Frobenius extension

Nov. 20

$\overset{alg}{A} \subset \overset{alg}{B}$

- B is free A-mod
- dual basis $\{b_i\}, \{b_i^*\}$
- trace $\partial : B \to A$

$$b_i b_j^* \mapsto \begin{cases} 1 & i=j \\ 0 & \end{cases}$$

Ex. (1) $k \subset \mathrm{End}\, V$

$\partial :=$ trace

$\{b_i\} = \{e_{ij}\}$ $\{b_i^*\} = \{e_{ji}\}$

(2) $kH \subset kG$

$$\partial : gH \mapsto \begin{cases} 1 & gH = H \\ 0 & \end{cases}$$

$\{b_i\} = \{gH\}$, $\{b_i^*\} = \{Hg^{-1}\}$

(3) Hopf alg. (Larson-Sweedler)

(4) $k \subset$ Hecke alg. Chapter 3

$$\partial : \delta_w \mapsto \begin{cases} 1 & w=1 \\ 0 & \end{cases}$$

$\{b_i\} = \{\delta_w\}$ $\{b_i^*\} = \{\delta_{w^{-1}}\}$

(5) $R^s \subset R = R^s \oplus R^s \alpha_s$ Chapter 4

$\partial := \partial_s$

$\{b_i\} = \{1, \frac{\alpha_s}{s}\}$, $\{b_i^*\} = \{\frac{\alpha_s}{2}, 1\}$

Frobenius reciprocity

$\mathrm{Res}_A^B : B\text{-mod} \to A\text{-mod}$

$\mathrm{Ind}_A^B : A\text{-mod} \to B\text{-mod}$

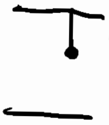
$M \mapsto B \otimes_A M$

$\mathrm{Coind}_A^B : A\text{-mod} \to B\text{-mod}$

$M \mapsto \mathrm{Hom}_A({}_AB_B, {}_AM)$

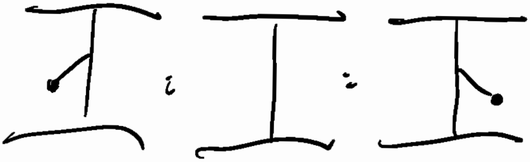
$\mathrm{Ind}_A^B \dashv \mathrm{Res}_A^B \dashv \mathrm{Coind}_A^B = \mathrm{Ind}_A^B$

Diagramatic Frobenius object



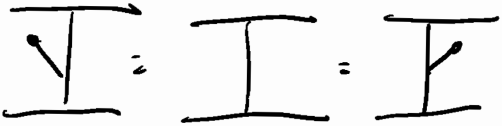
$$\begin{tikzcd} B \\ A \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} 1 \\ 1 \arrow[u, mapsto] \end{tikzcd}$$

Unit

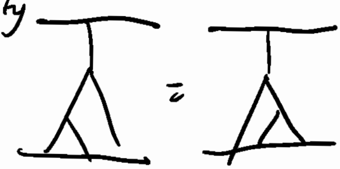


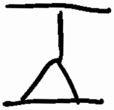
$$\begin{tikzcd} A \\ B \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \partial b \\ b \arrow[u, mapsto] \end{tikzcd}$$

Counit

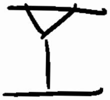


Associativity



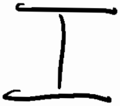


$$\begin{tikzcd} B \\ B \otimes_A B \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} 1 \\ 1 \otimes 1 \arrow[u, mapsto] \end{tikzcd}$$



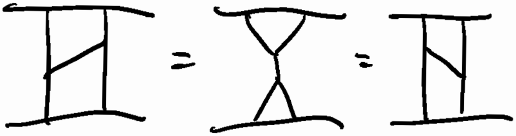
$$\begin{tikzcd} B \otimes_A B \\ B \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \sum b_i \otimes b_i^* \\ 1 \arrow[u, mapsto] \end{tikzcd}$$

