Let (W,S) be a Coxeter system. We have a real vector space V with basis given by symbols $\{\alpha_s \mid s\in S\}$ equipped with the symmetric bilinear form $(\alpha_s,\alpha_t)=\cos(\pi-\frac{\pi}{m_{st}})$ where $m_{st}>0$ is the minimal integer with $(st)^{m_{st}}=id$. On V for each $s\in S$ we have reflections given by $s.v=v-2(\alpha_s,v)\alpha_s$. This gives an action of W on V. To prove this, it suffices to check that the action of $(st)^{m_{st}}$ is trivial which reduces to the case of the dihedral group $I_2(m_{st})$.

Let $R=\mathbb{R}[V]$, also known as the ring of polynomials on V^*. Consider the grading on R for which $\mathbb{R}$ lives in degree 0 and any element in V has degree 2, so that R is a graded algebra $R=\bigoplus_{d\in\mathbb{Z}_{\geq 0}} R_d$. A graded R-module is an R-module M with grading $M=\bigoplus_{d\in\mathbb{Z}} M_d$ such that $x.m\in M_{d_1+d_2}$ whenever $x\in R_{d_1}$, $m\in M_{d_2}$. A (homo)morphism between graded R-module M,M' is a R-module morphism $\phi: M\to M'$ such that $\phi(M_d)\subset M'_d$. For any graded module M, we denote by $M(k)$ the shift such that $M(k)_d:=M_{k+d}$; whatever was in degree 0 in M will be shifted to degree $-k$ in $M(k)$.

We will also look at (graded) R-bimodules; at an abstract level it is nothing more than an (graded) $R\otimes R$-module. But more importantly, if M,M' are (graded) R-bimodules, we can form the tensor product $M\otimes_R M'$ which is again a (graded) R-bimodule.

Our main playground is the abelian category of graded R-bimodules which are finitely generated both as left and right R-modules. We denote this category by R-gbim. One can verify that this is a monoidal category, i.e. tensor product preserves the finiteness condition. We will frequently denote the tensor product by $\cdot$ or even omit the dot. So $M_1\otimes_R M_2$ in R-gbim will be abbreviated as $M_1\cdot M_2$ and eventually M_1M_2.

The actual toy today is $\mathbb{S}$Bim, a full additive subcategory (not abelian!) of R-gbim. To define it we need some invariant theory on R. We have seen that W acts on V and thus $R=\mathbb{R}[V]$ and the action obviously preserves grading. For any subset $I\subset S$ we denote by $W_I\subset W$ the sub-Coxeter group generated by I. We say I is **finitary** if $|W_I|<\infty$. We have a classical theorem

Theorem 1. *(Chevalley–Shephard–Todd theorem) Suppose I is finitary. Then the fixed subalgebra R^{W_I} is such that R is a finite free module over R^{W_I} with rank $|W_I|$.*

We will also denote by R^s the fixed subalgebra for $s\in S$.

Example 2. Suppose $(W,S)=I_2(m)$, $2\leq m<\infty$. Let $\{x,y\}$ be an orthonormal basis of the Euclidean plane on which the dihedral group $I_2(m)$ acts; say one of the reflection is about the x-axis. Then $R\cong\mathbb{R}[x,y]$, and

$$R^W\cong\mathbb{R}[x^2+y^2,\prod_{i=0}^{m-1}(\cos(\frac{2\pi i}{m})x+\sin(\frac{2\pi i}{m})y)].$$

Definition 3. *For any $s\in S$, the associated Bott-Samelson bimodule is the graded R-bimodule $B_s:=R\otimes_{R^s}R(1)$. In general, for any expression $\underline{w}=s_1s_2...s_k$ (not necessarily reduced) of (W,S), we put*

$$BS(\underline{w})=B_{s_1}\otimes_R B_{s_2}\otimes_R ...\otimes_R B_{s_k}(k).$$

As said it will be abbreviated as $B_{s_1}B_{s_2}...B_{s_k}(k)$ in the future.

