Let (W,S) coxeter group. $W=\langle S \mid \underbrace{s_1^2=\cdots=s_{|S|}^2=Id}_{\text{(quadratic rels)}},\ \underbrace{\{\underbrace{sts\cdots}_{m_{st}}=\underbrace{tst\cdots}_{m_{st}}\}}_{\text{(braid rels)}} \rangle$

Given $w\in W$,

$l(w):=$ minimum word length in S's

When achieved $l(w)$, called $\underline{w}$ reduced expr.

Th (Matsumoto) All reduced words are related by braid rels.

Th (Exchange Condition)

Given $\underline{w}=s_1s_2\cdots s_k$ reduced expr. Given $s\in S$.

If $l(ws)<l(w)$ then $\exists\, 1\le i\le k$ s.t. $ws=s_1s_2\cdots\hat{s_i}\cdots s_k$.

Cor (Useful) In above case, $w=wss=s_1\cdots\hat{s_i}\cdots s_k s$ is reduced & ending at s

i.e. $\forall s\in S$, either $l(ws)=k+1$ or $\exists\, i$ s.t. $w=s_1\cdots\hat{s_i}\cdots s_k s$

Th (Deletion Condition)

Given $x=s_1\cdots s_k$ any expres

If $l(x)<k$, then $\exists\, 1\le i<j<k$ s.t. $x=s_1\cdots\hat{s_i}\cdots\hat{s_j}\cdots s_k$.

Rmk In view of exchange condition, Not every s_j can be eliminated by $(\ )\cdot s$, $s\in S$

However, can be achieved by $(\ )\cdot t\in T$, $T=\{z\in W / z\overset{\text{conj}}{\sim}\text{ some } s\in S\}$ T are all reflections!

Namely, let $t=s_k\cdots s_{j+1}s_js_{j+1}\cdots s_k \in T$

Then $wt=s_1\cdots\hat{s_j}\cdots s_k$ eliminates s_j for arbitrary j

Def (Bruhat order)

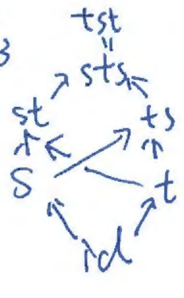
We write $x\to y$ if $\begin{cases} l(x)<l(y) \\ x=yt \text{ (equiv, } xt=y) \text{ for some } t\in T. \end{cases}$

and $w\le z$ if $\exists\ w\to a\to b\to\cdots\to y\to z$

Def' (Bruhat order)

$x\le y$ means some substring of some reduced exprs of y is a reduced exprs of x.

Eg $A_2=S_3$



Rmk It does not matter to use left or right T-multiplication in the def. of Bruhat order (Bruhat order is intrinsic)

$\because\ x=yt=\underbrace{(yty^{-1})}_{\in T}\,y$ (but for Coxeter grp always on the right!)

Given $W=(W,S) \rightsquigarrow \mathbb{Z}[W]$ its group alg $\xrightarrow{\text{deform}}$ $H=H(W)$ Hecke alg over $\mathbb{Z}[v,v^{-1}]$

$v=1$

$\boxed{\text{Def}}$ (Hecke alg)

H is the unital ($1=\delta_{id}$) assoc. alg. $/\mathbb{Z}[v,v^{-1}]$ gen. by $\{\delta_s \mid s\in S\}$ w/ relations

$$\begin{cases} \text{(quadratic rels)} & \delta_s^2=(v^{-1}-v)\delta_s+1 \\ \text{(braid rels)} & \underbrace{\delta_s\delta_t\delta_s\cdots}_{m_{st}} = \underbrace{\delta_t\delta_s\delta_t\cdots}_{m_{st}} \end{cases}$$

may think $v\in S^1\subset\mathbb{C}$ then $v^{-1}-v\in i\mathbb{R}$. only when $v=1$ gives $0\in\mathbb{R}$.

by Matsumoto, well-defined

$\boxed{\text{Standard Basis}}$.

$\forall x\in W$, $x=s_1s_2\cdots s_m$ reduced define $\delta_x:=\delta_{s_1}\delta_{s_2}\cdots\delta_{s_m}\in H$

Standard basis of Hecke alg H: $\{\delta_x \mid x\in W\}$.

