Sep. 25, 2020

Learning Seminar: Soergel bimodules

I Type A

Permutations

$$Sym_n = Aut(\{1,2,\cdots,n\})$$

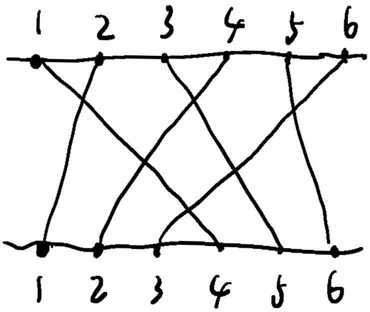
$$Sym_n \curvearrowright \{1,2,\cdots,n\}$$

Two line notation: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 3 & 5 \end{pmatrix}$ mod 7

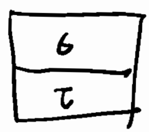
One line notation; $(2\,4\,6\,1\,3\,5)$

Cycle notation $(1\,2\,4)(3\,6\,5)$

Diagram:

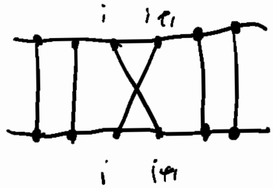


Convention:

- read from right to left
- read from bottom to top
- $\sigma \cdot \tau =$ 

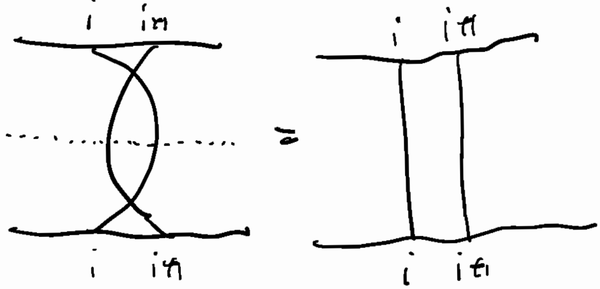
Presentation | diagram | formula

Gen.



s_i

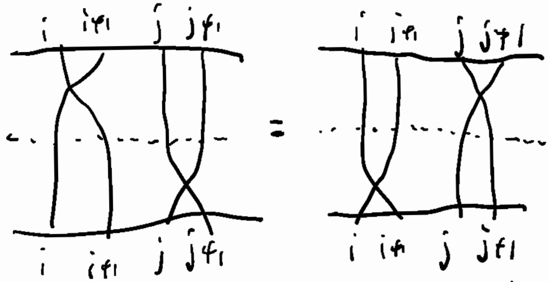
Rel.



$s_i^2 = 1$

$\uparrow$

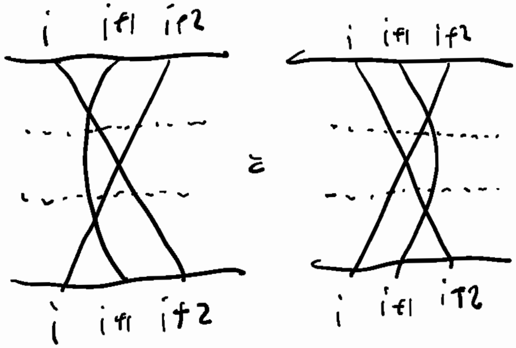
quadratic relation



$s_i s_j = s_j s_i$

if $|i-j| > 1$

braid relation



$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

Thm $\mathrm{Sym}_n = \langle s_i \mid \text{quadratic, braid} \rangle$

Length

The length of $w \in S_n$ is

$\ell(w) :=$ # crossings in reduced diagram of w

reduced # crossing is minimal

Lemma $\ell(w) := \#\{\text{inversions}\}$

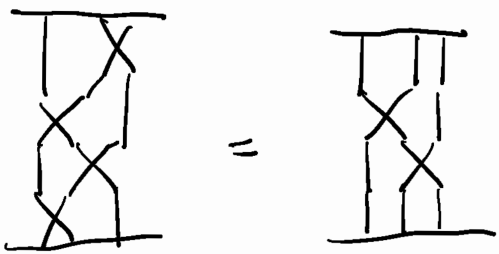
$= \#\{ i < j : w(i) > w(j) \}$

Cor There exists a longest element w_0.

$\ell(w_0) = \binom{n}{2}$

Deletion condition

If two strands cross twice, the we can remove the crossing.



Observation: a diagram is reduced if no two strands cross at least twice.

Expression

An *expression* of w is a sequence $\underline{w}=(s_1, \cdots, s_\ell)$ s.t. $s_1 s_2 \cdots s_\ell = w$.