Coassociativity



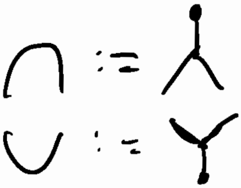
$$\begin{tikzcd} B \\ B \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} 1 \\ 1 \arrow[u, mapsto] \end{tikzcd}$$

Frobenius associativity

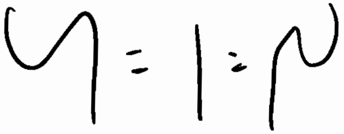


Derived axioms

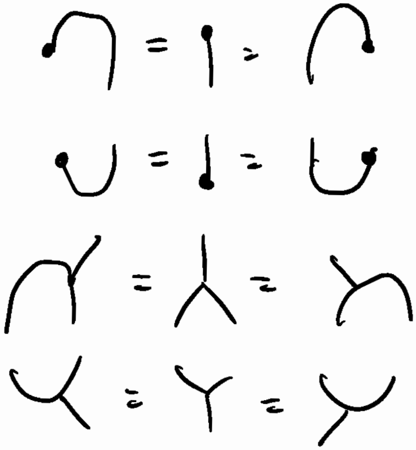
Cap Cup

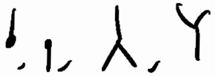


Isotopy



Rotation



Def A <u>Frobenius object</u> consists of  s.t. all axioms are satisfied.

Thm/Prop Isotopic diagrams are the same.

Retraction

Prop. $B\otimes_A B \supset B$ Frob. $\Longleftarrow$ $B \supset A$ Frob.

 $B \to B\otimes_A B$:=  $B \to B\otimes_A B$ $\quad 1 \mapsto \Sigma b_i \otimes b_i^*$

 $B\otimes_A B \to B$:=  $B\otimes_A B \to B$ $\quad f\otimes g \mapsto fg$

 $B\otimes_A B \to B\otimes_A B\otimes_A B$:=  $f\otimes g \mapsto f\otimes 1\otimes g$

 $B\otimes_A B\otimes_A B \to B\otimes_A B$:=  $f\otimes g\otimes h \mapsto \partial g(f\otimes h)$

Cor. $B_s \supset R \supset R^s$ Frob.

deg 1  $R \to B_s$ $\quad 1 \mapsto c_1$

deg 1  $B_s \to R$ $\quad c_0 \mapsto 1 \quad c_1 \mapsto \alpha_s$

deg -1  $B_s \to B_s B_s$

deg -1  $B_s B_s \to B_s$

Recall

$B_s = R \otimes_{B_s} R(1)$

B_s has R-basis $\quad c_0 = 1\otimes 1 \quad$ deg -1

$c_1 = \frac{1}{2}(\alpha_s\otimes 1 + 1\otimes\alpha_s) \quad$ deg 1

$$0 \to B_s(1) \to B_s B_s \to B_s(-1) \to 0$$

Chapter 12

1-color diagrammatic Hecke category $\mathcal{H}$

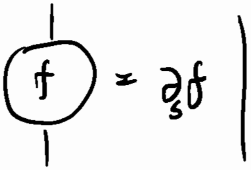
gen:

deg 0, 1, 1, -1, -1, deg f

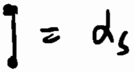
rel : Frob. rel.