$\boxed{\text{Product Rule}}$

$$\delta_x\delta_s=\begin{cases}\delta_{xs}, & x<xs\\ (v^{-1}-v)\delta_x+\delta_{xs}, & x>xs.\end{cases} \qquad \delta_s\delta_x=\begin{cases}\delta_{sx}, & x<sx\\ (v^{-1}-v)\delta_x+\delta_{sx}, & x>sx\end{cases}$$

(pf) $x=s_1\cdots s_n$ reduced expr. if $x<xs \Rightarrow xs=s_1\cdots s_n\cdot s$ is reduced $\Rightarrow \delta_x\delta_s=\delta_{s_1}\delta_{s_2}\cdots\delta_{s_n}\delta_s=\delta_{xs}$

If $x>xs$, by $\boxed{\text{useful cor}}$ rewrite $x=s_1\cdots s_{n-1}s$ & $xs=s_1\cdots s_{n-1}$ reduced.

$\Rightarrow \delta_x\delta_s=\delta_{s_1}\cdots\delta_{s_{n-1}}\delta_s\cdot\delta_s=\delta_{s_1}\cdots\delta_{s_{n-1}}\left[(v^{-1}-v)\delta_s+1\right]=(v^{-1}-v)\delta_x+\delta_{xs}$

$\boxed{\text{Inversion Rule}}$ $\delta_s^2=(v^{-1}-v)\delta_s+1 \Rightarrow \delta_s=(v^{-1}-v)+\delta_s^{-1}\Rightarrow \delta_s^{-1}=\delta_s+(v-v^{-1})$

Let $w=s_1\cdots s_m$ reduced.

$\Rightarrow w^{-1}=s_m\cdots s_1 \Rightarrow \delta_{w^{-1}}=\delta_{s_m}\cdots\delta_{s_1}\Rightarrow (\delta_{w^{-1}})^{-1}=\delta_{s_1}^{-1}\cdots\delta_{s_m}^{-1}$

algebra (when it comes to inverse, the product is opposite to group product

$\Rightarrow (\delta_{w^{-1}})^{-1}=[\delta_{s_1}+(v-v^{-1})]\cdots[\delta_{s_m}+(v-v^{-1})]=\sum_{e\in\{0,1\}^m} A_e(v,v^{-1})\cdot\delta_{s_1}^{e_1}\cdots\delta_{s_m}^{e_m}$

Want to show $\delta_{s_1}^{e_1}\cdots\delta_{s_m}^{e_m}\in\sum_{x\le w}\mathbb{Z}[v,v^{-1}]\cdot\delta_x$.

Write $\delta_{s_1}^{e_1}\cdots\delta_{s_m}^{e_m}=\delta_{r_1}\delta_{r_2}\cdots\delta_{r_k}$ by picking those $e_j=1$. If reduced $\Rightarrow \delta_{s_1}^{e_1}\cdots\delta_{s_m}^{e_m}=\delta_{r_1r_2\cdots r_k}$ where $r_1r_2\cdots r_k\le s_1\cdots s_m = w$

Subexpr reduced $\Rightarrow\le$

If not reduced, let $r_1r_2\cdots r_j$ reduced $\Rightarrow (r_1r_2\cdots r_j)r_{j+1}$ not reduced. (j max s.t.)

By $\boxed{\text{Deletion Condition}}$ $\exists 1\le a<b\le j+1$ s.t. $r_1r_2\cdots r_{j+1}=r_1\cdots\hat{r}_a\cdots\hat{r}_b\cdots r_{j+1}$ reduced.

$\Rightarrow$ By $\boxed{\text{Product Rule}}$ $(\delta_{r_1}\cdots\delta_{r_{j+1}})\delta_{r_{j+2}}\cdots\delta_{r_k}=[(v^{-1}-v)\delta_{r_1\cdots r_j}+\delta_{r_1\cdots\hat{r}_a\cdots\hat{r}_b\cdots r_{j+1}}]\delta_{r_{j+2}}\cdots\delta_{r_k}$.

