Convention

- Read from right to left.
- Read from bottom to top.

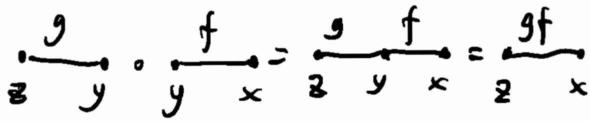
Compatibility conditions

1-category

obj (0-mor)

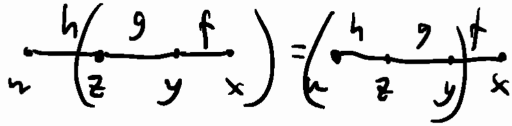
mor (1-mor)

horizontal composition



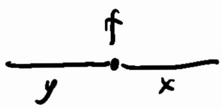
identity

associativity

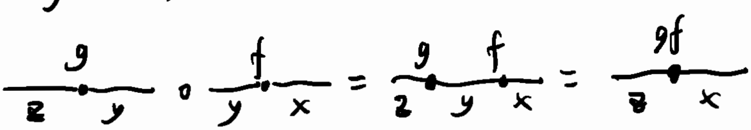


Dualizing the picture

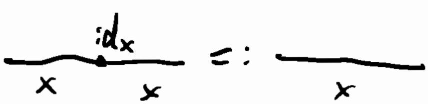
mor



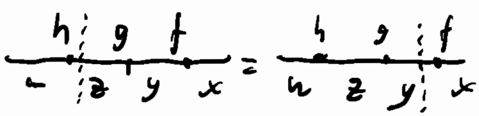
comp



identity



associativity

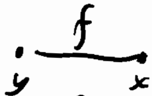


compatibility condition $\simeq$ all parsings of the dual diagram are the same.

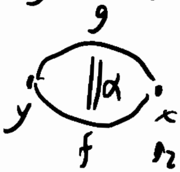
2-category

0-mor

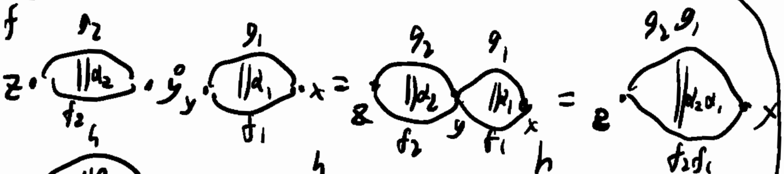
1-mor



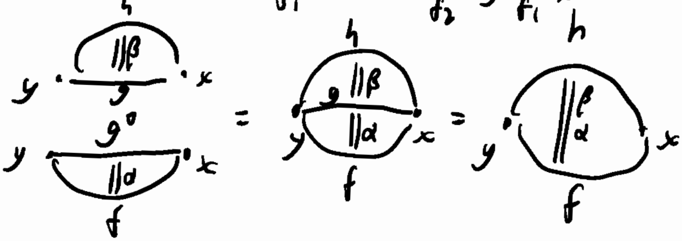
2-mor



horizontal composition



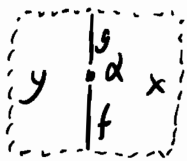
vertical composition



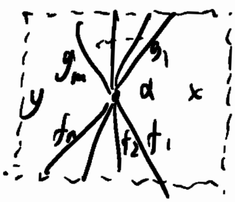
compatibility conditions

Dualizing the picture

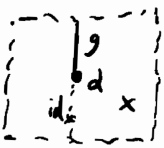
2-mor



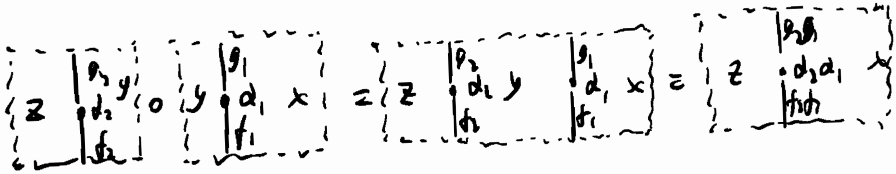
$\alpha : (f : x \to y) \to (g : x \to y)$

