~~The category $\mathcal{H}_{BS}$.~~

Fix (W, S) and ~~Kac-Moody realization~~ ~~representation which~~ $W \curvearrowright V_{KM}$ that is reflection faithful; Soergel constructed one when $\# k = \infty$ and $\mathrm{char}\, k \neq 2$. $R = k[V_{KM}]$.

§ Monoidal Category $\mathcal{H}_{BS}$

Objects: expression $\underline{w} = s_1 s_2 \cdots s_k$, $k \in \mathbb{Z}_{\geq 0}$, $s_i \in S$

Monoidal struc: $\underline{w_1} \otimes \underline{w_2} = \underline{w_1}\,\underline{w_2}$

Morphisms: An R-linear combination of things like  $\in \mathrm{Hom}(ssst, ttstss)$

modulo a list of relations in §10.2.2 (or §10.2.4).

- Monoidal category $BSBim$: the monoidal category of Bott-Samelson bimodules.
- $\exists$ functor $\mathcal{F}: \mathcal{H}_{BS} \to BSBim$, an R-bilinear monoidal functor wi
  $\underline{w} \mapsto BS(\underline{w})$

$\mathcal{F}($  $) = (1 \mapsto f)$

$\mathcal{F}($  $) = (f \otimes g \mapsto fg$

$\mathcal{F}($  $) = (1 \mapsto \frac{1}{2}(1 \otimes \alpha_s + \alpha_s \otimes 1))$

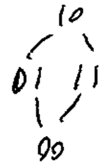
$\mathcal{F}($  $) = (1 \otimes g \otimes 1 \mapsto \partial_s g \otimes 1$

$\mathcal{F}($  $) = (f \otimes g \mapsto f \otimes 1 \otimes g)$

$\mathcal{F}($  $) = BS(sts\ldots) \to B_{sts\ldots} = B_{ts\ldots} \to BS(tst\ldots)$

$\underline{w} := ss.$

$$BS(ss)\otimes Q = Q_{\underline{w}^{00}} \oplus Q_{\underline{w}^{01}} \oplus Q_{\underline{w}^{10}} \oplus Q_{\underline{w}^{11}}$$



$$BS(s)\otimes Q = Q_e \oplus Q_s$$

$$Q_e = \langle \alpha_s|1 + 1|\alpha_s\rangle$$

$$Q_s = \langle \alpha_s|1 - 1|\alpha_s\rangle$$

$$Q_{\underline{w}^{00}} = \langle \alpha_s|\alpha_s|1 + 1|\alpha_s^2|1 + \alpha_s|1|\alpha_s + 1|\alpha_s|\alpha_s\rangle$$

$$Q_{\underline{w}^{01}} = \langle \quad '' \quad + \quad '' \quad - \quad '' \quad - \quad '' \quad \rangle$$

$$Q_{\underline{w}^{10}} = \langle \quad '' \quad - \quad '' \quad + \quad '' \quad - \quad '' \quad \rangle$$

$$Q_{\underline{w}^{11}} = \langle \quad '' \quad - \quad '' \quad - \quad '' \quad + \quad '' \quad \rangle$$

$LL_{\underline{w},01} =$  $:$

$Q_{\underline{w}^{00}} \xrightarrow{\sim} Q_e$

$Q_{\underline{w}^{01}} \xrightarrow{\sim} Q_s$

$Q_{\underline{w}^{10}} \longrightarrow 0$

$Q_{\underline{w}^{11}} \longrightarrow 0$

$(f\otimes g\otimes h \mapsto fg\otimes h)$

$LL_{\underline{w},11} =$  $:$

$Q_{\underline{w}^{00}} \xrightarrow{\sim} Q_e$

$Q_{\underline{w}^{01}} \longrightarrow 0$

$Q_{\underline{w}^{10}} \longrightarrow 0$

$Q_{\underline{w}^{11}} \xrightarrow{\sim} Q_e$

$(f\otimes g\otimes h \mapsto (\partial_s g)fh)$

Goal Convince ourselves that $\mathcal{F}$ is an equivalence and understand $\mathrm{Hom}_{\mathcal{H}_{BS}}(\underline{w},\underline{v})$.

In particular, we need to show that $\mathrm{Hom}_{\mathcal{H}_{BS}}(\underline{w},\underline{v}) \xrightarrow{\sim} \mathrm{Hom}_{BSBim}(BS(\underline{w}), BS(\underline{v}))$ and understand either.

Fact (Soergel Hom formula) $\mathrm{Hom}_{BSBim}(BS(\underline{w}), BS(\underline{v}))$ is finite free as either a left R-mod or a right R-mod.

Let Q = field of fractions of R.