$\delta_{r_1\cdots r_j}=\delta_{r_1}\cdots\delta_{r_j}$, $\delta_{r_1\cdots\hat{r}_a\cdots\hat{r}_b\cdots r_{j+1}}=\delta_{r_1}\cdots\hat{\delta}_{r_a}\cdots\hat{\delta}_{r_b}\cdots\delta_{r_{j+1}}$

$= (v^{-1}v)\delta_{r_1}\cdots\hat{\delta}_{j+1}\cdots\delta_k + \delta_{r_1}\cdots\delta_{r_a}\cdots\hat{\delta_b}\cdots\delta_k.$

again, choose max l s.t. ... max i s.t. ....

repeat the ~~s~~ procedure we finally get reduced expr. for each terms.

therefore $\delta_{s_1}^{e_1}\cdots\delta_{s_m}^{e_m} \in \sum_{x\le w} \mathbb{Z}[v,v^{-1}]\cdot\delta_x$.

So we prove

$\boxed{\text{Prop}}$ Each of The standard basis δ_w is invertible and.

$$\delta_{w^{-1}}^{-1} = \delta_w + \sum_{x<w} a_x\cdot\delta_x \quad , a_x\in\mathbb{Z}[v,v^{-1}]$$

$\boxed{\text{Kazhdan-Lusztig Basis}}$

Observe that $\delta_s^{-1} = \delta_s + (v-v^{-1}) \Rightarrow \underline{\delta_s^{-1}+v^{-1}} = \delta_s + v$.) self dual. $b_s := \delta_s + v$ is

Define bar involution $H \xrightarrow{\bar{\ }} H$ by $\bar{v} = v^{-1}$, $\bar{\delta_s} := \delta_s^{-1}$ ~~and extend covariantly~~

extends to product $(\overline{ab} = \bar{a}\bar{b}) \Rightarrow \overline{\delta_{s_1s_2}} = \overline{\delta_{s_1}\delta_{s_2}} = \overline{\delta_{s_1}}\,\overline{\delta_{s_2}} = \delta_{s_1}^{-1}\delta_{s_2}^{-1}$

$= (\delta_{s_2}\delta_{s_1})^{-1} = \delta_{s_2s_1}^{-1} = \delta_{(s_1s_2)^{-1}}^{-1} \Rightarrow$ on standard basis $\overline{\delta_x} := (\delta_{x^{-1}})^{-1}$.

$\boxed{\text{Def}}$ (KL-basis) $\{b_x \mid x\in W\}$ basis of H satisfying

① (self-duality) $\overline{b_x} = b_x$

② (degree bound) $b_x = \delta_x + \sum_{y<x} h_{y,x}\cdot\delta_y$, for some $h_{y,x}\in v\mathbb{Z}[v]$

$\boxed{\text{Prop}}$ KL-basis is unique. Say suppose $\{b_x \mid x\in W\}$ & $\{C_x \mid x\in W\}$ are KL-basis.

$\Rightarrow$ write $b_x = \delta_x + \sum_{y<x} \underset{v\mathbb{Z}[v,v^{-1}]}{h_{y,x}}\delta_y$, $C_x = \delta_x + \sum_{y<x} \underset{v\mathbb{Z}[v,v^{-1}]}{g_{y,x}}\delta_y$. as in algebra

$\Rightarrow \sum_{y<x}(h_{y,x}-g_{y,x})\delta_y = b_x - C_x = \overline{b_x - C_x} = \sum_{y<x}\left(\overline{h_{y,x}-g_{y,x}}\right)\cdot\overline{\delta_y}$

pick highest ~~max~~ order y s.t. $h_{y,x}-g_{y,x}\neq 0$. One has $\overline{\delta_y} = \delta_{y^{-1}}^{-1} = \delta_y + \sum_{z<y} a_z\delta_z$

So by comparing the highest term on both side

$\Rightarrow \underset{\in v\mathbb{Z}[v,v^{-1}]}{h_{y,x}-g_{y,x}} = \overline{h_{y,x}-g_{y,x}} \in v^{-1}\mathbb{Z}[v,v^{-1}] . \Rightarrow$ must be zero.