$\{\text{exp. of } w\} \xleftrightarrow{\sim} \{\text{diagrams of } w\}$

The *length* of expression $s_1, \cdots, s_\ell$ is ℓ.

An expression of w is *reduced* if the length is minimal among all expressions of w.

$\ell(\text{expression}) = \ell(\text{diagram})$

Let rex := reduced exp.

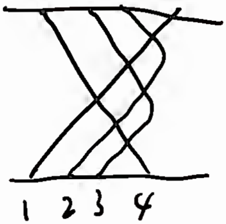
Rex graph

Consider all rex of w.

A *rex move* is a single application of a braid relation to a rex.

The *rex graph* of w is the graph with rexes as vertices and rex moves as edges.

<u>Ex.</u> Let w_0 be the longest element of Sym_4.



$w_0 = s_3 s_2 s_1 s_3 s_2 s_3$ — $s_3 s_2 s_3 s_1 s_2 s_3$

$s_3 s_2 s_1 s_2 s_3 s_2$ — $s_2 s_3 s_2 s_1 s_2 s_3$

$s_3 s_1 s_2 s_1 s_3 s_2$ — $s_2 s_3 s_1 s_2 s_1 s_3$

$s_1 s_3 s_2 s_1 s_3 s_2$ — $s_3 s_1 s_2 s_3 s_1 s_2$ — $s_2 s_1 s_3 s_2 s_1 s_3$ — $s_2 s_3 s_1 s_2 s_3 s_1$

$s_1 s_3 s_2 s_3 s_1 s_2$ — $s_2 s_1 s_3 s_2 s_3 s_1$

$s_1 s_2 s_3 s_2 s_1 s_2$ — $s_2 s_1 s_2 s_3 s_2 s_1$

$s_1 s_2 s_3 s_1 s_2 s_1$ — $s_1 s_2 s_1 s_3 s_2 s_1$

Thm (Matsumoto)

The rexes of w are unique up to rex moves.

not unique in general

Later

exp. $\underline{w}$ $\rightsquigarrow$ Bott-Samelson bimodules, BS

rex. $\underline{w}$ $\rightsquigarrow$ Soergel bimodules $B_{\underline{w}}$

w $\rightsquigarrow$ Soergel bimodule B_w
up to iso.

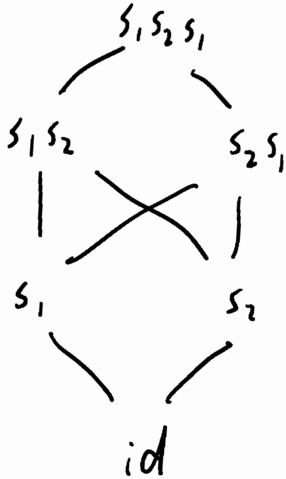
$\rightsquigarrow$ Soergel bimodule B_w
up to unique iso.

Bruhat order

$\underline{x} < \underline{y}$ if $\underline{x}$ is a subexp. of $\underline{y}$.

$x < y$ if $\underline{x}$ is a subexp of rex $\underline{y}$
for some $\underline{x}$.

Ex. $\mathrm{Sym}_3 = \{ id, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1 = s_2 s_1 s_2 \}$

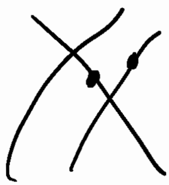


II Type B, D, I

Type B $\quad \mathrm{SignSym}_n :=$ perms of $\{\pm 1, \cdots \pm n\}$
s.t. $\sigma(i) + \sigma(-i) = 0$

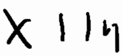
I_t is generated by $\{ \underbrace{(1\,2)(-1\,-2)}_{= s_1}, \cdots, \underbrace{(1, -1)}_{t} \}$

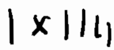
Dotted diagram



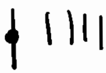
• means negation

$1 \mapsto 2$
$2 \mapsto -3$
$3 \mapsto -1$

gen:  $= s_1$

 $= s_2$

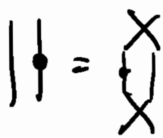
$\vdots$

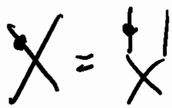
 $= t$

(new) rel: 

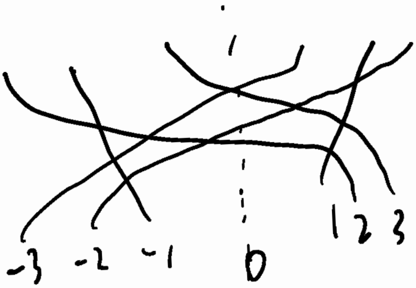
Pro: easy to use

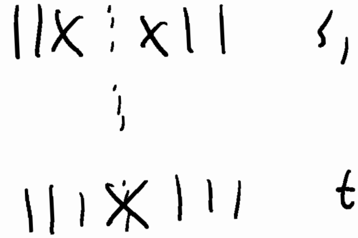
Con: length is not obvious

 length 3

 length 2

Mirrored diagram



gen: 

