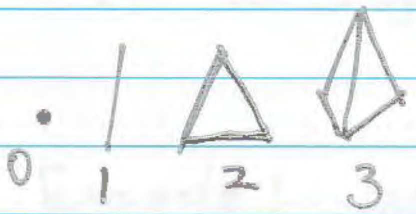
The Coxeter complex

- Construction.
- Examples: A_2, A_3, $I_2(m)$, $I_2(\infty)$ and $\tilde{A}_2$.
- Coxeter group action.



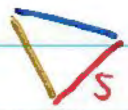
A standard ℓ-simplex
$$\Delta^{\ell} := \{(t_0, t_1, \dots, t_\ell) \in \mathbb{R}^{\ell+1} \mid \textstyle\sum t_i = 1,\ t_i \geq 0, \forall i\}.$$

Definition:

- Let (W,S) be a Coxeter group of rank n.

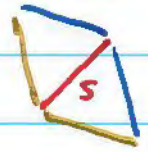
$(W,S) \rightsquigarrow$ (n-1)-standard simplex Δ^{n-1} with coloring of the n faces by S.

Example: $|S| = 3$. A 2-simplex associated with (W,S).

Δ^2 : 

- For $s \in S$, the s-glueing is given by the refl'n along the face colored by s:

Example:
(A s-glueing)



$\Sigma(W,S)$

of (W,S)

- The Coxeter complex is a simplicial complex of dimension $n-1$ with faces colored by S.

  It is constructed as follows:

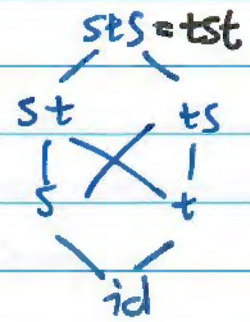
  (1). Take one copy Δ_w of Δ for each $w \in W$.

  (2). For each $w \in W$, $s \in W$, glue Δ_w to Δ_{ws} via an s-glueing.

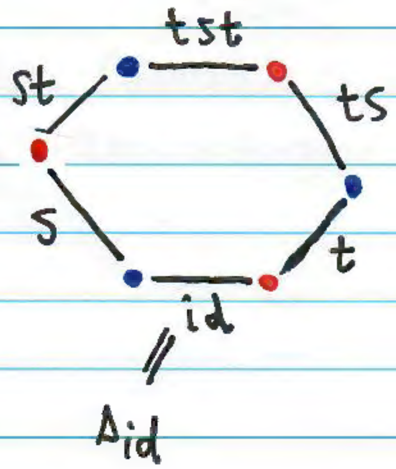
$m_{st} = 3$

Example: Let $(W, S) = (\underset{I_2(3)}{\underset{\parallel}{A_2}}, \{s,t\})$.

Bruhat graph :



Δ^2 :



$\Sigma(A_2, \{s,t\})$

A 1-dim simplicial complex.

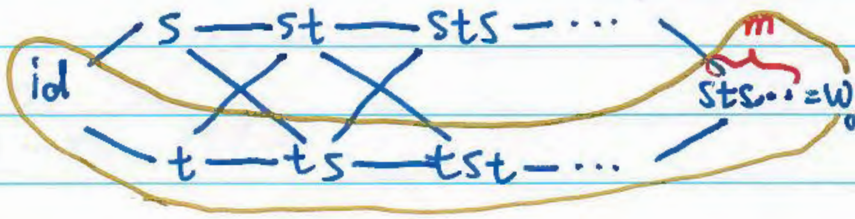
The Coxeter complex of $(A_2, \{s,t\})$ is a 6-polygon.