It suffices to study $\mathrm{Hom}_{BSBim_Q}(BS(\underline{w})\otimes Q, BS(\underline{v})\otimes Q)$. But this is much easier: recall

$$0 \to R_s(-1) \to B_s \to R(1) \to 0$$
$$0 \to R(-1) \to B_s \to R_s(1) \to 0$$

where R_s is the sed module which is R as a right R-mod, but with $a\alpha = \alpha(s.a)$ for $\alpha \in R_s$, $a \in R$. Tensoring Q, the two sequences split each other, the grading is forgotten, and $B_s \otimes Q \cong Q \oplus Q_s$ where $Q_s := Q \otimes R_s = R_s \otimes Q$

We also write $Q_w := Q \otimes R_w$ $\forall w \in W$. They are bi-Q-modules.

Def^n A subexpression $\underline{e}$ of an expression $\underset{s_1 \cdots s_m}{\underline{w}}$ of (W,S), written $\underline{e} \subset \underline{w}$, is a string $\underline{e} = e_1 \cdots e_m$, $e_i \in \{0,1\}$. (We write $\underline{e} \subset \underline{w}$). Also $\underline{w}^{\underline{e}} := s_1^{e_1} \cdots s_m^{e_m} \in W$.

Lemma $BS(\underline{w}) \otimes Q \cong \bigoplus_{\underline{e} \subset \underline{w}} Q_{\underline{w}^{\underline{e}}}$

pf $BS(\underline{w}) \otimes Q = (B_{s_1} \otimes Q) \otimes_Q (B_{s_2} \otimes Q) \otimes \cdots \otimes_Q (B_{s_m} \otimes Q)$ $\square$

In the proof we see that for any $\underline{e} \subset \underline{w}$, $BS(\underline{w}) \otimes Q$ has a canonical component $\cong Q_{\underline{w}^{\underline{e}}}$, ~~By abuse of notation we'll~~ which we'll denote by $Q_{\underline{w}^{\underline{e}}}$ by abuse of notation,

Cor $\mathrm{Hom}_{BSBim_Q}(BS(\underline{w})\otimes Q,\ BS(\underline{v})\otimes Q)$

$$= \bigoplus_{\substack{\underline{e}\subset\underline{w}\\ \underline{f}\subset\underline{v}}} \mathrm{Hom}(Q_{\underline{w}^{\underline{e}}},\ Q_{\underline{v}^{\underline{f}}}) \cong \bigoplus_{\substack{\underline{e}\subset\underline{w}\\ \underline{f}\subset\underline{v}\\ \underline{w}^{\underline{e}}=\underline{v}^{\underline{f}}\in W}} Q.$$

Cor $\mathrm{Hom}_{BSBim}(BS(\underline{w}), BS(\underline{v}))$ has rank $= \#\{\underline{e}\subset\underline{w}, \underline{f}\subset\underline{v} \mid \underline{w}^{\underline{e}}=\underline{v}^{\underline{f}}\}$

if we knew its finite free

Goal 2 ~~$\underline{e}\subset\underline{w}, \underline{f}\subset\underline{v}$ with $\underline{w}^{\underline{e}}=\underline{v}^{\underline{f}}$~~, indexed by $\{\underline{e}\subset\underline{w}, \underline{f}\subset\underline{v} \mid \underline{w}^{\underline{e}}=\underline{v}^{\underline{f}}\}$

Construct $\mathbb{LL}_{\underline{w},\underline{v},\underline{e},\underline{f}} \in \mathrm{Hom}_{\mathcal{H}_{BS}}(\underline{w},\underline{v})$

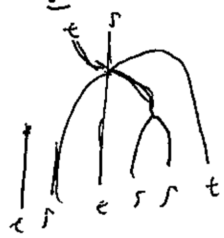
that serves as a basis (as both left R-mod and right R-mod)

~~Example $W=S_3$, $\underline{w}=tstsst$, $\underline{e}=011101$, $\underline{v}=tstsst$, $\underline{f}\in\{0,1\}^6$, $\underline{w}^{\underline{e}}=stst=ts$, $\underline{v}^{\underline{f}}=$~~

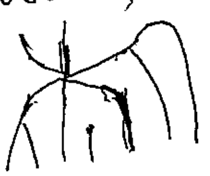
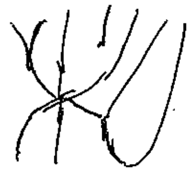
Structure: Fix $\underline{x}$ an reduced expression of $x:=\underline{w}^{\underline{e}}=\underline{v}^{\underline{f}}$. Then

$$\mathbb{LL}_{\underline{w},\underline{v},\underline{e},\underline{f}} = \underbrace{\overline{LL_{\underline{v},\underline{f}}}}_{\mathrm{Hom}(\underline{x},\underline{v})} \circ \underbrace{LL_{\underline{w},\underline{e}}}_{\mathrm{Hom}(\underline{w},\underline{x})}$$