Pro: length = #{crossings ≥ 0}

Con: Hard to use

Type D

EvenSignSym$_n$:= subgp. of SignSym$_n$ with $\#\{i>0 : \sigma(i)<0\}$ is even

Type I

Dihedral group

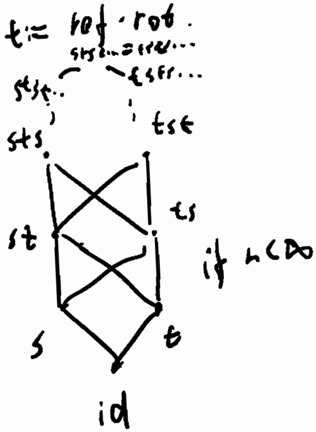
Type I ⇝ Two pts property / Three pts properties } ⇝ Coxeter system

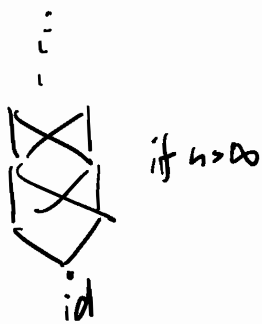
= Sym(Regular polygon)

$= \langle rot, ref : rot^n = 1,\ ref^2 = 1,\ (ref \cdot rot)^2 = 1 \rangle$

Let $s := ref$, $t := ref \cdot rot$.

Bruhat order





III Coxeter group.

Thm (Coxeter) Any finite subgp. of $O(\mathbb{R}^n)$ generated by reflection has a presentation

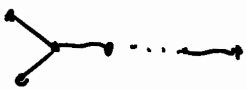
$$W = \langle s \in S : s^2 = 1, (st)^{m_{st}} = 1 \rangle \qquad sts\cdots = tst\cdots$$

simple refl. — quad. — braid — $2 \le m_{st} < \infty$

Moreover, they are products of

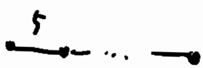
A_n 

B_n 

D_n 

E_6, E_7, E_8 

F_4 

H_3, H_4 

I_n 

Def Coxeter system (W, S), S finite

$$W = \langle s \in S : s^2 = 1,\ (st)^{m_{st}} = 1 \rangle$$

$m_{st} \in \{2, 3, \dots\} \cup \{\infty\}$

All Coxeter systems are "limits" of type I.

Geometric representation

$$V := \bigoplus_{s \in S} k\alpha_s \qquad R = \operatorname{Sym} V$$

$$\langle\ ,\ \rangle : V \times V \to k$$

$$\alpha_s,\ \alpha_t \mapsto -\cos\frac{\pi}{m_{st}} \qquad (m_{st} = \infty)$$

action $W \times V \to V$

$$s\lambda = \lambda - 2\langle \lambda, \alpha_s \rangle \alpha_s$$

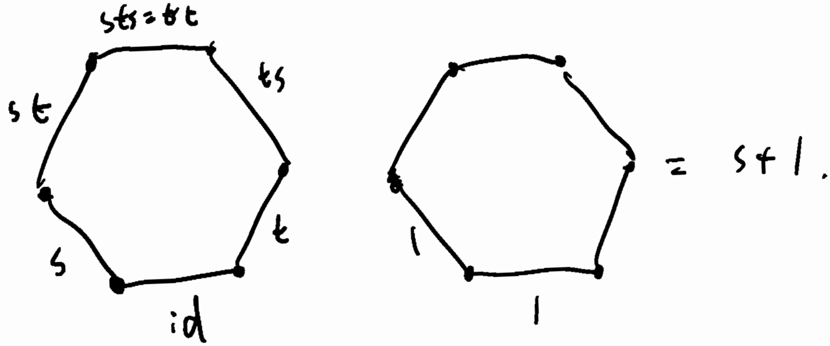
Prop The geometric repr. is faithful.

Cor $W_{m,\infty}$ $B_{\underline{stst\cdots}} \to B_{\underline{tsts\cdots}}$. "$1 \mapsto 1$"

Faithful $V \Rightarrow$ such good map exists.

Coxeter complex of type I

Geometric way of writing $\sum_{w\in W} c_w w \in kW$.



finite Coxeter complex ~ |S| dim sphere