Now one pick $b_s = \delta_s + v$ for $s\in S$. The construction of KL basis explicitly by examples.

$b_{id} = \delta_{id} = 1$

2.5

Coxeter cplx of (W,S): Coxeter gp of rank $n=|S|$ $\rightsquigarrow$ optain in $\mathbb{A}^{n-1}$.

① Take one copy of Δ_w: $(n-1)$-simplex for each w.

② $\forall w\in W$, $s\in S$, glue Δ_w to Δ_{ws} via s-gluing
act on $\widetilde{\text{right}}$.

$$\delta_x b_s = \begin{cases} \delta_{xs} + v\,\delta_x, & x<xs \\ \delta_{xs} + v^{-1}\delta_x, & x>xs \end{cases}$$

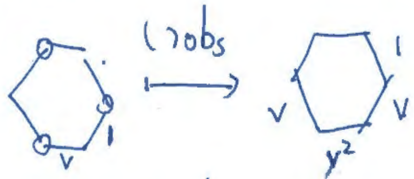
quite Product rule.

$\boxed{\text{Eg}}$ (Type A_2)
$W = \langle s,t \mid s^2=t^2=\mathrm{Id},\ sts=tst\rangle$.

Bruhat Order

$$\begin{cases} b_{Id} = \delta_{Id} = 1. \\ b_s = \delta_s + v \\ b_t = \delta_t + v. \end{cases}$$

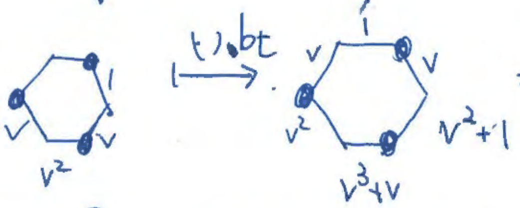
try $b_t\cdot b_s$. $b_t = \delta_t + v$



$= \delta_{ts} + v\delta_t + v\delta_s + v^2 = b_{ts}$ by deg bdd & self-dual

$b_t b_s$ is apriori self-dual

try $b_{ts}\cdot b_t$



$= \delta_{tst} + v(\delta_{st}+\delta_{ts}) + v^2(\delta_s+\delta_t) + (\delta_t) + (v^3+v)$

$\Rightarrow\ b_{ts}\cdot b_t - b_t = \delta_{tst} + v(\delta_{st}+\delta_{ts}) + v^2(\delta_s+\delta_t) + v^3$.

$b_t b_s b_t - b_s$ $\Rightarrow$ $\overset{\text{deg bdd}}{\underset{\text{self-dual}}{=}}$ b_{tst} $\underset{\text{by symm}}{=}$ $b_{st}b_s - b_s = b_s b_t b_s - b_s$

Same way $\Rightarrow$ $b_{st} = \delta_{st} + v(\delta_t+\delta_s) + v^2$.

$$\Rightarrow \begin{pmatrix} 1\\ b_s\\ b_t\\ b_{st}\\ b_{ts}\\ b_{sts}\end{pmatrix} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&v&0&0&0\\ 0&0&1&v&0&0\\ &&&1&& \end{pmatrix}\begin{pmatrix} 1\\ \delta_s\\ \delta_t\\ \delta_{st}\\ \delta_{ts}\\ \delta_{sts}\end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} b_{sts}\\ b_{st}\\ b_{ts}\\ b_s\\ b_t\\ 1\end{pmatrix} = \begin{bmatrix} 1&v&v&v^2&v^2&v^3\\ 0&1&0&v&v&v^2\\ 0&0&1&v&v&v^2\\ 0&0&0&1&0&v\\ 0&0&0&0&1&v\\ 0&0&0&0&0&1 \end{bmatrix}\begin{bmatrix} \delta_{sts}\\ \delta_{st}\\ \delta_{ts}\\ \delta_s\\ \delta_t\\ 1\end{bmatrix}.$$