Coxeter graph: $\overset{s}{\circ}\overset{m}{\text{——}}\overset{t}{\circ}$ (i.e. $m_{st}=m$)

Example: Let $(W, S) = (I_2(m), \{s,t\})$

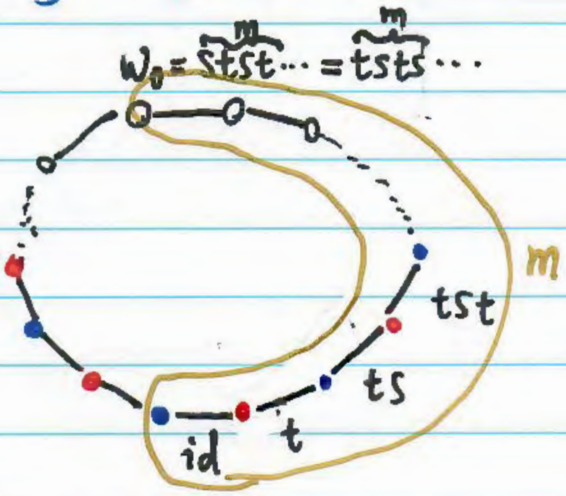
diheral group of order $2m$.

Bruhat graph



Δ^2:

$\Sigma(I_2(m),\{s,t\})$:

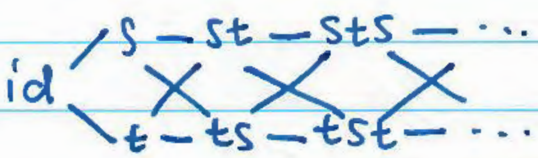


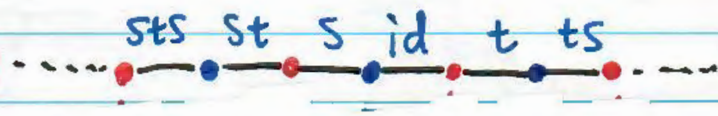
A 1-dim simplicial complex.

The Coxeter complex of $(I_2(m), \{s,t\})$ is a $2m$-polygon.

Example : Let $(W, S) = (I_2(\infty), \{s,t\}) = \tilde{A}_1$ (i.e. $m_{st}=\infty$).

(infinite diheral group).

Bruhat graph 

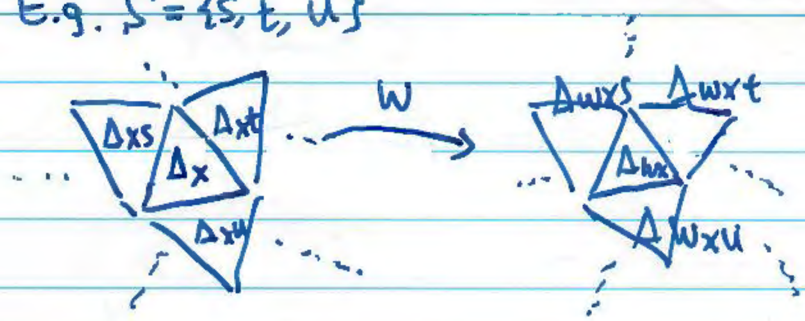
$\Sigma(I_2(\infty), \{s,t\})$: 

(W,S) : Coxeter group

Then W acts faithfully on $\Sigma(W,S)$:

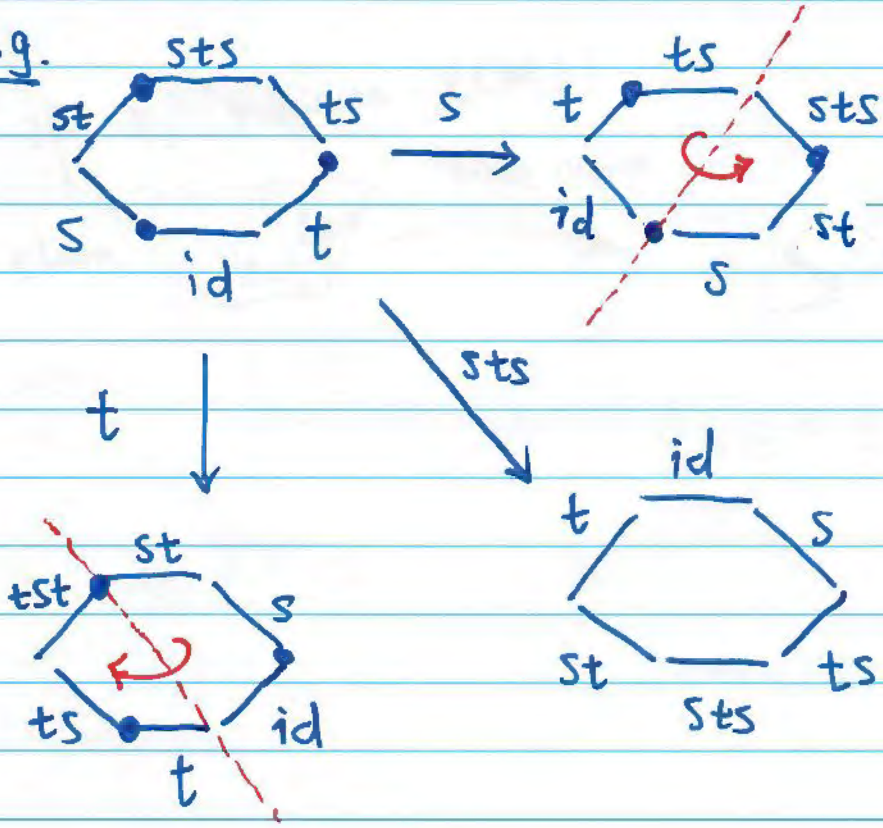
$$w: \Delta_x \mapsto \Delta_{wx}$$

E.g. $S=\{s,t,u\}$



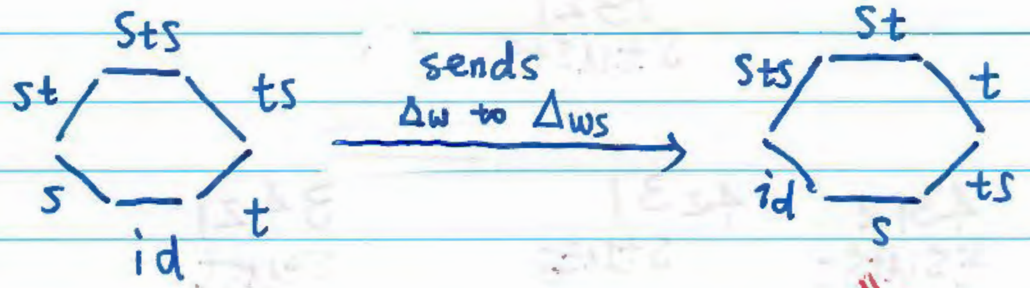
preserve the Coxeter complex

E.g.



- W can not act on $\Sigma(W,S)$ "on the right".

e.g.

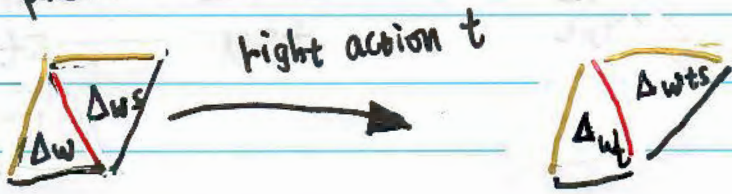


$\neq \Sigma(W,S)$

because $wts \neq wst$ in general.

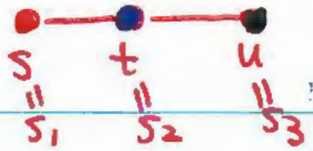
i.e. If r_t preserve $\Sigma(W,S)$

then



$\Rightarrow \Delta_{wst} = \Delta_{wts}$

⑦



$s = s_1$, $t = s_2$, $u = s_3$

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1432 utu   4123 uts   2413 sut   3142 tsu   3214 sts   2341 stu

1423 ut   1342 tu   2143 su   3124 ts   2314 st

1243 u   1324 t   2134 s

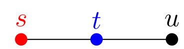
1234 (id)

