0. Prerequisites

- Grad alg: module theory, linear alg, $\otimes$ ... etc
- Basic caty theory: equiv. of catries, isom. of functors, adjunctions ... etc
- Basic homological alg: projectives, Ext, derived catries ... etc

Helpful to know but not required:

- Basic Lie theory: Lie alg, root systems, ... etc
- Cohomology: of $\mathbb{CP}^1$ ... etc
- Algebraic geometry: proper maps, fibers of a morphism of varieties ... etc

1. Motivation

Study Rep($\mathfrak{g}$), $\mathfrak{g}$: simple Lie alg

- 1970's: BGG caty $\mathcal{O}$ (⊆ Rep($\mathfrak{g}$)) is the "right" caty to study $\boxed{\text{Chp 13-14}}$
  - ⇒ suffices to study its principal block $\mathcal{O}_0$ ∋ M_x Verma module (∞-dim'l), $x \in W$: Weyl group $\boxed{\text{Chp 1-2}}$ (special case of Coxeter grp)
    - ← Verma's 1966 thesis
  - → characterize simple mod $L_y \in \mathcal{O}_0$
  - → suffices to compute Jordan Hölder multy $[M_x : L_y]$
- 1979–1981: Kazhdan–Lusztig theory $\boxed{\text{Chp 15-16}}$

World	$\mathfrak{g}$-mod	⟷	Perverse sheaves	⟷	Hecke alg (of Weyl grp)	
"bases"	Verma M_x	⟷	Perv_x	⟷	Std basis T_x	Change-of-basis KL polyn $P_{y,x}(v)$ ← Computable using combinatorics
	Simples L_y	⟷	IC sheaf IC_y ← Intersection cohomology	⟷	KL basis $\underline{H}_y$	

- 1990: Soergel: reformulation of KL theory using Soergel modules + decomposition thm (deep/difficult thm in geometry)
- 1992–2006: Soergel establish theory of (now called) Soergel bimodules so its cat $\mathbb{S}\mathrm{Bim}$ categorifies Hecke alg $\boxed{\text{Chp 3-5}}$
  - Argument goes through for Coxeter grp (not nec Weyl grps) provided an analog of decompn thm
- 2016: Elias–Williamson: diagrammatic theory $\boxed{\text{Chp 7-12}}$

  $\mathbb{S}\mathrm{Bim} \Rightarrow$ diag. Hecke caty

  - reduces complicated polyn computations in $\mathbb{S}\mathrm{Bim}$ to doable manipulations of diagrams
  - Hodge theory ⇒ alg pf of decomp thm (for Coxeter grps)

    $\boxed{\text{Chp 17-20}}$ ⟶ alg pf of KL theory (for Coxeter grps)
  - over $\mathbb{Z}$ $\xrightarrow[\text{mod } p]{\text{reduction}}$ modular repn

    ⇒ counterexample to Lusztig's Conjecture $\boxed{\text{Chp 27}}$
  - Applications in
    - Knot theory: Jones polyn, HOMFLY polyns $\boxed{\text{Chp 21}}$
    - Koszul duality $\boxed{\text{Chp 25-26}}$
    - positivity of $P_{y,x}$