Example $W=S_3$. $\underline{w}=tstsst$, $\underline{e}=011101$. $\underline{w}^{\underline{e}}=stst\sim ts=\underline{x}$

$LL_{\underline{w},\underline{e}} =$ 

$\underline{v}=sstsstt$, $\underline{f}=1010101$, $\underline{v}^{\underline{f}}=x$

$LL_{\underline{v},\underline{f}} =$ , $\overline{LL_{\underline{v},\underline{f}}} =$ 

(Note: $\overline{}$ is an anti-equivalence that's identity on objects)

Fix $\underline{w}, \underline{v}$.
~~Now~~ We want to prove $\mathbb{LL}_{\underline{w},\underline{v},\underline{e},\underline{f}} = \overline{LL}_{\underline{v},\underline{f}} \circ LL_{\underline{w},\underline{e}}$ is a basis

for $\mathrm{Hom}_{\mathcal{H}_{BS}}(\underline{w}, \underline{v})$ as a left R-mod (also right).

Prop $\mathbb{LL}$'s span $\mathrm{Hom}_{\mathcal{H}_{BS}}(\underline{w}, \underline{v})$.

The book says this is diagrammatic but scary. See [EW16, §7].

Thm ~~Prop~~ $\mathbb{LL}$'s are linearly indep.

~~Pf~~ It suffices to show that $\mathcal{F}(\mathbb{LL}_{\underline{w},\underline{v},\underline{e},\underline{f}}) \otimes Q : \mathrm{Hom}_{BSBim_Q}(BS(\underline{w}) \otimes Q, BS(\underline{v}) \otimes Q)$

are linearly independent.

$\boxed{\text{Def For } \underset{e_1\cdots e_n}{\underline{e}}, \underset{f_1\cdots f_m}{\underline{f}} \subset \underset{s_1\cdots s_m}{\underline{w}}, \text{ say } \underline{e} \leq_{PD} \underline{f} \text{ if } s_1^{e_1} \cdots s_k^{e_k} \leq s_1^{f_1} \cdots s_k^{f_k}}$

Write $x = \underline{w}^{\underline{e}}$

Prop $\mathcal{F}(LL_{\underline{w},\underline{e}}) \otimes Q \in \mathrm{Hom}_{BSBim_Q}(BS(\underline{w}) \otimes Q, BS(\underline{x}) \otimes Q)$ has non-zero projection to $\mathrm{Hom}(Q_{\underline{w}^{\underline{e}}}, Q_x)$. Moreover, ~~its~~ its projection to $\mathrm{Hom}(Q_{\underline{w}^{\underline{f}}}, Q_x)$ is zero $\forall\, \underline{f} \not\leq_{PD} \underline{e}$.

~~Fix $\underline{w} = \ldots$~~ 10 00 1 11

The theorem follows from the prop, as the latter says ~~the~~ when we write

~~matrix from~~ $(\mathcal{F}(\mathbb{LL}_{\underline{w},\underline{v},\underline{e},\underline{f}}) \otimes Q)$ in ~~Hom~~ $\bigoplus_{\substack{\underline{e} \subset \underline{w} \\ \underline{f} \subset \underline{v} \\ \underline{w}^{\underline{e}} = \underline{v}^{\underline{f}}}} Q$, the matrix

is strictly upper ~~triangular~~ triangular

Goal 3: Study the Karoubi closure $\mathcal{H}$ of $\mathcal{H}_{BS}$.

Firstly, instead of $\mathcal{H}_{BS}$ we want $\mathcal{H}_{BS}^{sh,\oplus}$. That means objects in $\mathcal{H}_{BS}^{sh,\oplus}$ are direct sums of graded shifts of objects in $\mathcal{H}_{BS}$, and morphisms are now required to have ~~gr~~ degree 0.

Then we want to say that the Karoubi closure

$Kar(\mathcal{H}_{BS}^{sh,\oplus})$ is such that

Thm

(A) ~~Every object is a unique direct~~ Every indecomposable object in $Kar(\mathcal{H}_{BS}^{sh,\oplus})$ is ~~of the form~~ associated to some

$w \in W$; for any reduced expression $\underline{w}$ of w, ie is of the form $(\underline{w}, e_{\underline{w}})$

for some idempotent $e_w \in Hom(\underline{w}, \underline{w})$ (deg 0). Let's call it B_w.