Let us analyze B_s as a graded left R-module, we have $R=\mathbb{R}[V]=\mathbb{R}[\langle\alpha_s\rangle\oplus\langle\alpha_s\rangle^\perp]$. Since s acts on $\langle\alpha_s\rangle\oplus\langle\alpha_s\rangle^\perp$ by sending α_s to $-\alpha_s$ and preserving the rest, we have $R^s=$

$\mathbb{R}[\alpha_s^2] \otimes_{\mathbb{R}} \mathbb{R}[\langle \alpha_s \rangle^{\perp}]$. In particular as a graded R^s-module R is generated by 1 and α_s and that gives $R \cong R^s \oplus R^s(-2)$ as a graded R^s-module. Consequently

$$\begin{aligned} & BS(ss) := B_s B_s = R \otimes_{R^s} R \otimes_R R \otimes_{R^s} R(2) = R \otimes_{R^s} R \otimes_{R^s} R(2) \\ = \quad & R \otimes_{R^s} (R^s \oplus R^s(-2)) \otimes_{R^s} R(2) = R \otimes_{R^s} R(2) \oplus R \otimes_{R^s} R = B_s(1) \oplus B_s(-1). \end{aligned}$$

Recall that from (W, S) we define the Hecke algebra $\mathcal{H}$ and Kazhdan-Lusztig basis. It has elements $b_s \in \mathcal{H}$ such that $b_s b_s = (v + v^{-1}) b_s$. Let us highlight the two identities:

$$B_s B_s = B_s(1) \oplus B_s(-1), \quad b_s b_s = (v + v^{-1}) b_s.$$

Recall that in general Kazhdan-Lusztig basis b_w is "part" of products of b_{s_i}. This motivates us to define

Definition 4. *The category of Soergel bimodule* $\mathbb{S}\mathrm{Bim}$ *is the full additive subcategory of* $R-\mathrm{gbim}$ *consists of direct sums of graded shifts of direct summands of* $BS(\underline{w})$*.*

Example 5. The category $\mathbb{S}\mathrm{Bim}$ is not abelian. For example, when $S = \{s\}$, we have $R \cong \mathbb{R}[x]$, $R^W = R^s \cong \mathbb{R}[x^2]$. Both $B_s(-1) = R \otimes_{R^s} R \cong \mathbb{R}[x, y]/(x^2 - y^2)$ and $B_e = R = \mathbb{R}[x, y]/(x - y)$ are indecomposable R-bimodules, and thus $\mathbb{S}\mathrm{Bim}$ is the category of direct sums of shifts of these two. It is a bit tedious to verify that in $\mathbb{S}\mathrm{Bim}$ the natural morphism $B_s(-1) \to B_e$ (as from $R \otimes_{R^s} R \to R$) has no kernel.

Lemma 6. *Tensor products of Soergel bimodules are Soergel bimodules. Moreover, any Soergel bimodule is finite free as either a graded left-R-module or a graded right-R-module.*

There is an obvious morphism $R \to \mathbb{R}$ by sending anything of higher degree to 0. For any R-bimodule M we can then consider $M \otimes_R \mathbb{R}$, which is a left R-module. Let R-Mod be the category of finitely generated graded (left) R-modules. We denote by $\mathbb{S}\mathrm{Mod}$ the strictly full subcategory of R-Mod consisting of direct sums of graded shifts of direct summands of $BS(\underline{w}) \otimes_R \mathbb{R}$; objects in $\mathbb{S}\mathrm{Mod}$ are called Soergel modules and in particular $BS(\underline{w}) \otimes_R \mathbb{R}$ the Bott-Samelson modules. By constructing we have a functor $\otimes_R \mathbb{R} : \mathbb{S}\mathrm{Bim} \to \mathbb{S}\mathrm{Mod}$. Objects in $\mathbb{S}\mathrm{Mod}$ are usually easier for direct computation. We need the following commutative algebra fact:

Proposition 7. *The functor* $\otimes_R \mathbb{R} : \mathbb{S}\mathrm{Bim} \to \mathbb{S}\mathrm{Mod}$ *sends indecomposables to indecomposables. Moreover, it sends non-isomorphic indecomposables to non-isomorphic indecomposables.*