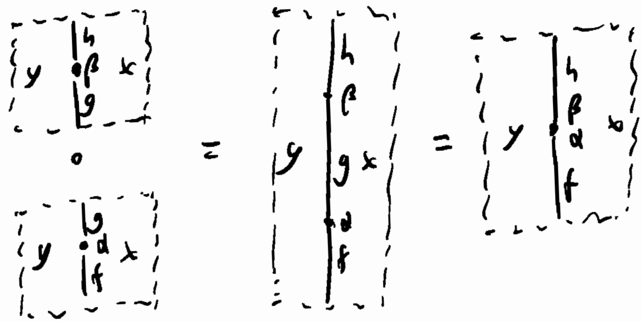


$\alpha : (f_n \cdots f_1 : x \to y) \to (g_m \cdots g_1 : x \to y)$



$\alpha : (id_x : x \to x) \to (g : x \to x)$

horizontal comp 

vertical comp 

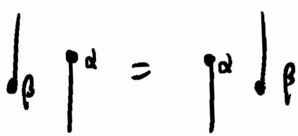
Compatibility conditions

- all parsings are the same

e.g. 

- rectilinear isotopic diagrams are the same.

$\sim$ no rotations

e.g. 

Examples

(1) 0-mor {cats}

1-mor {functors}

2-mor {nat. trans.}

(2) X: top. sp.

0-mor {pts}

1-mor {paths}

2-mor {homotopy}

(3) 0-mor {rings}

1-mor {R-S-bimodules}

$${}_R M_S \circ {}_S N_T := {}_R(M \otimes_S N)_T$$

not strict $(M \otimes N) \otimes L \cong M \otimes (N \otimes L)$

2-mor $\mathrm{Hom}_{R\text{-}S\text{-}\mathrm{bimod}}(M, M')$

(4) C: cat with pullbacks

0-mor {objs in C}

1-mor {spans $\swarrow\searrow$ in C}

2-mor {mor between spans}

(5) C: monoidal cat

0-mor {C}

1-mor {objs in C}

2-mor {mor in C}

Temperley-Lieb category

0-mor : $\{*\}$

1-mor : gen by  $=\{$  $\}$

2-mor : $\{$gen by  $\}$ / relations.

$\{$gen by  $\}$

A representation

$\bullet := V := \mathbb{C}e_1 \oplus \mathbb{C}e_2$ $= V^{\otimes 0} = \mathbb{C}$

$=$ standard rep of $sl_2 / U_q(sl_2)$  $= V^{\otimes 3}$

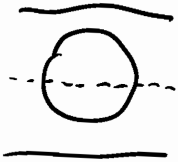
$\approx B_s$

$$V \to V \qquad e_i \mapsto e_i$$

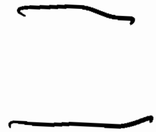
$$V \otimes V \to \mathbb{C} \qquad e_1e_1 \mapsto 0 \quad e_2e_2 \mapsto 0 \quad e_1e_2 \mapsto -1 \quad e_2e_1 \mapsto 1$$

$$\mathbb{C} \to V \otimes V \qquad 1 \mapsto e_1e_2 - e_2e_1$$

Ex. (1)

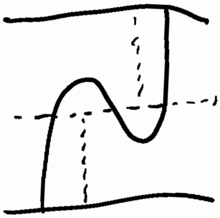


$$\begin{tikzcd} \mathbb{C} \\ V\otimes V \arrow[u] \\ \mathbb{C} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} -2 \\ e_1e_2-e_2e_1 \arrow[u, mapsto] \\ 1 \arrow[u, mapsto] \end{tikzcd}$$