Multiplication $f \ g = fg$

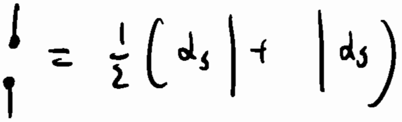
Keyhole



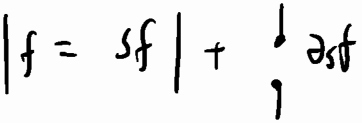
Barbell



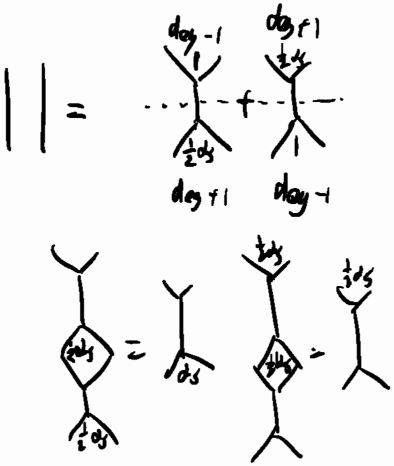
Fusion



Forcing



Ex. $B_sB_s = B_s(1) \oplus B_s(-1)$

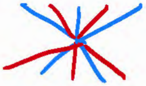


Thm $\mathcal{H}(A_1) \longrightarrow \mathrm{BSBim}$ is fully faithful

$s \longmapsto B_s$

2-color diagramatic Hecke category $\mathcal{H}$

gen : 1-color gen



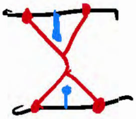
t s t s t

s t s t s

deg 0

Ex.

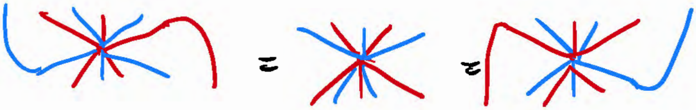
$$\begin{tikzcd}
B_t B_s B_t B_s B_t \\
B_{tstst} = B_{tstst} \arrow[u] \\
B_s B_t B_s B_t B_s \arrow[u]
\end{tikzcd}$$



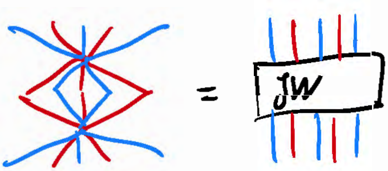
$$\begin{tikzcd}
B_s B_t B_s & \frac{1}{2}\partial_t(fg)(||\alpha_t|||| + |||t|(\alpha_t|)) \\
B_s B_s \arrow[u] & \partial_t(fg)|||| \arrow[u] \\
B_s \arrow[u] & \partial_s(fg)||| \arrow[u] \\
B_s B_s \arrow[u] & ||\partial_s g|| \arrow[u] \\
B_s B_t B_s \arrow[u] & ||f|g|| \arrow[u]
\end{tikzcd}$$

rel : 1-color rel

2-color rotation :

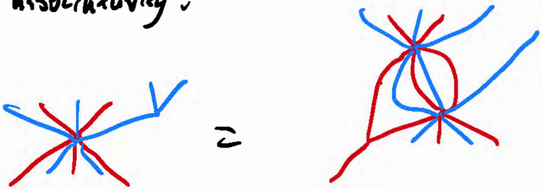


Elias-Jones-Wenzl :



$$\text{JW} : \begin{tikzcd}
B_s B_t B_s \\
B_{sts} \arrow[u] \\
B_s B_t B_s \arrow[u]
\end{tikzcd}$$

2-color associativity :



Thm $\mathcal{H}(I_2) \to BSBim$ is fully faithful.

3-color diagrammatic Hecke category

gen: 1-color gen, 2-color gen

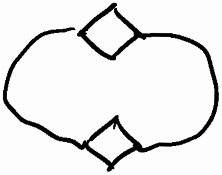
rel: 1-color rel, 2-color rel.

3-col rel:

one for each finite type rank 3

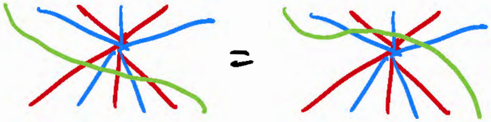
Chapter 1

Recall Rex graph for A_3



Want B_w unique up to unique iso.

$I_2 \times A_1$



Diagrammatic Hecke category

gen: 1-color gen, 2-color gen

rel: 1-color rel, 2-color rel, 3-color rel

A_1 $\quad I_2 \quad I_2 \times A_1, A_3, B_3, H_3$

Thm $\mathcal{H} \xrightarrow{\sim} BSBim$

$Kar\, \mathcal{H} \xrightarrow{\sim} SBim$