Chp 5 The classical theory of Soergel bimodules

Last time:

$MN = M \otimes_R N$ for short

Defined Bott-Samelson bimod $BS(\underline{w}) = B_{s_1} \cdots B_{s_\ell} = R \otimes_{R^{s_1}} \cdots \otimes_{R^{s_\ell}} R(\ell(\underline{w}))$

where $\underline{w} = s_1 \cdots s_\ell$ is an expr.,

$SBim = \{ M \in R\text{-gbim} \mid M \mid \bigoplus_{i=1}^{n} BS(\underline{w}_i)(m_i) \}$

In particular, $BS(\underline{e}) = R$ $\quad$ are indec. Soergel bimod which we call B_e

$BS(\underline{s}) =: R \otimes_{R^s} R(1)$ $\quad$ B_s

Goal 1. Classify indecomposable Soergel bimod

2. State Soergel's cat'n theorem and its appl'ns.

to be elaborated

Recall $R = Sym(V)$. For now we assume V satisfies some technical cond ($\star$)

Defn For each $x \in W$, we define a standard module $R_x \in R\text{-gbim}$ s.t.

$R_x = R$ with the same left action

a twisted right action $m \cdot_{new} r = m \cdot_{old} (xr)$

$\Rightarrow R_x R_y = R_{xy}$

Let StdBim = smallest strictly full subcat'y of R-gbim containing R_x, $x \in W$

full + closed under iso

Fact (1) StdBim is monoidal, closed under taking $\oplus$ summands.

(2) $[\text{StdBim}]_\oplus \cong \mathbb{Z}[v^{\pm 1}][W]$

indec $[R_x] \mapsto x$

$(\pm 1) \mapsto v^{\pm 1}$

(3) $\exists$ SES's in R-gbim:

$$0 \to R_s(-1) \to B_s \to R(1) \to 0 \qquad (\Delta)$$

$$0 \to R(-1) \to B_s \to R_s(1) \to 0 \qquad (\nabla)$$

$(\Delta) \Leftrightarrow$ fil'n $0 \subset R_s(-1) \subset B_s$ w/ subquotients $R_s(-1)$, $R_e(1)$

$(\nabla) \Leftrightarrow$ — $\subset R_e(-1) \subset B_s$ ——— $R_e(-1)$, $R_s(1)$

Example (fil'n for $BS(\underline{ss}) = B_s B_s$) $\triangle$ modules are free $\Rightarrow$ tensoring is exact

$(\Delta) B_s: \quad 0 \to R_s B_s(-1) \to B_s B_s \to R B_s(1) \to 0$

$\cong B_s(1)$

$R_s(-1)(\Delta): \quad 0 \to R_s R_s(-2) \to R_s B_s(-1) \to R_s R \to 0$

$\cong R_e(-2) \qquad \cong R_s$

$R(1)(\Delta): \quad 0 \to R R_s \to R B_s(1) \to R R(2) \to 0$

$\cong R_s \qquad \cong R_e(2)$

$\Rightarrow$ fil'n on $BS(\underline{ss})$ w/ subquot. R_e, $R_e(2)$, R_s, R_s.

$\Rightarrow \cdots \Rightarrow$ fil'n on $BS(\underline{w})$ w/ subquot. $\cong \otimes$ of $R(1)$'s and $R_{s_i}(-1)$'s

StdBim (up to shift)

$\triangle$ $\exists$ fil'n on $BS(\underline{w})$ using (∇) or a mixture of $(\Delta), (\nabla)$.

Now we define two fil'ns for $B \in SBim$

Defn Fix $W = \{x_0, x_1, \dots\}$ s.t. $x_i \le x_j \Rightarrow i \le j$

A Δ-filtration of B is $0 = B^k \subset B^{k-1} \subset \cdots \subset B^0 = B$

s.t. $B^i / B^{i+1} \cong R_{x_i}^{\oplus h_{x_i}}$

where $h_{x_i} \in \mathbb{Z}_{\ge 0}[v^{\pm 1}]$ tracks the shift e.g. $R_s^{\oplus v + v^{-1}} = R_s(1) \oplus R_s(-1)$

Fact (4) [Soergel]

Δ-fil'n is unique. Moreover, h depends only on x and B, not on the choice of enumeration.