The Coxeter complex $\Sigma(A_3, \{s,t,u\})$ of A_3

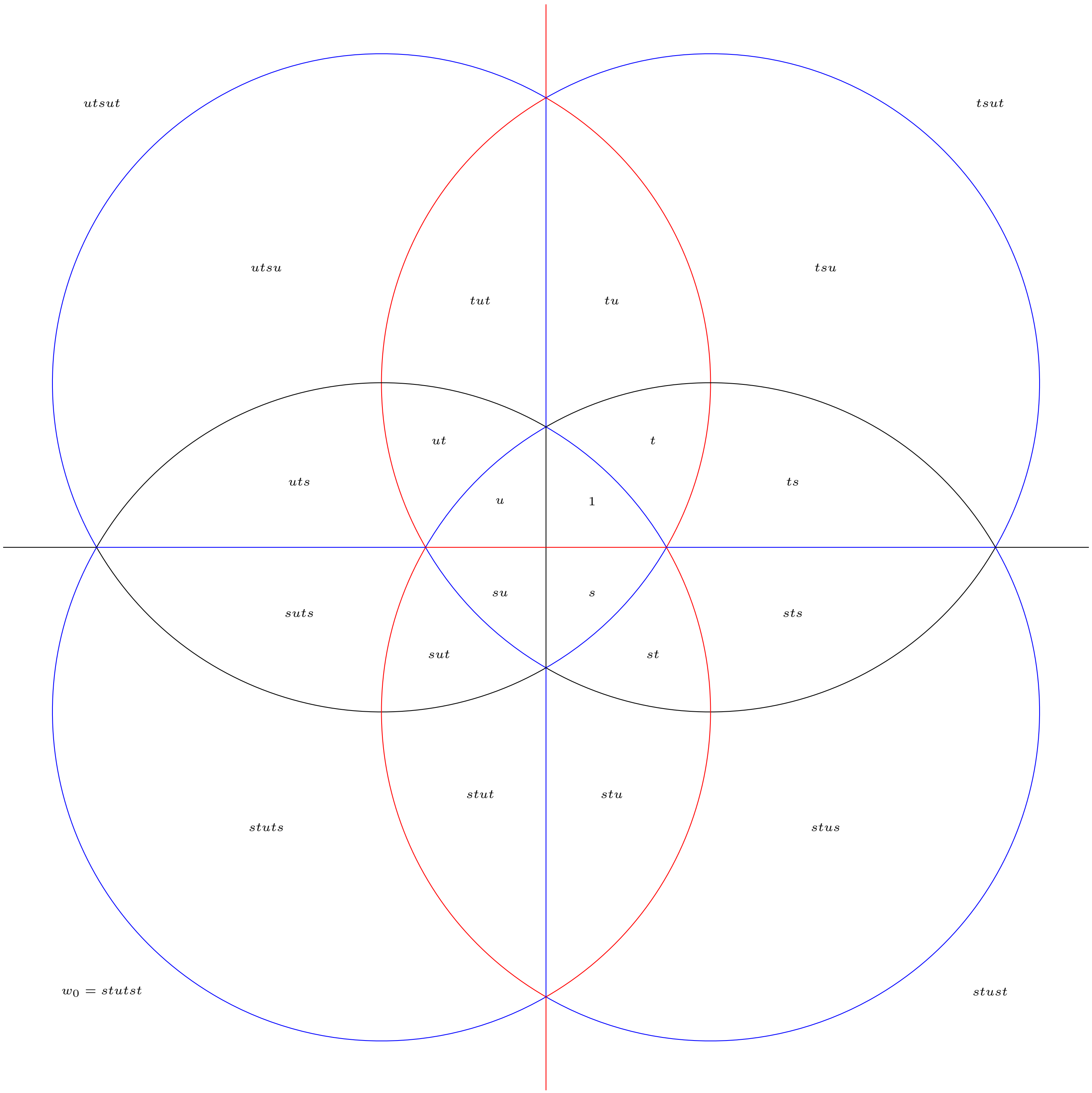
The Coxeter complex $\Sigma(\tilde{A}_2, \{s,t,u\})$ of $\tilde{A}_2$.