We have seen that objects in $\mathbb{S}\mathrm{Bim}$ are $R \otimes_{R^W} R$-modules, and consequently objects in $\mathbb{S}\mathrm{Mod}$ are $R \otimes_{R^W} \mathbb{R}$-modules. What is $R \otimes_{R^W} \mathbb{R}$? By definition $\mathbb{R} = R/R_+$ where R_+ is the ideal of strictly positive degree elements. So $R \otimes_{R^W} \mathbb{R} = R/I_W$ where $I_W := (R_+^W)$ is the ideal generated by $R_+^W := R^W \cap R_+$. The quotient $C := R/I_W$ is called the coinvariant algebra of W and plays some important role.

Now consider the 2-generators case $W = I_2(m) = \langle s, t \mid s^2 = t^2 = (st)^m = e \rangle$. As in Example 2, one has $R \cong \mathbb{R}[x, y]$ and $I_W = (x^2 + y^2, Z)$ where $Z = \prod_{i=0}^{m-1} (\cos(\frac{2\pi i}{m}) x + \sin(\frac{2\pi i}{m}) y)$. The quotient R/I_W is an $\mathbb{R}$-algebra of graded dimension one at degree 0, two at degree $2, 4, ..., 2m - 2$, one at degree $2m$ and zero elsewhere. In particular it has a unique element in degree $2m$. This element can be represented by

$$\mathbb{L} := \prod_{i=0}^{m-1} (\cos(\frac{\pi i}{m}) x + \sin(\frac{\pi i}{m}) y).$$

It satisfies $s.\mathbb{L} = t.\mathbb{L} = -\mathbb{L}$ in R. Consider the so-called **BGG-Demazure operator** on R defined by

$$\partial_s x := \frac{x - s.x}{\alpha_s}, \ \ \forall s \in S$$

This operator induces an operator on $C = R/I_W$ which we still denote by ∂_s. In fact, because $\partial_s(xy) = \partial_s(x)y$ for any $x \in R$, $y \in I_W$, the action of ∂_s on R can be determined by the action on C. It is easy to see that $\partial_s^2 = 0$. More importantly it's easy to verify in C the braid relation

$$\partial_s\partial_t\partial_s... = \partial_t\partial_s\partial_t...$$

where both sides are compositions of m operators. One can then derive from C the same relation in R. This shows that, for any **reduced** expression $w = s_1...s_k \in W$, we can define $\partial_w := \partial_{s_1}...\partial_{s_k}$ on R. Thanks to Matsumoto's theorem, this in fact shows we have well-defined ∂_w on R for arbitrary Coxeter group.

Back to the $W = I_2(m)$ case, one can verify that $\{\partial_w\mathbb{L}\}_{w\in W}$ is a basis for C. Moreover, let $w_0 = sts... = tst...$ (m elements for both expressions) be the longest element in W. One shows that $(a, b) \mapsto \partial_{w_0}(ab)$ is a perfect pairing so that the dual basis of $\{\partial_w\mathbb{L}\}_{w\in W}$ under the pairing is still itself (though with some permutation). Using these, it is possible to show (note W is just our finite dihedral group):

Proposition 8. *Every object in category* $\mathbb{S}\mathrm{Mod}$ *is isomorphic to a direct sum of graded shifts of* $\langle\partial_w\mathbb{L}\rangle \subset C$*.*

The category $\mathbb{S}\mathrm{Bim}$ *consists of direct sums of graded shifts of indecomposables* B_w *indexed by* $w \in W$ *for which* $B_w \otimes_R \mathbb{R} \cong \langle\partial_w\mathbb{L}\rangle$*. They satisfy*

$$\begin{cases} B_sB_t = B_{st} & \\ B_sB_w = B_{sw} \oplus B_{tw} & \text{, if } \ell(sw) > \ell(w) > \ell(tw) > 0 \\ B_sB_w = B_w(1) \oplus B_w(-1) & \text{, if } \ell(sw) < \ell(w) \end{cases}$$