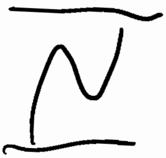
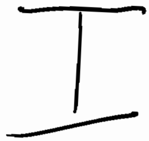
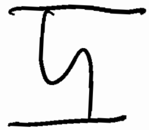
$= -2$ 

(first TL relation)

(2)



$$\begin{tikzcd} V \\ V\otimes V\otimes V \arrow[u] \\ V \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} e_1 \\ e_1(e_1e_2-e_2e_1) \arrow[u, mapsto] \\ e_1 \arrow[u, mapsto] \end{tikzcd} \qquad \begin{tikzcd} e_2 \\ e_2(e_1e_2-e_2e_1) \arrow[u, mapsto] \\ e_2 \arrow[u, mapsto] \end{tikzcd}$$

 =  = 

(second TL relation)

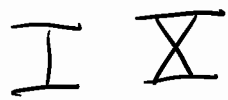
Isotopic diagrams are the same in TL.

$TL_q = \{\text{generators}\} / \{\bigcirc = -[2]_q,$  $= | =$  $\}$

Symmetric category

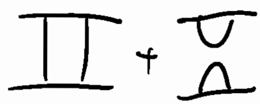
0-mor $\{*\}$

1-mor gen by 

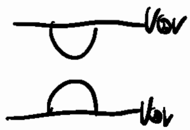
2-mor gen by  /rel

Fact

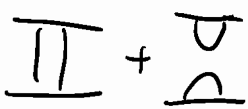
$$\mathrm{SymCat} \longrightarrow TL_1$$

 $\longmapsto$ 

Sketch of proof



$e_1e_1 \mapsto 0$ $\quad$ $e_2e_2 \mapsto 0$ $\quad$ $e_1e_2 \mapsto -e_1e_2+e_2e_1$ $\quad$ $e_2e_1 \mapsto e_1e_2-e_2e_1$

 $\quad e_ie_j \mapsto e_je_i$

Soergel bimodules

$$BS_{\underline{w}} = B_w \oplus \bigoplus_{x<w} B_x^{\oplus h_x}$$

$$V^{\otimes n} = V_n \oplus \bigoplus_{m<n} V_m^{\oplus h_m}$$

Direct summand

$$M = M_1 \oplus M_2$$

$$\begin{tikzcd}
M_1 \arrow[r, bend left, "i_1"] & M \arrow[l, bend left, "p_1"] \arrow[r, bend right, "p_2"'] & M_2 \arrow[l, bend right, "i_2"']
\end{tikzcd}$$

$e_1 := i_1 p_1 \qquad e_1^2 = e_1 \qquad e_1 + e_2 = id$

$e_2 := i_2 p_2 \qquad e_2^2 = e_2$

Karoubian Envelope

$\mathrm{Kar}\, C$: (X, e), $X \in \mathrm{Ob}\, C$

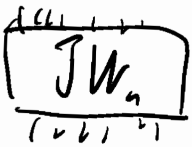
$e^2 = e \in \mathrm{Mor}(X, X)$

Idempotents in TL

Ex. (1) $e := -\frac{1}{[2]_q}$  $e^2 = e$

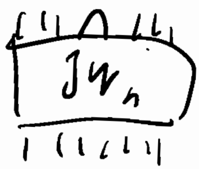
(2) $f := ||- \frac{1}{[2]_q}$  $f^2 = f$

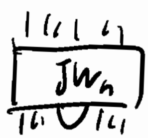
Jones-Wenzl operators



n

$JW_n \approx$ symmetrizer $\approx$ highest wt in $V^{\otimes n}$ is the unique element in TL s.t.

