Erratum to [Tsa20].

[Tsa20, Thm. 4.2] is wrong. Roughly speaking, it is wrong for the reason that a space (however nice) X and a discrete group A acting on it freely, we don't have $H^*(X/A) = H^*(X)^A$ when A is infinite. In ordinary topology, the identity fails for example when $A = \mathbb{Z}$ acting by translation on $X = \mathbb{R}$ with $X/A \cong S^1$.

The mistake does not affect the main theorems or the corollaries in §8. In fact it does not affect the rest of [Tsa20], because we only use the incorrect statement for the top degree H^{2d} when X is a d-dimensional ind-variety, in which case the statement is correct. Alternatively, one can replace every use of [Tsa20, Thm. 4.2] by the correct version it cites, which is the $\kappa = 1$ case of [GKM04, Thm. 15.8]. In the end [Tsa20] is concerned with the components of the ind-variety X, and the A-orbits of components of X is the same as the components of X/A.

For more detail, the incorrect [Tsa20, Thm. 4.2] asserts an identity

$$I^{st}_\gamma(1_{\mathfrak{g}_{x,\geq 0}}) = \frac{|G_{x,0}|}{|T_0|} \cdot q^{(-v(D(\gamma)) - \dim \mathbf{G}_{x,0} + \dim \mathbf{T}_0)/2} \cdot \mathrm{Tr}(\mathrm{Frob}\,;\, H^*(\mathcal{X}_\gamma \times \mathrm{Spec}\,\bar{k})^{\mathcal{T}}) \tag{0.1}$$

where

(1) $\gamma \in \mathfrak{g} = (\mathrm{Lie}\,\mathbf{G})(k((t)))$ is regular semisimple.
(2) $\mathcal{X}_\gamma$ is the affine Springer fiber.
(3) $\mathcal{T}$ is the centralizer of γ in the loop group.
(4) $1_{\mathfrak{g}_{x,\geq 0}}$ is the characteristic function of some parahoric subalgebra.
(5) $I^{st}_\gamma(1_{\mathfrak{g}_{x,\geq 0}})$ is the stable orbital integral of the aforementioned function on the stable orbit of γ.

The group $\mathcal{T}$ acts on $\mathcal{X}_\gamma$ and the induced action on $H^*(\mathcal{X}_\gamma \times \mathrm{Spec}\,\bar{k})$ factors through some quotient A, which is a discrete (smooth of dimension 0, though not finite) group scheme over k that fits into an exact sequence of group schemes over k:

$$1 \to \Lambda \to A \to B \to 1$$

where Λ (this is the same Λ in [GKM04, §15]) is such that its base change to $\mathrm{Spec}\,\bar{k}$ is a constant group scheme corresponding to a free abelian group of finite rank, and B is a finite group scheme. The trouble is that in general $H^*((\mathcal{X}_\gamma/A) \times \mathrm{Spec}\,\bar{k})$ is **NOT** equal to $H^*(\mathcal{X}_\gamma \times \mathrm{Spec}\,\bar{k})^A = H^*(\mathcal{X}_\gamma \times \mathrm{Spec}\,\bar{k})^{\mathcal{T}}$. The correct object that should appear in the RHS of (0.1), or rather [Tsa20, Thm. 4.2], is

$$\mathrm{Tr}(\mathrm{Frob}\,;\, H^*((\mathcal{X}_\gamma/A)\times\mathrm{Spec}\,\bar{k})) = \mathrm{Tr}(\mathrm{Frob}\,;\, H^*((\mathcal{X}_\gamma/\Lambda)\times\mathrm{Spec}\,\bar{k})^B = \mathrm{Tr}(\mathrm{Frob}\,;\, H^*((\mathcal{X}_\gamma/\Lambda)\times\mathrm{Spec}\,\bar{k})^{\mathcal{T}})$$

which is essentially the RHS of [GKM04, Thm. 15.8] for trivial κ. (I tried to avoid the need to define Λ in [Tsa20, Thm. 4.2]. Apparently I failed terribly.)

References

[GKM04] Mark Goresky, Robert Kottwitz, and Robert Macpherson, *Homology of affine Springer fibers in the unramified case*, Duke Math. J. **121** (2004), no. 3, 509–561. MR 2040285

[Tsa20] Cheng-Chiang Tsai, *Components of affine Springer fibers*, Int. Math. Res. Not. IMRN (2020), no. 6, 1882–1919. MR 4089436