SOERGELBOOK APPENDIX:
COXETER COMPLEXES FOR TYPES A_3, B_3, $\widetilde{A}_2$

1. Type A_3



Date: November 13, 2018.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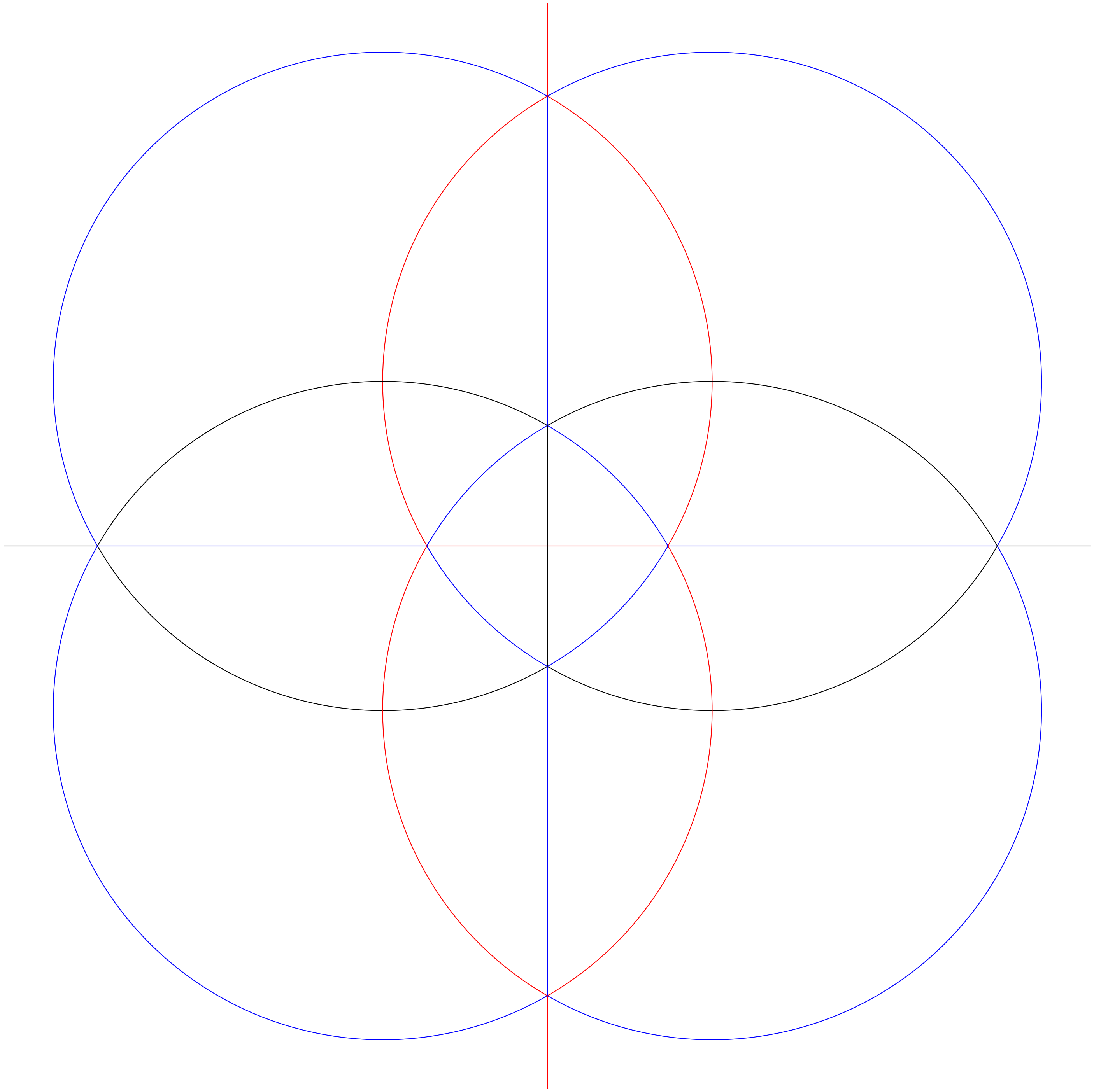
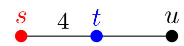
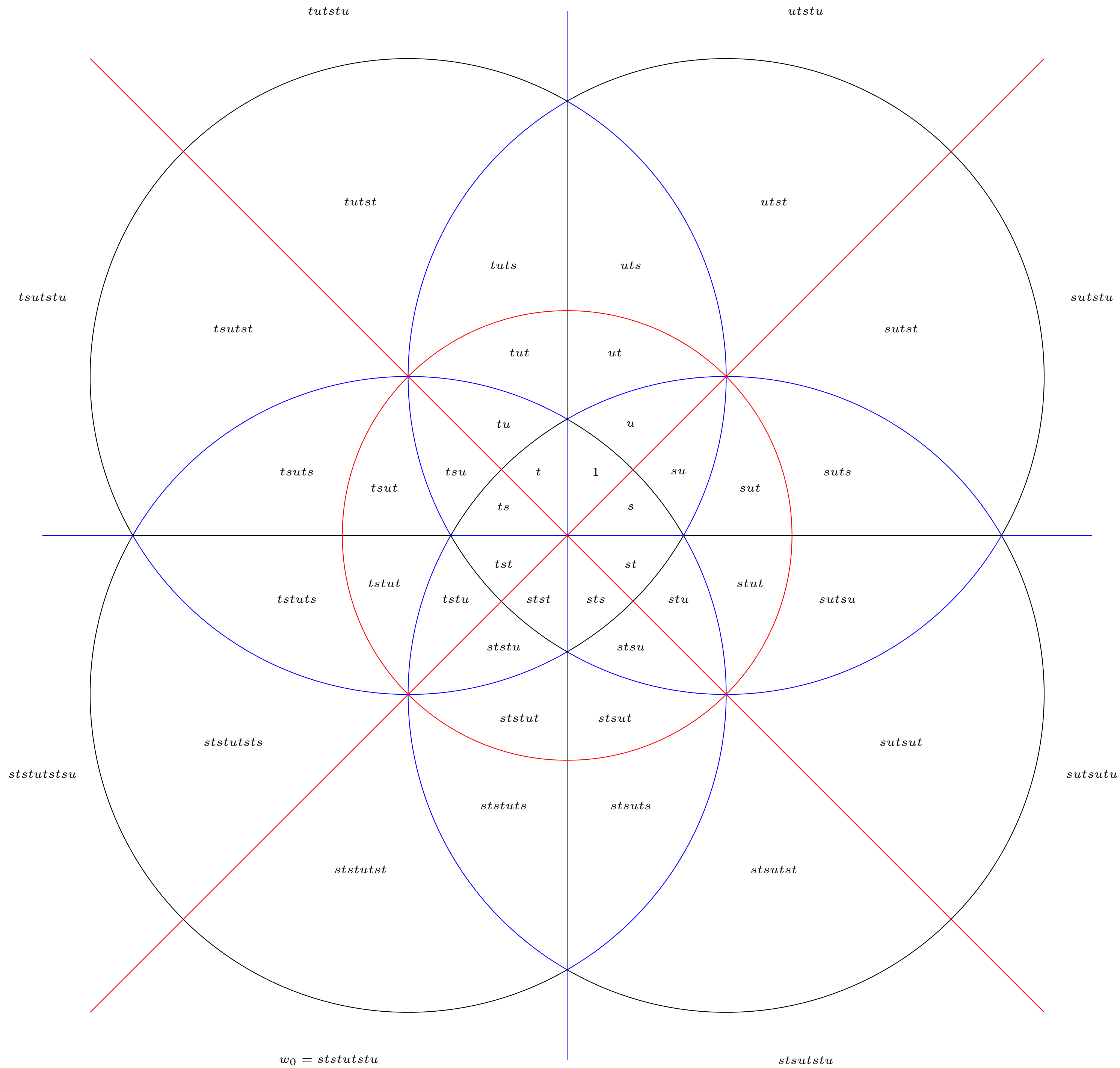


