I. Lie Theory

II. Category $\mathcal{O}$

III. KL-Conjecture

$\mathfrak{g}$: semisimple Lie algebra, f.d. $/\mathbb{C}$. $\mathfrak{h}$: Cartan subalg.

$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \lambda(h) x \ \forall h \in \mathfrak{h}\}$.

$\mathfrak{h}^* \supset \Phi := \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0,\ \mathfrak{g}_\alpha \neq 0\} \supset \Phi^+ \supset \Pi$, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$

The Killing form $K(x, y) = tr(ad\, x \circ ad\, y)$ is nondegenerate on $\mathfrak{h}$

$\rightsquigarrow$ inner product $(\cdot,\cdot)$ on $span_{\mathbb{R}} \Phi = E \subset \mathfrak{h}^*$.

Define the reflection $S_\alpha : E \to E$ by $S_\alpha(\beta) := \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \alpha$.

The Weyl group $W := \langle S_\alpha \mid \alpha \in \Pi \rangle$. $S = \{S_\alpha \mid \alpha \in \Pi\}$,

(W, S) is a finite crystallographic Coxter system. i.e. $\forall s, t \in S$,

$m_{st} \in \{2, 3, 4, 6\}$.

$\mathfrak{g}$-mod $\longleftrightarrow$ $\mathcal{U}(\mathfrak{g})$-mod, $\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g}) / \langle x \otimes y - y \otimes x - [x, y] \rangle$.

II. Def. The BGG category $\mathcal{O}$ is the full subcat. of $\mathcal{U}(\mathfrak{g})$-mod whose objects are modules satisfying:

($\mathcal{O}$1). M is a f.g. $\mathcal{U}(\mathfrak{g})$-mod.

($\mathcal{O}$2). $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$, where $V_\lambda := \{v \in M \mid h \cdot v = \lambda(h) v \ \forall h \in \mathfrak{h}\}$.

($\mathcal{O}$3). $\forall v \in M$, $\mathcal{U}(\mathfrak{n}) \cdot v$ is f.d. $wt(M) := \{\lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0\}$.

Fact: $\{$f.d. $\mathcal{U}(\mathfrak{g})$-mod$\} \subset \mathcal{O}$.

Def. Let $M \in \mathcal{O}$. A nonzero vector $v^+ \in M$ is called a maximal vector of wt. $\lambda \in \mathfrak{h}^*$ if $v^+ \in M_\lambda$ and $\mathfrak{n} \cdot v^+ = 0$.

Def. A module $M \in \mathcal{O}$ is called a highest weight module of wt. λ if $M = \mathcal{U}(\mathfrak{g}) \cdot v^+$ for some max. vector $v^+ \in M_\lambda$.

Prop. M: ht. wt. mod. of wt. λ.

(a) $wt(M) \subseteq \lambda - \mathbb{N}\Phi^+$.

(b) $\forall \mu \in \mathfrak{h}^*$, $\dim M_\mu < \infty$. $\dim M_\lambda = 1$.

(c) M has a unique max. submod.

unique simple quotient.

(d) All simple ht. wt. mod. of wt. λ are iso.

$\mathfrak{h}^* \longleftrightarrow$ {simple ht. wt. mod.}.

Thm. $\forall M \in \mathcal{O}$, M has a finite filtration s.t. each quotient is a ht. wt. mod.

pf. Let S be a finite generating set of M, consisting of weight vectors. By Lie's thm, as $\mathcal{U}(\mathfrak{b})$-mod., $\exists\, 0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathcal{U}(\mathfrak{b})\cdot S$ s.t. V_i/V_{i-1} is 1-dim. Then each quotient of $0 = \mathcal{U}(\mathfrak{n}_-)V_0 \subset \mathcal{U}(\mathfrak{n}_-)V_1 \subset \cdots \subset \mathcal{U}(\mathfrak{n}_-)V_n = \mathcal{U}(\mathfrak{g})\cdot S = M$ is a ht. wt. m.