~~(2) B_w is not a direct summand of any $\underline{v}$ such~~

(B) B_w is a direct summand of a shift of $\underline{v}$ only if $w = \underline{v}^{\underline{f}}$ for some $\underline{f} \subset \underline{v}$.

In particular, $B_{w_1} \cong B_{w_2}(-)$ iff $w_1 = w_2$

Sketch

$\forall$ object $\underline{w}$ in $\mathcal{H}_{BS}$ and $x \in W$, define

$$End_{<x}(\underline{w}) = \langle \mathbb{LL}_{\underline{w},\underline{w},\underline{e},\underline{f}} \mid \underline{w}^{\underline{e}} = \underline{w}^{\underline{f}} < x \rangle_{R\text{-span}}.$$

Lemma Suppose $\underline{x}$ is a reduced expression of x $End(\underline{x})/End_{<x}(\underline{x}) = R$.

(spanned by $\mathbb{LL}_{\underline{w},\underline{w},\underline{1},\underline{1}}$ >. Go To Page 5

The whole algebra $End(\underline{w})$, on the other hand, is a finitely generated R-algebra. If R was a field, (or a Henselian local ring) then any such algebra is a direct ~~product~~ sum of local rings $End(\underline{w}) \cong A_1 \oplus A_2 \oplus \cdots \oplus A_n$ and if R was a field,

$1_{\underline{w}} \mapsto e_1 + e_2 + \cdots + e_n$

~~non-zero image in $End(\underline{w})/End_{<x}(\underline{w})$,~~ $End(\underline{w})/End_{<w}(\underline{w}) = R$. Then there is a unique A_i that maps surjective to

A_i defines our $B_w \subset \underline{w}$.

The algebra $\mathrm{End}(\underline{x})^0$ is a finitely generated k-algebra, which is then $\mathrm{End}(\underline{x})^0 = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ a finite direct sum of local rings. Each local ring corresp. to an indecomp. component of $\underline{x}$

We also have $\mathrm{End}(\underline{x})^0 / \mathrm{End}_{<x}(\underline{x})^0 = \left(\mathrm{End}(\underline{x}) / \mathrm{End}_{<x}(\underline{x})\right)^0 = R^0 = k$.

The map $A_1 \oplus \cdots \oplus A_n \to k$ then gives a unique component, say A_1, which maps ~~isomorphically~~ surjectively onto k. Then A_1 defines an indecomposable of $\underline{x}$ which we call B_x.

~~Now~~ For arbitrary idempotent $i \in \mathrm{End}(\underline{w})$ that gives an indecomposable ~~B_x~~, we may write

$$i = \sum_{\substack{\underline{e}, \underline{f} \subset \underline{w} \\ \underline{w}^{\underline{e}} = \underline{w}^{\underline{f}}}} r_{\underline{e}, \underline{f}} \, \mathbb{LL}_{\underline{w}, \underline{w}, \underline{e}, \underline{f}} \quad , \quad r_{\underline{e}, \underline{f}} \in R.$$

Look at the non-empty set $\{ \underline{w}^{\underline{e}} = \underline{w}^{\underline{f}} \mid r_{\underline{e}, \underline{f}} \neq 0 \}$. ~~This set~~ Let $x = \underline{w}^{\underline{e}} = \underline{w}^{\underline{f}}$ be a maximal elt under Bruhat order. We claim that ~~i defines~~ the idempotent given by i is isomorphic to B_x. In short, using $i^2 = i$ one shows

$$r_{\underline{e}, \underline{f}} = \sum_{\substack{\underline{e}', \underline{f}' \subset \underline{w} \\ \underline{w}^{\underline{e}'} = \underline{w}^{\underline{f}'} = x}} r_{\underline{f}, \underline{e}'} \, r_{\underline{f}', \underline{e}'} \, \mathrm{LL}_{\underline{w}, \underline{e}'} \circ i \circ \overline{\mathrm{LL}}_{\underline{w}, \underline{f}'}.$$

From which we know some $\mathrm{LL}_{\underline{w}, \underline{e}'} \circ i \circ \overline{\mathrm{LL}}_{\underline{w}, \underline{f}'}$ has non-zero image in ~~Hom~~ $\mathrm{End}(\underline{x}) / \mathrm{End}_{<x}(\underline{x}) = R$. Same grading trick shows that ~~its image in R~~ it's actually ~~can't be non-zero.~~ of degree 0 & thus non-zero in $R^0 = k$. Then $\underline{x} \xrightarrow{\overline{\mathrm{LL}}_{\underline{w}, \underline{f}'}} \underline{w} \circlearrowleft e$, $\underline{w} \xrightarrow{\mathrm{LL}_{\underline{w}, \underline{e}'}} \underline{x}$ (for some $\underline{e}', \underline{f}'$) gives the desired isom.