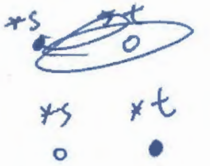
What forms the relation in $W=A_2$ is

$$\begin{cases} b_s b_s = (v+v^{-1})\,b_s \\ b_s b_t b_s - b_s = b_t b_s b_t - b_t \end{cases}$$

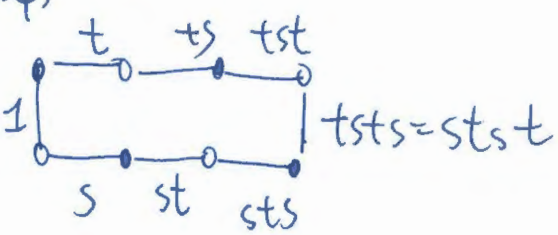
in terms of KL basis!

Kazhdan–Lusztig ~~term~~ polynomials.

$\boxed{Eg}$ (Type $I_2(m)$) $\overset{m}{\underset{s}{\circ}\!-\!\underset{t}{\bullet}}$ means $Id = s^2 = t^2 = (st)^m$



($m=4$)

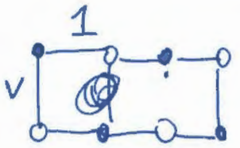


Start with $\begin{cases} b_{Id} = \delta_{Id} = 1 \\ b_s = \delta_s + v \\ b_t = \delta_t + v \end{cases}$

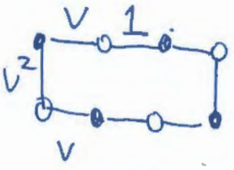
Bruhat order

try $b_t \cdot b_s$

$b_t = \delta_t + v$



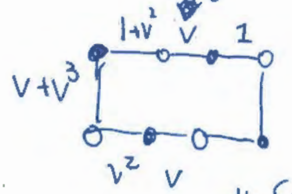
$\xrightarrow{(\ )\circ b_s}$



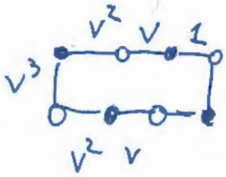
selfadj by deg

$b_t b_s = b_{ts} = \delta_{ts} + v(\delta_t + \delta_s) + v^2.$

$\Big\downarrow (\ )\cdot b_t = \delta_{tst} + v(\delta_{ts} + \delta_{st}) + v^2(\delta_t + \delta_s) + v^3 + \underbrace{[\delta_t + v]}_{b_t}.$

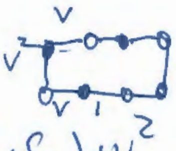


⇓ Subtract



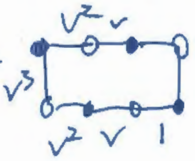
$= b_{tst} = b_{ts} \cdot b_t - b_t$

In the symm way

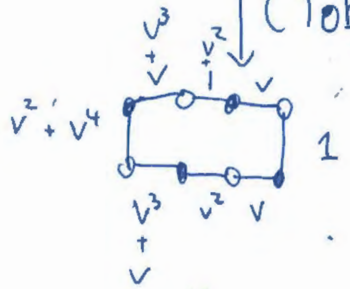


$b_{st} = b_s \cdot b_t = \delta_{st} + v(\delta_s + \delta_t) + v^2$

$b_{sts} = b_{st} \circ b_s - b_s$

$=$ 

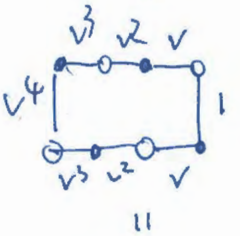
$\Big\downarrow (\ )\circ b_s$



$= \delta_{tsts} + v(\delta_{tst} + \delta_{sts}) + v^2(\delta_{ts} + \delta_{st}) + (v^3)(\delta_t + \delta_s) + (v^4)$