Def. $\forall \lambda \in \mathfrak{h}^*$, define an 1-dim $\mathcal{U}(\mathfrak{b})$-mod. $\mathbb{C}_\lambda = \mathbb{C}\cdot v_\lambda$ by $h\cdot v_\lambda = \lambda(h)\, v_\lambda$, $\forall h \in \mathfrak{h}$, $\mathfrak{n}\cdot v_\lambda = 0$.

Set $\Delta(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$, called the Verma module.

$\Delta(\lambda) \cong \mathcal{U}(\mathfrak{g})/I$, $\quad I = \langle \mathfrak{n},\ h - \lambda(h)\cdot 1,\ h \in \mathfrak{h} \rangle$.

If $M = \mathcal{U}(\mathfrak{g})\cdot v^+$, with $v^+ \in M_\lambda$, then $I\cdot v^+ = 0$. Hence $\Delta(\lambda) \twoheadrightarrow M$.

Denote $L(\lambda)$ the unique simple quotient of $\Delta(\lambda)$.

$M = \mathcal{U}(\mathfrak{g})\cdot v^+$, $v^+ \in M_\lambda$, $z \in Z(\mathfrak{g})$, center of $\mathcal{U}(\mathfrak{g})$

$h\cdot(z\cdot v^+) = z\cdot(h\cdot v^+) = \lambda(h)\, z\cdot v^+ \Rightarrow z\cdot v^+ \in M_\lambda$

$\Rightarrow z\cdot v^+ = \chi_\lambda(z)\, v^+$ for some $\chi_\lambda(z) \in \mathbb{C}$.

This defines an alg. hom. $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$, called the central character.

Thm (Harish-Chandra): $\chi_\lambda = \chi_\mu \iff \mu = w\cdot\lambda$, $w \in W$

$= w(\lambda+\rho) - \rho$, where

$\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \rho$.

For $M \in \mathcal{O}$, define $M^{\chi_\lambda} := \left\{ v \in M \;\middle|\; \begin{array}{l} (z - \chi_\lambda(z))^{n_z}\cdot v = 0,\ n_z \in \mathbb{N}, \\ \forall z \in Z(\mathfrak{g}) \end{array} \right\}$

Define $\mathcal{O}_\lambda = \mathcal{O}_{\chi_\lambda}$ to be the full subcat. of $\mathcal{O}$ consisting of $M \in \mathcal{O}$ s.t. $M = M^{\chi_\lambda}$.

$\Delta(w\cdot\lambda) \in \mathcal{O}_\lambda$. $\forall w \in W$.

We have the decomp. $\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*/(W,\cdot)} \mathcal{O}_\lambda$.

If λ is integral, i.e., $(\lambda, \alpha^\vee) \in \mathbb{Z}$ $\forall \alpha \in \Phi$, then $\mathcal{O}_\lambda$ is indecomp.

Def. We call $\mathcal{O}_0 = \mathcal{O}_{\chi_0}$ the principal block.

Thm. $\mathcal{O}$ is Artinian.

pf. $M \in \mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*/(W,\cdot)} \mathcal{O}_\lambda$. Assume $M = \Delta(\lambda)$.

$M = M_0 \supset M_1 \supset \cdots$, $M_i / M_{i+1} \cong L(w\cdot\lambda)$, $w \in W$.

$\dim M_{w\cdot\lambda} < \infty \Rightarrow$ DCC.

If λ is dominant, i.e. $(\lambda, \alpha^\vee) \in \mathbb{Z}_{\geq 0}$, then for $M \in \mathcal{O}_\lambda$,

$\mathrm{Hom}_{\mathcal{O}}(\Delta(\lambda), M) \cong \mathrm{Hom}_{\mathcal{U}(\mathfrak{b})}(\mathbb{C}_\lambda, M) \cong M_\lambda$.

$\Rightarrow \mathrm{Hom}_{\mathcal{O}}(\Delta(\lambda), \cdot)$ exact $\Rightarrow \Delta(\lambda)$ is proj.