At the same time, the Kazhdan-Lusztig basis $\{b_w\}$ for a dihedral group satisfies

$$\begin{cases} b_sb_t = b_{st} & \\ b_sb_w = b_{sw} + b_{tw} & \text{, if } \ell(sw) > \ell(w) > \ell(tw) > 0 \\ b_sb_w = (v + v^{-1})b_w & \text{, if } \ell(sw) < \ell(w) \end{cases}$$

Let $[\mathbb{S}\mathrm{Bim}]_\oplus$ be the $\mathbb{Z}[v, v^{-1}]$-algebra with generators $[B]$ for Soergel bimodules B, multiplication rule $[B][B'] := [BB']$ and $[B(k)] = v^k[B]$, and relations $[B \oplus B'] = [B] + [B']$. Since in general $\mathcal{H}$ is defined by the quadratic and braid relation that involves at most two reflections, the above comparison shows that (need to extend the above example if some $m_{st} = \infty$)

Proposition 9. *For arbitrary Coxeter group there exists a* $\mathbb{Z}[v, v^{-1}]$*-algebra morphism* $c : \mathcal{H} \to [\mathbb{S}\mathrm{Bim}]_\oplus$ *satisfying* $c(b_w) = [B_w]$ *for any* $w \in W_{s_1,s_2}$ *for any* $s_1, s_2 \in S$*.*

In general, it is possible to prove that for any $w \in W$ there is an indecomposable Soergel bimodule B_w with the following property: for any reduced expression $\underline{w}$ of w it is the unique indecomposable that appears in $BS(\underline{w})$ but not (nor its graded shifts) in $BS(\underline{w}')$ for $w' < w$ in the Bruhat order. And any indecomposable Soergel bimodule is isomorphic to some B_w. The big shots of the theory are

Theorem 10. *(Soergel categorification theorem)* $\mathbb{S}\mathrm{Bim}$ *categorizes* $\mathcal{H}$*. That is,* c *is an isomorphism.*

Theorem 11. *(Soergel's Conjecture; theorem of Elias-Williamson) We have* $c(b_w) = [B_w]$ *for any* b_w *in the Kazhdan-Lusztig basis.*

1. Geometric motivation

Here we explain the origin of Bott-Samelson bimodules. Suppose G is a complex semisimple Lie group with (W, S) its Weyl group; it has a Borel subgroup B and maximal torus T such that $N_G(T) \cong W$ has generators S produced by simple roots with respect to B; that is, each simple root a_s corresponds to a parabolic $B \subset P_s \subset G$ such that $P_s/B \cong \mathbb{P}^1$ and $N_{P_s}(B) \subset N_G(B)$ is a group of order 2 which is then identified with $\langle s \rangle$.

In this case as W is finite, Chevalley–Shephard–Todd theorem applies and R is finite free over R^W. For all Bott-Samelson modules the left action of R^W is always the same with the right action of R^W. That is, our R-bimodules are the same as $R \otimes_{R^W} R$-modules. One has the equivariant cohomology (with $\mathbb{R}$-coefficients)

$$H^*_T(pt) = \mathbb{R}[\mathfrak{h}^*] = R$$

where $\mathfrak{h}^*$ is the dual of the Cartan algebra of the split real form, and

$$H^*_T(G/B) = R \otimes_{R^W} R.$$

Now for any $\underline{w} = s_1 ... s_k$ the **Bott-Samelson variety** is the quotient $\mathcal{BS}(\underline{w}) := P_{s_1} \times ... \times P_{s_k}/B^k$ where the B^k acts by $(b_1, ..., b_k).(p_1, ..., p_k) = (p_1 b_1^{-1}, b_1 p_2 b_2^{-1}, ..., b_{k-1} p_k b_k^{-1})$. This is a compact complex manifold of dimension k. The highlight is then

$$H^*_T(\mathcal{BS}(\underline{w})) \cong R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} ... \otimes_{R^{s_k}} R = BS(\underline{w})(-k)$$

as a graded module over $H^*_T(G/B)$. The shift (k) is natural because in Poincaré-Verdier duality of a compact complex manifold of dimension k it is desirable to shift the degree by k so that the full cohomology group becomes self-dual.