$\boxed{+\delta_{ts} + v(\delta_t + \delta_s) + v^2}$ b_{ts}

⇓



||

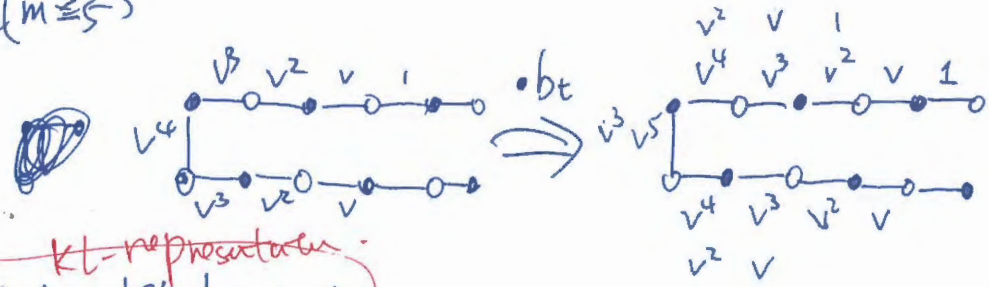
$b_{tsts} = b_{tst} \circ b_s - b_{ts}$

$$\begin{bmatrix} 1 & v & v & v^2 & v^2 & v^3 & v^3 & v^4 \\ & 1 & 0 & v & v & v^2 & v^2 & v^3 \\ & & 1 & v & v & v^2 & v^2 & v^3 \\ & & & 1 & 0 & v & v & v^2 \\ & & & & 1 & v & v & v^2 \\ & & & & & 1 & 0 & v \\ & & & & & & 1 & v \\ & & & & & & & 1 \end{bmatrix} \begin{pmatrix} \delta_{tsts} \\ \delta_{tst} \\ \delta_{sts} \\ \delta_{ts} \\ \delta_{st} \\ \delta_t \\ \delta_s \\ 1 \end{pmatrix}$$

In fact the only KL polynomial is "1", up to the difference of orders

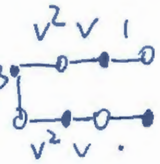
On the other hand, $\boxed{Thm}$ (Polo 1999)

$I_2(m \geqq 5)$



$b_{tsts} \bullet b_t$

$\Downarrow$ subtract



b_{tst}.

~~$b_{tstst} - b_{tst}$~~.

$b_{tstst} = b_{tsts} \bullet b_t - b_{tst}$.

~~KL-representation~~

Let's use KL-basis to represent itself:

$b_{Id} = 1$

b_s

b_t

- $b_{st} = b_s \bullet b_t$

  $b_{ts} = b_t \circ b_s$

- $b_{sts} = b_{st} \circ b_s - b_s = b_s b_t b_s - b_s$

  $b_{tst} = b_{ts} \bullet b_t - b_t = b_t b_s b_t - b_t$

- $b_{stst} = b_{sts} \bullet b_t - b_{st} = (b_s b_t b_s - b_s) b_t - b_s b_t = b_s b_t b_s b_t - 2 b_s b_t$

  $b_{tsts} = b_{tst} \circ b_s - b_{ts} = b_t b_s b_t b_s - 2 b_t b_s$

  $b_{tstst} = b_{tsts} \bullet b_t - b_{tst} = b_t b_s b_t b_s b_t - 3 b_t b_s b_t + b_t$

- $b_{ststs} = b_{stst} \circ b_s - b_{sts} = b_s b_t b_s b_t b_s - 3 b_s b_t b_s + b_s$

- $b_{ststst} = b_{ststs} \bullet b_t - b_{stst} = b_s b_t b_s b_t b_s b_t - 4 b_s b_t b_s b_t + 3 b_s b_t$

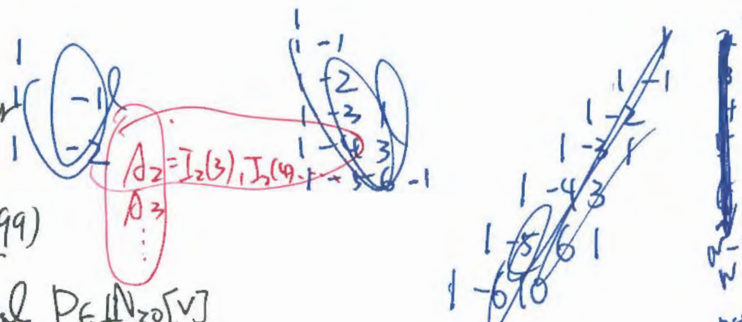
  $b_{tststs} = \qquad = b_t b_s b_t b_s b_t b_s - 4 b_t b_s b_t b_s + 3 b_t b_s$

- $b_{stststs} = b_{ststst} \circ b_s - b_{ststs} = b_s b_t b_s b_t b_s b_t b_s - 5 b_s b_t b_s b_t b_s + 6 b_s b_t b_s - b_s$

On the other hand,

Thm (Polo 1999)

Any polynomial $P \in \mathbb{N}_{\geq 0}[v]$ w/ const. $= 1$. appears in KL-polynomials for some S_n.