$\Rightarrow$ Δ-character $ch_\Delta(B) = \sum_{x \in W} v^{\ell(x)} h_x(B) \delta_x \in \mathcal{H}(W)$

is well-defined

(5) $\exists$ ∇-variant for (4): for $W = \{x_0, x_1, \dots\}$ as above,

$\exists!$ ∇-fil'n of B i.e. $0 \subset B^1 \subset \cdots \subset B^k = B$ w/

$B^{i+1}/B^i \cong R_{x_i}^{\oplus h_{x_i}}$

⇒ The ∇-character $ch_\nabla(B) = \sum_{x\in W} v^{\ell(x)}\, \overline{h'_x(B)}\, \delta_x \in \mathcal{H}(W)$ is well-defined.

__Example__

(Δ) is the Δ-filt'n of B_s wrt $W = \{e = x_0,\ s = x_1, \dots\}$ by

$$\begin{aligned} B^0 &= B_s \\ B^1 &= R_s(-1) \\ B^2 &= 0 \end{aligned} \;\Rightarrow\; \begin{aligned} B^0/B^1 &\simeq R(1) = R_{x_0}^{\oplus v} \\ B^1/B^2 &\simeq R_s(-1) = R_{x_1}^{\oplus v^{-1}} \end{aligned}$$

$$\Rightarrow ch_\Delta(B_s) = v\,\delta_e + v\cdot v^{-1}\cdot \delta_s = \delta_s + v$$

(∇) is the ∇-filt'n of B_s wrt $W = \{e = x_0,\ s = x_1, \dots\}$ by

$$\begin{aligned} B^0 &= 0 \\ B^1 &= R(-1) \\ B^2 &= B_s \end{aligned} \;\Rightarrow\; \begin{aligned} B^1/B^0 &\simeq R(-1) = R_{x_0}^{\oplus v^{-1}} \\ B^2/B^0 &\simeq R_s(1) = R_{x_1}^{\oplus v} \end{aligned}$$

$$\Rightarrow ch_\nabla(B_s) = \overline{v}^{-1}\delta_e + v\,\overline{v}\,\delta_s = \delta_s + v = ch_\Delta(B_s)$$

__Fact__ (6) $ch_\Delta(B\oplus B') = ch_\Delta(B) + ch_\Delta(B')$ ⇒ $ch_\Delta : [SBim]_\oplus \to \mathcal{H}$ are $\mathbb{Z}$-linear

$ch_\nabla(B\oplus B') = ch_\nabla(B) + ch_\nabla(B')$, $ch_\nabla : [SBim]_\oplus \to \mathcal{H}$

However, $ch_\Delta(B(1)) = v\, ch_\Delta B$ → ch_Δ is $\mathbb{Z}[v, v^{-1}]$-linear

$ch_\nabla(B(1)) = v^{-1} ch_\nabla B$ ∴ while ch_∇ isn't

⇒ We prefer ch_Δ and set $ch := ch_\Delta$ from now on.

(7) $ch_\Delta(B) = ch_\nabla(B)$ if B indecomposable w/o "grading shift" does not hold in general. (See §18)

__Thm__ (Soergel's categorification thm)

Under (★), we have:

(1) $\mathrm{Indec}(SBim)/\simeq, (1) \xleftrightarrow{1:1} W$, where $B_x \mid BS(\underline{x})$.

$B_x \longmapsto x$

Moreover, $BS(\underline{x}) = \bigoplus_{y\le x} B_y^{\oplus h_y}$ with $h_x = 1$

(2) $ch : [SBim]_\oplus \to \mathcal{H}$ is a $\mathbb{Z}[v^{\pm 1}]$-alg isom with inverse $c : \mathcal{H} \to [SBim]_\oplus$ sending b_s to B_s.