The morphism $H^*_T(G/B) \to H^*(G/B)$ dropping the equivariant structure is exactly $R \otimes_{R^W} R \to R \otimes_{R^W} \mathbb{R} = R/(R^W_+)$; that is, the passage from Bott-Samelson (resp. Soergel) bimodules to Bott-Samelson (resp. Soergel) modules is given by dropping the equivariant structure.

By the standard machinery of sheaves, $H^*_T(\mathcal{BS}(\underline{w}))$ is also (shift of) the equivariant cohomology of the push-forward to G/B of the constant sheaf on $\mathcal{BS}(\underline{w})$. The machinery of perverse sheaves allows to further decompose these (complexes of) sheaves on G/B into the so-called simple perverse (equivariant) sheaves. The indecomposable Soergel bimodules inside $BS(\underline{w})$ are then the equivariant hypercohomology of these simple perverse sheaves. Historically these simple perverse sheaves were the only main tool to study Kazhdan-Lusztig basis until the purely algebraic theory of Soergel bimodules came in place.

2. Proof of Proposition 8

Recall that we are in the setting $(W, S) = I_2(m)$. We have to prove inductively that as $C := R/I_W$-modules where $I_W := (R^W_+)$, we have

$$(2.1) \qquad \begin{cases} R \otimes_{R^s} \langle \partial_t \mathbb{L} \rangle \cong \langle \partial_{st} \mathbb{L} \rangle & \\ R \otimes_{R^s} \langle \partial_w \mathbb{L} \rangle \cong \langle \partial_{sw} \mathbb{L} \rangle \oplus \langle \partial_{tw} \mathbb{L} \rangle & , \text{if } \ell(sw) > \ell(w) > \ell(tw) > 0 \\ Rx \otimes_{R^s} \langle \partial_w \mathbb{L} \rangle \cong \langle \partial_w \mathbb{L} \rangle(1) \oplus \langle \partial_w \mathbb{L} \rangle(-1) & , \text{if } \ell(sw) < \ell(w) \end{cases}$$

We need a bit preparation about C and the basis.

Lemma 12. *We have*

(1) The image of R^s in C is C^s the invariant subalgebra and we have $C \cong C^s \oplus \alpha_s C^s$ as graded C^s-modules.
(2) Suppose $x_1, x_2 \in R^s$ are such that $x_1 + \alpha_s x_2 \in I_W$. Then $x_1, x_2 \in I_W$.
(3) Denote by $s^{(k)}$ the element sts... with length k and $t^{(k)}$ likewise. Then for any $1 \leq k \leq m$ and $x \in R^s$ we have $x\partial_{s^{(k)}}\mathbb{L} \in I_W \Leftrightarrow x\partial_{t^{(k-1)}}\mathbb{L} \in I_W \Leftrightarrow x\alpha_s\partial_{s^{(k)}}\mathbb{L}$.

The C-module $R \otimes_{R^s} \langle \partial_w \mathbb{L} \rangle$ can be generated by $1 \otimes \partial_w \mathbb{L}$ and $1 \otimes \alpha_s \partial_w \mathbb{L}$. When $w = e$ or $w = t$ we have $\alpha_s \partial_w \mathbb{L} \in I_W$ in which case we only need the former generator; this explains the first case of (2.1). Otherwise $\alpha_s \partial_w \mathbb{L} \notin I_W$ and the two generators are minimal. When $w = s^{(k)}$ with $1 \leq k \leq m$, we have that $\partial_w \mathbb{L}$ and $\alpha_s \partial_w \mathbb{L}$ have the same annihilator A' in C^s such that the annihilator of $\partial_w \mathbb{L}$ in C is $A' \oplus \alpha_s A'$. This proves the third case of (2.1).

When $w = t^{(k)}$, $2 \leq k \leq m$, the C-module $R \otimes_{R^s} \langle \partial_w \mathbb{L} \rangle$ can also be generated by $1 \otimes \partial_w \mathbb{L}$ and $1 \otimes \partial_{tw} \mathbb{L}$. The annihilator of $\partial_w \mathbb{L}$ in C^s is equal to that of $\partial_{sw} \mathbb{L}$. This explain the second case of (2.1).