1	q	q^2	q^3	q^4	...
1					
1	-1				
1	-2				
1	-3	1			
1	-4	3			
1	-5	6	1		
1	-6	10	-2		
1	-7	15	-8	-1	
1	-8	21	-18	1	
1	-9	28	-33	9	1
1	-10	36	-54	27	

$P_{N+1} = P_N - q P_{N-1}$

The Standard form on H.

Let $H \xrightarrow{\varepsilon} \mathbb{Z}[v,v^{-1}]$ trace. $\delta_{id} \mapsto 1$
$\delta_x \mapsto 0$, $x \neq id$.

Let $H \xrightarrow{\bar{\ }} H$, $\delta_x \mapsto \delta_{x^{-1}}^{-1}$, $\overline{ab} = \bar{a}\bar{b}$

$H \xrightarrow{\omega} H$, $\delta_x \mapsto \delta_{x}^{-1}$, $\omega(ab) = \omega(b)\omega(a)$ exactly

$\boxed{\text{Define}}$ $H \times H \xrightarrow{(-,-)} \mathbb{Z}[v,v^{-1}]$

$(a,b) \mapsto \varepsilon(\omega(a)\cdot b)$

(x,y) is like $\bar{x}\cdot y$.

$\boxed{\text{Prop}}$
$$\begin{cases} (ax,y) = (x, \omega(a)y), & a \in H \\ (v^{\pm 1}x, y) = v^{\mp 1}(x,y), & \\ (x, v^{\pm 1}y) = v^{\pm 1}(x,y) & \end{cases}$$
$(vx,y) = \bar{v}(x,y)$
$(x,vy) = v(x,y)$

$\boxed{\text{Prop}}$ (bar-~~orthogonality~~ orthonormality)

$\{\overline{\delta_x} \mid x \in W\}$ & $\{\delta_x \mid x \in W\}$ are dual.

$\because$ $(\overline{\delta_x}, \delta_y) = \begin{cases} 1, & x = y \\ 0, & \text{otherwise}. \end{cases}$

$\boxed{\text{Cor}}$ $\because$ $\omega(b_s) = b_s$ $\quad$ b_s's are "self-adj".

$\therefore$ $(b_s x, y) = (x, b_s y)$

$\boxed{\text{Thm}}$ (asymptotic orthonormality) For KL-basis,

$$(b_x, b_y) \in \begin{cases} 1 + v\mathbb{Z}[v], & x = y \\ v\mathbb{Z}[v], & \text{otherwise}. \end{cases}$$

(ε & $(-,-)$ are natural for $\mathbb{Z}[W]$)

KL-basis is the correct choice of basis for v-deformation of $\mathbb{Z}[W]$

Deodhar's formula.

$W = S_3 = A_2$, $S = \{s,t\}$. longest elemt $x = sts$

$\underline{e}$	111	110	101	100	011	010	001	000
sub expr $x^{\underline{e}}$	sts	st	id	s	ts	t	s	id
defect(e)	0	1	1	0	1	2	2	3

$\leftarrow$ it could repeat.

$\Rightarrow b_s b_t b_s \neq 0$ $\left(b_{s_1} \cdots b_{s_m} = \sum_{e \subset x} v^{\text{defect}(e)} \delta_{x^e} \right)$ $\quad$ $\text{defect}(e) \overset{\Delta}{=} \#U(0) - \#D(0)$

$\delta_{sts} + v\delta_{st} + v + \delta_s + v\cdot\delta_{ts} + v^2\delta_t + v^2\delta_s + v^3$