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(1) ⇒ algorithm finding B_x (will be elaborated in later chapters)

(1) ⇒ $(B_x : R_x(-\ell(x))) = 1$

$(B_x : R_y^{\oplus h_y}) = 0$ unless $y < x$

↳ proof requires Deodhar formula in §3.

__Conj__ (Soergel) ← Now proved by [Elias-Williamson], see Part IV

$ch(B_x) = b_x \quad \forall\, x\in W.$

Moreover, $h_x(B_y) = h_{x,y}(v)$ is the KL polyn.

⇒ $h_{x,y}(v) \in \mathbb{Z}_{\ge 0}[v^{\pm 1}]$: KL positivity conj

(no known proof w/o using cat'n)

Recall the (sesquilinear) standard form $(-,-) : \mathcal{H}\times\mathcal{H} \to \mathbb{Z}[v^{\pm 1}]$ from §3

We had $(\overline{\delta_x}, \delta_y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otw} \end{cases}$

__Thm__ ⇒ Soergel Hom formula: for $B, B' \in SBim$,

$$(ch(B), ch(B')) = \text{graded rank}\left(\mathrm{Hom}^\bullet_{SBim}(B, B')\right).$$

⚠ sesquilinear = antilinear in 1st var

linear in 2nd var

The assumption (☆):

<u>Defn</u>

A W-mod V is <u>reflection faithful</u> if

- V is faithful, (o/w Δ-filt'n is not well-defined)
- x is a refl'n (i.e., $x = gsg^{-1}$ for $g \in W$, $s \in S$) $\leftrightarrow$ codim $V^x = 1$

<u>Fact</u> (7) V_{geom} on which $W \curvearrowright$ by $s(\lambda) = \lambda - (\lambda, \alpha_s)\alpha_s$ is always faithful

V_{geom} is NOT refl'n faithful when $W = I_2(m)$ or affine Weyl group

(8) Soergel constructed a Kac-Moody repn V_{KM} s.t.

V_{KM} is reflection faithful when $k = \mathbb{R}$.

(9) If $|W| = \infty$ then $\nexists$ faithful f.d. repn of W over $k = \mathbb{F}_p$ or $\overline{\mathbb{F}}_p$

(faithfulness is restrictive!)

(☆) $\Leftrightarrow \begin{cases} V \text{ is reflection faithful} \\ |k| = \infty \text{ and } \operatorname{char} k \neq 2. \end{cases}$

<u>Rmk</u> Soergel's "classical" approach doesn't work for finite fields

Later we will remove this restriction by using a diagrammatic category defined by generators/relations coming from a given realization of Coxeter sys.

<u>Defn</u> Let k = comm. integral domain. A <u>realization</u> of (W, S) over k is a triple $(\mathfrak{h}, \{\alpha_s\}, \{\alpha_s^\vee\})$ where

$\mathfrak{h}$ is a free k-mod of finite rank

$\{\alpha_s^\vee \in \mathfrak{h}\}_{s \in S}$ and $\{\alpha_s \in \mathfrak{h}^*\}_{s \in S}$ are subsets satisfying

1. $\alpha_s(\alpha_s^\vee) = 2 \quad \forall\ s \in S$
2. $S \times \mathfrak{h} \to \mathfrak{h}$, $(s, v) \mapsto v - \alpha_s(v)\alpha_s^\vee$ extends to a W-action
3. technical conditions to rule out unusual poss. say in char 2.
   $\approx$ quantum #

$\Rightarrow$ Geom. real'n. $V_{geom} = \mathfrak{h}^*$ w/ $s.\lambda = \lambda - \lambda(\alpha_s^\vee)\alpha_s \in \mathfrak{h}^*$

$\Rightarrow R = Sym(\mathfrak{h}^*) = k[\alpha_s : s \in S]$

From now on, we assume that $(\mathfrak{h}, \{\alpha_s\}, \{\alpha_s^\vee\})$ satisfies the <u>Demazure surjectivity</u>, i.e.,

$$\alpha_s : \mathfrak{h} \to k, \quad \alpha_s^\vee : \mathfrak{h}^* \to k \text{ are surjective } \forall\ s \in S$$

⚠ This is true if $2 \in k^\times$.

