Y_{a'}$ is another placid presentation of X, so we apply the above lemma. In particular it follows that $Z_a \times_{Y_a} Y_{a'}$ are smooth and thus the $Y_{a'}$ are as well. □

0.4. **Placid Stacks.** We define the class of n-placid stacks. A stack $\mathcal{X}$ is said to be 0-placid if it admits a decomposition $\mathcal{X} = \sqcup_c \mathcal{X}_c$ into placid affine schemes. A morphism $f : \mathcal{X} \to \mathcal{Y}$ is said to be 0-smooth if for all placid Y and $Y \to \mathcal{Y}$, the pull-back $X = Y \times_{\mathcal{Y}} \mathcal{X}$ admits a decomposition $\sqcup X_c$ where each X_c is placid and the map $X_c \to Y$ is sps. Now we assume the definition of the classes of n-placid stacks and n-smooth morphisms. We say that $\mathcal{X}$ is $(n+1)$-placid if it admits an n-smooth cover $X \to \mathcal{X}$ with X 0-placid. A morphism of $\mathcal{X} \to \mathcal{Y}$ is called $(n+1)$-smooth if for every $Y \to \mathcal{Y}$, with Y 0-placid, the pullback $\mathcal{X} \times_{\mathcal{Y}} Y$ is $(n+1)$-placid and there is a n-smooth covering $X \to \mathcal{X} \times_{\mathcal{Y}} Y$ such that the composition down to Y is 0-smooth.

Remark. This is a standard inductive definition, due to Simpson. We start with classes Obj_0 and Mor_0. We assume that Mor_0 is closed under composition and pullbacks. We need an extra assumption as well - if $f : x \to y$ is in Mor_0 and $y \in Obj_0$, then $x \in Obj_0$. Then assume given the classes Mor_n and Obj_n.

(i) We denote by Obj_{n+1} the class of objects x so that there exists a covering $f : z \to x$ with $z \in Obj_0$ and $f \in Mor_n$.

(ii) We denote by $Mor_{n+1,0}$ the class of morphisms $x \to y$ with $x \in Obj_{n+1}$ and $y \in Obj_0$ so that there exists a cover $z \to x$ in Mor_n so that $z \to y$ is in Mor_0.

(iii) We denote by Mor_{n+1} the class of morphisms $x \to y$ so that for all $u \to y$ (with $u \in Obj_0$) the pull-back $u \times_y x \to u$ belongs to $Mor_{n,0}$.

Example.

- Let X be a 0-placid affine scheme and let H be 0-smooth, then $\mathcal{X} := X/H$ is 1-placid. Indeed $X \to \mathcal{X}$ is a cover by 0-placid X so it remains to see that $X \to \mathcal{X}$ is 0-smooth. It suffices to show that $X \times_{\mathcal{X}} X \to X$ is sps, which it is as it is the projection $X \times H \to X$.
- Let $X \to Y$ be fp morphism of schemes so that Y admits a placid presentation. Then X admits one as well.
- Let $\mathcal{X} \to \mathcal{Y}$ be representable by fp-schematic morphisms, then if $\mathcal{Y}$ is placid so too is $\mathcal{X}$.

0.5. **Reduced substacks.** Let $Red \subset Aff$ be the category of reduced affine schemes. Then the inclusion has a right adjoint $X \mapsto X_{red}$, given on affines by $spec(A) \mapsto spec(A/Nil)$. Let $\mathcal{Y} \in \mathrm{St}$ be a stack, and consider the restriction

$$i^*\mathcal{Y} \in \mathrm{St}_{red} := Fun(Red^{op}, Ani).$$

Then i^* has a fully faithful left adjoint $i_!$. Fully faithful means that the natural map $1 \to i^*i_!$ is an equivalence. Thus we can consider St_{red} as a full subcategory of St . If $\mathcal{Y}$ is a stack then we call $i_!i^*\mathcal{Y} \to \mathcal{Y}$ the reduced stack of $\mathcal{Y}$, denoted $\mathcal{Y}_{red}$. If $\mathcal{Y}_{red} \to \mathcal{Y}$ is an equivalence we call $\mathcal{Y}$ reduced.

Remark. $i_!$ is given by a left Kan extension, so if $\mathcal{Y} \in \mathrm{St}_{red}$ then $i_!\mathcal{Y}(S) = colim_{S' \to S}\mathcal{Y}(S')$. If $\mathcal{X}$ is a scheme then $\mathcal{X}_{red}$ is the classical reduced subscheme corresponding to $\mathcal{X}$.

Here are basic properties of this construction. Recall $i_!i^*\mathcal{Y} = \mathcal{Y}_{red}$. Now we have a canonical equivalence

$$i^*\mathcal{Y}_{red} = i^*i_!i^*\mathcal{Y} = i^*\mathcal{Y}$$

as $i_!$ is fully faithful. From this we deduce an equivalence

$$(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}_{red})_{red} \simeq (\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})_{red}.$$

We use only that i^* preserves limits and tahat $i^*i_! \simeq id$.

$$i_!i^*(\mathcal{X} \times_{\mathcal{Z}} i_!i^*\mathcal{Y})$$

$$= i_!(i^*\mathcal{X} \times_{i^*\mathcal{Z}} i^*i_!i^*\mathcal{Y})$$

$$= i_!(i^*\mathcal{X} \times_{i^*\mathcal{Z}} i^*\mathcal{Y})$$
$$= (\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})_{red}.$$

Iterating this we have an equivalece

$$(\mathcal{X}_{red} \times_{\mathcal{Z}_{red}} \mathcal{Y}_{red}) \simeq (\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})_{red}.$$

Now there are a bunch of theorems proven about interaction of reduced part with smoothness. The model such theorem in the finite case is as follows - let $X \to Y$ be a smooth morphism. Then the natural map $X_{red} \to Y_{red} \times_Y X$ is an equivalence. We want versions of this theorem in the placid/ sps context. The following seems to be the main lemma -

Lemma 0.6. *Let $g : \mathcal{X} \to \mathcal{Y}$ be a smooth morphism of placid stacks. Then for every $\mathcal{Y}' \to \mathcal{Y}$, we have an equivalence*

$$(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}')_r \to (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'_r).$$

In particular (take $\mathcal{Y}' = \mathcal{Y}$) the natural morphism $\mathcal{X}_r \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_r$ is an equivalence.

Corollary 0.7. *(a) If $f : \mathcal{X} \to \mathcal{Y}$ is a smooth morphism/ cover of placid stacks, then so is the function on reduced substacks.*
(b) If $\mathcal{X}$ is a placid stack then so is $\mathcal{X}_r$ and the natural morphism $\mathcal{X}_r \to \mathcal{X}$ is fp-closed.

Proof. (a) Smooth preserved by pull-back, but by above lemma f_r is pulled back from f under $\mathcal{Y}_r \to \mathcal{Y}$.
(b) Suffices by some result above to prove that the map is fp-closed. Reduce to case of $\mathcal{X}$ 0-placid affine scheme by compat with colimits. In this case there is an sps morphism $\mathcal{X} \to X$ (take one of the terms in a presentation). Then $\mathcal{X}_r \simeq \mathcal{X} \times_X X_r$ by above lemma and get the fp-result from the finite case. □