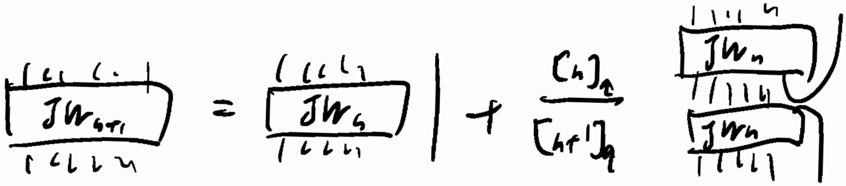
(1)  $= 0$

(2)  $= 0$

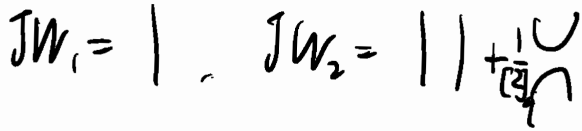
(3) $JW_n = ||||| +$ linear comb of others.

Moreover, $JW_n^2 = JW_n$.

Recursive formula

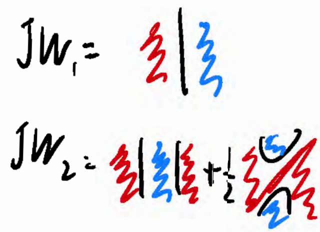


$\boxed{\text{Check that this is } JW_2}$



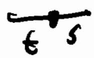
2-color TL

Ex



(W, S) Coxeter system $|S| = 2$

0-mor S

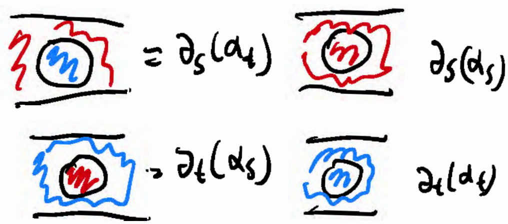
1-mor generated by 

2-mor gen. by 

"diagrammatic Hecke category"

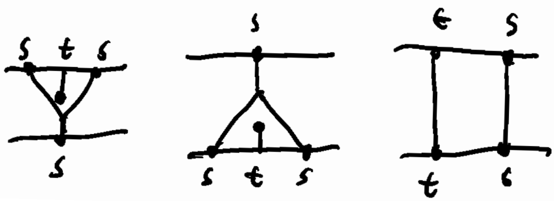
↓

shrink



0-mor $\{*\}$

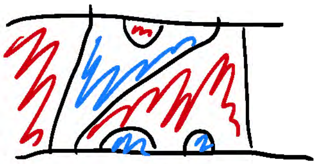
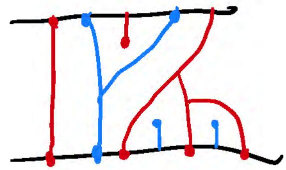
1-mor gen by 

2-mor gen by 

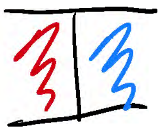
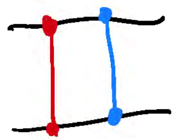
(other gen. rel)

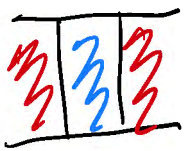
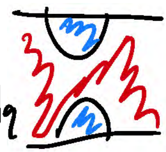
Ex.

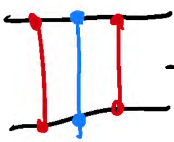
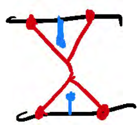
2TL $\longrightarrow$ Hecke

 $\mapsto$ 

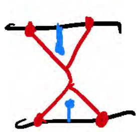
$$\begin{array}{c} B_s B_t \;\; B_s B_t B_s \\ \uparrow \\ B_s B_t B_s B_t B_s B_t B_s \end{array}$$

JW_1  $\mapsto$ 

JW_2  $- \frac{1}{[2]_q}$  $\mapsto$

 $- \frac{1}{[2]_q}$ 

$$\begin{array}{c} B_s B_t B_s \\ \uparrow \\ B_{sts} \\ \uparrow \\ B_s B_t B_s \end{array}$$



$$\begin{array}{c} B_s B_t B_s \\ \uparrow \\ B_s \\ \uparrow \\ B_s B_t B_s \end{array}$$