2. Type B_3





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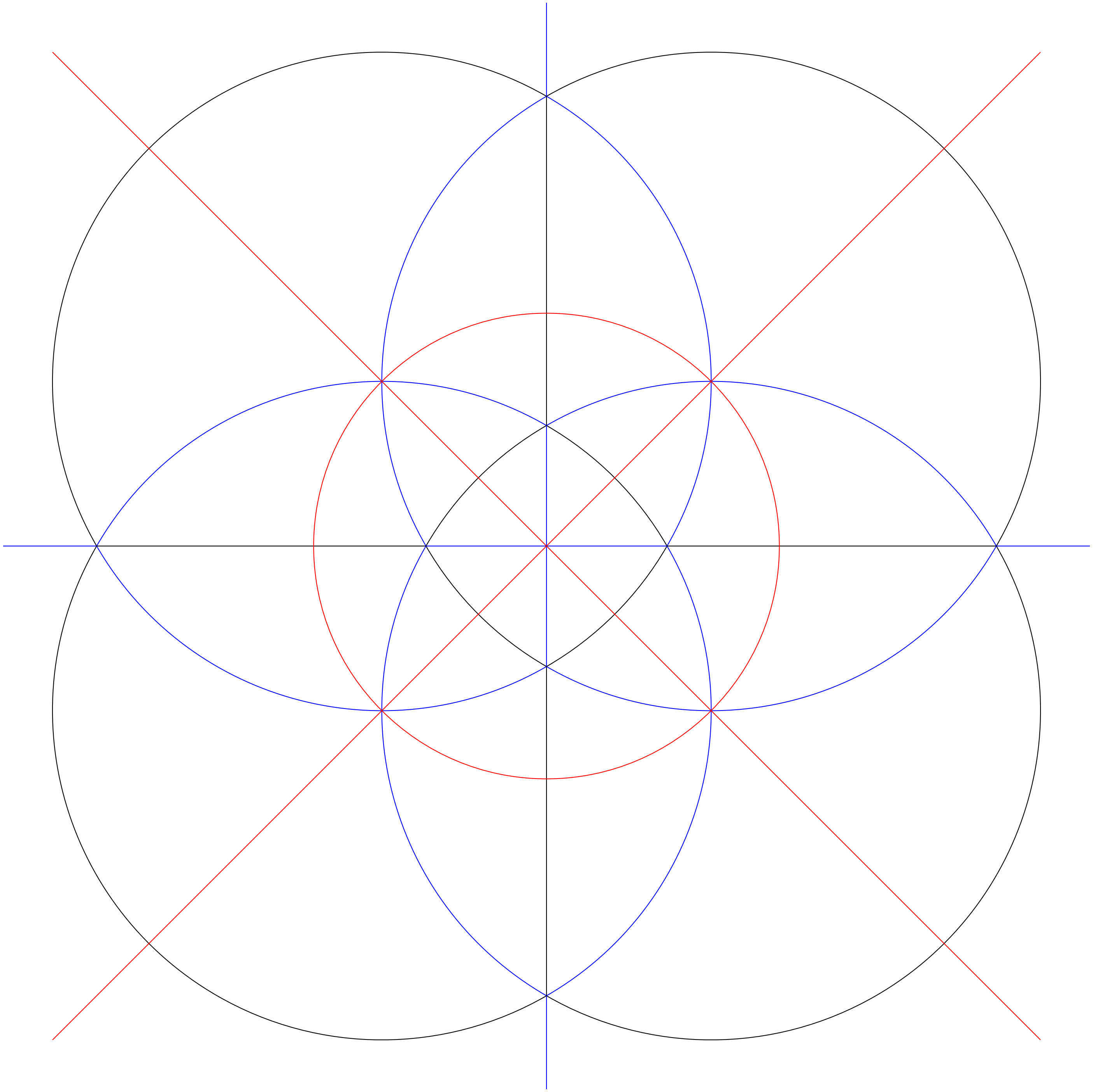
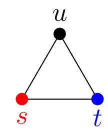