We need this assumption in §8 to assert that

$R^s \subset R$ is a Frobenius extension $\forall\ s \in S$.

Finally, we need an assumption on the <u>Cartan matrix</u>

$A = (a_{ij})_{i,j \in S}$ where $a_{ij} = \alpha_j(\alpha_i^\vee) \in k$

↖ no integrality conditions like in Lie theory

We require that A is <u>balanced</u>, i.e.,

$$a_{ij} = a_{ji} = -[2]_q \Rightarrow [m_{ij} - 1]_q = 1$$

where $[n]_q = q^{n-1} + q^{n-3} + \dots + q^{1-n}$ for $n \geq 1$, $q \in k^\times$

Appendix: Demazure operators

Defn For $s \in S$, the Demazure operator is the graded map

$$\partial_s : R \longrightarrow R^s(-2) \qquad \text{well-defined since } R = R^s \oplus R^s \alpha_s$$
$$f \longmapsto \frac{f - s(f)}{\alpha_s}$$

antiinv't divisible by α_s

e.g. $\partial_{s_1}(\alpha_2) = \frac{\alpha_2 - s_1\alpha_2}{\alpha_1} = \frac{\alpha_2 - (\alpha_1+\alpha_2)}{\alpha_1} = -1$

Fact (1) The iso $R \to R^s \oplus R^s(-2)$ as R^s-mod is given by

$$f \mapsto \left(\partial_s\left(f\tfrac{\alpha_s}{2}\right), \partial_s(f)\right) \quad \text{with inverse}$$
$$g + h\tfrac{\alpha_s}{2} \longleftarrow (g, h)$$

(2) Consider $B_s \simeq R(1) \oplus R(-1)$ as graded R-mod and elts

$$c_e := 1 \otimes 1, \quad c_s := \tfrac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \in B_s.$$

We have, $\forall f \in R$, $\begin{cases} f c_s = c_s \cdot f \\ f c_e = c_e \cdot s(f) + c_s \cdot \partial_s(f) \\ c_e f = s(f). c_e + c_s \cdot \partial_s(f) \end{cases}$

$\Rightarrow \{c_e, c_s\}$ forms a basis of B_s whether as a left or right R-mod

$\Rightarrow R \cdot c_s = \{x \in B_s \mid fx = xf \ \forall f \in R\} \subseteq B_s$: submod

$\quad \| s$

$\quad R(-1)$

Hence $\begin{array}{c} R(-1) \hookrightarrow B_s \\ 1 \longmapsto c_s \end{array}$ and it can be checked that

$$0 \to R(-1) \longrightarrow B_s \longrightarrow R_s(1) \longrightarrow 0 \quad \text{is a SES}$$
$$1 \mapsto c_s \qquad f \otimes g \mapsto f.s(g) \qquad \text{in } R\text{-gbim}$$

(3) Consider elt $d_s = \frac{1}{2}(\alpha_s \otimes 1 - 1 \otimes \alpha_s) \in B_s$

We have, $\forall f \in R$, $\begin{cases} f. d_s = d_s \cdot s(f) \\ f \cdot c_e = c_e \cdot f - d_s \cdot \partial_s(f) \\ c_e \cdot f = s(f). c_e - d_s \cdot \partial_s(f) \end{cases}$

$\Rightarrow \{c_e, d_s\}$ forms a basis of B_s as R-mod or mod-R

$\Rightarrow R_s(-1) \cong R. d_s \subseteq B_s$

$$\Rightarrow 0 \to R_s(-1) \longrightarrow B_s \longrightarrow R(1) \longrightarrow 0 \quad \text{is a SES}$$
$$1 \mapsto d_s \qquad f \otimes g \longmapsto fg \qquad \text{in } R\text{-gbim}$$

Rmk $\{\partial_s\}$ satisfy the Braid relations

$\Rightarrow \partial_w \ (w \in W)$ is well-defined

In fact, $\{\partial_w \mid w \in R\}$ forms a basis of so-called nil Coxeter alg in $\mathrm{End}_{\mathbb{R}}(R)$.