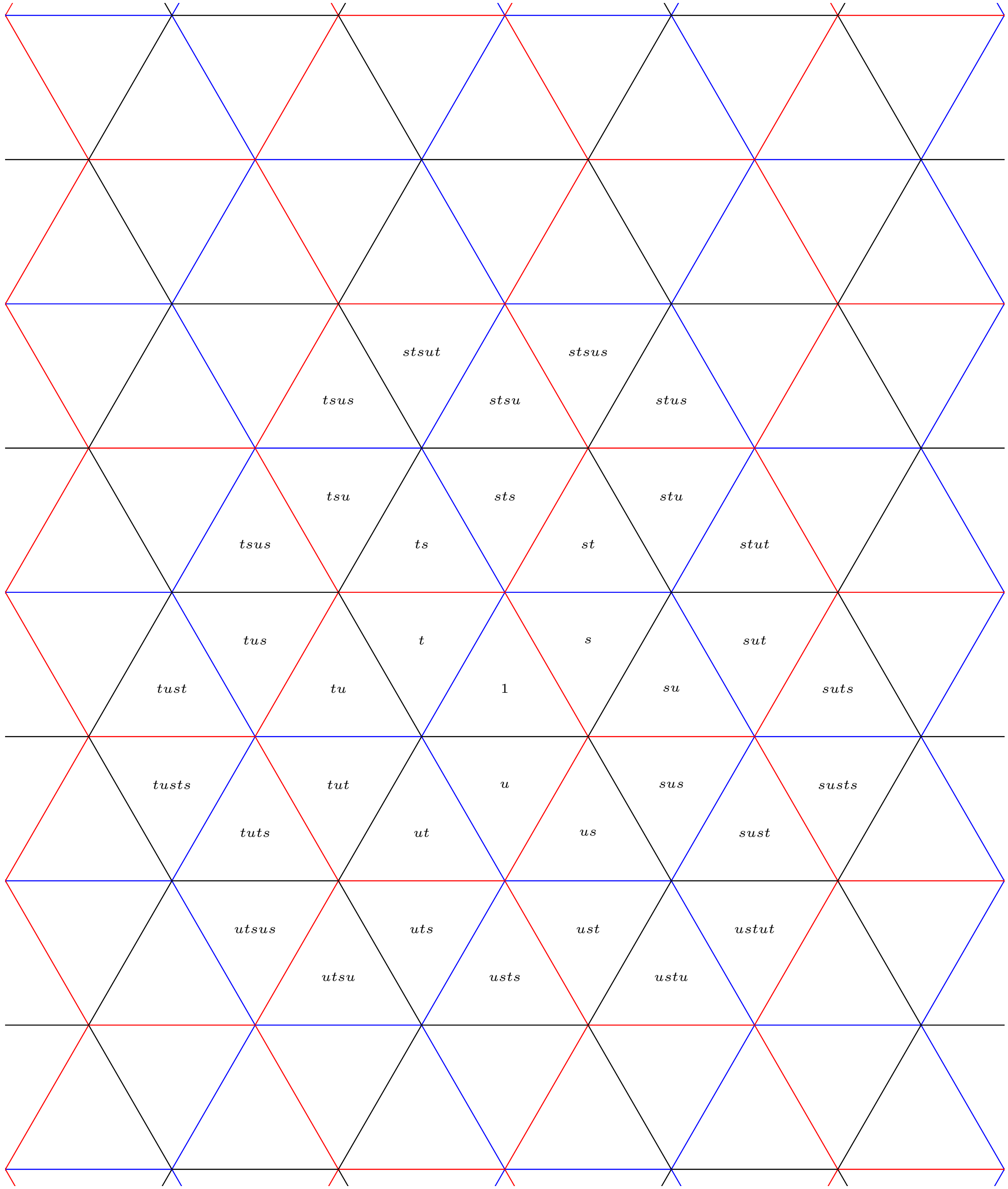


3. Type $\widetilde{A}_2$

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