If $E \in \mathcal{O}$ is f.d., then $\mathrm{Hom}_{\mathcal{O}}(\Delta(\lambda) \otimes E, \cdot) \cong \mathrm{Hom}_{\mathcal{O}}(\Delta(\lambda), E^* \otimes \cdot)$ is exact. $\Rightarrow \Delta(\lambda) \otimes E$ is proj.

For every $\lambda \in \mathfrak{h}^*$, we can choose a dominant wt. μ & a f.d. $\mathcal{U}(\mathfrak{g})$-mod. E s.t. $\mathrm{Hom}_{\mathcal{O}}(\Delta(\mu) \otimes E, L(\lambda)) \neq 0$.

$\Rightarrow \mathcal{O}$ has enough proj. (Since $L(\lambda)$'s are all the simple objects in $\mathcal{O}$).

Every $M \in \mathcal{O}$ has a proj. cover $\pi : P_M \to M$, where π is surj. essential, i.e. no proper submod. of P_M is mapped onto M.

When $M = L(\lambda)$, we write $P_M = P(\lambda)$.

$\pi_\lambda : P(\lambda) \twoheadrightarrow L(\lambda)$ is essential $\Rightarrow \mathrm{Ker}\,\pi$ is the unique max. submod. of $P(\lambda) \Rightarrow P(\lambda)$ is indecomp.

$\Delta(\mu) \otimes E \twoheadrightarrow L(\lambda)$. $\Rightarrow \Delta(\mu) \otimes E = P(\lambda) \oplus \sim$

Def. $M \in \mathcal{O}$. Recall that M has a finite filtration with quotients being ht. wt. mod. If each quotient is a Verma mod., then we call the filtration a Verma flag (or a std. filtration).

Denote the flag multiplicity by $(M : \Delta(\lambda))$.

$\Delta(\mu) \otimes E = (\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\mu) \otimes_{\mathbb{C}} E \cong \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} (\mathbb{C}_\mu \otimes E)$.

$\Rightarrow \Delta(\mu) \otimes E$ has a Verma flag.

$\Rightarrow P(\lambda)$ has a Verma flag

$\Rightarrow$ every proj. has a Verma flag. ($P(\lambda)$ are all indecomp. proj. in $\mathcal{O}$)

BGG reciprocity: $(P(\lambda) : \Delta(\mu)) = [\Delta(\mu) : L(\lambda)]$.

III Define $\mathrm{ch}\, M = \sum_{\mu \in \mathfrak{h}^*} (\dim M_\mu) e^\mu$, called the (formal) character

e.g. $\mathrm{ch}\, \Delta(\lambda) = \dfrac{e^\lambda}{\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})}$.

Question: compute $\mathrm{ch}\, L(\lambda)$.

When λ is **dominant integral**, $L(\lambda)$ is f.d., and we have the Weyl character formula: $\mathrm{ch}\, L(\lambda) = \dfrac{\sum_{w \in W} (-1)^{l(w)} e(w \cdot \lambda)}{\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})}$

Thm (Verma). $\lambda \in \mathfrak{h}^*$, **dominant**. Suppose $x, y \in W$ s.t. $x \geq y$ in the Bruhat order. Then $\mathrm{Hom}_{\mathcal{O}}(\Delta(x \cdot \lambda), \Delta(y \cdot \lambda)) \cong \mathbb{C}$.

$\Rightarrow [\Delta(y \cdot \lambda) : L(x \cdot \lambda)] \neq 0$.

In general, $\dim \mathrm{Hom}_{\mathcal{O}}(\Delta(\lambda), \Delta(\mu)) \leq 1$.

Thm (BGG) If $\mathrm{Hom}_{\mathcal{O}}(\Delta(\mu), \Delta(\eta)) = 1$, then $\exists \lambda$ dominant s.t. $\mu = x \cdot \lambda$, $\eta = y \cdot \lambda$ with $x \geq y$ in the Bruhat order.

Note: $[\Delta(y \cdot \lambda) : L(x \cdot \lambda)] \neq 0 \Rightarrow x \geq y$.

Since $[\Delta(y \cdot \lambda) : L(y \cdot \lambda)] = 1$, we have $\{[\Delta(y \cdot \lambda)] \mid y \in W\}$ and $\{[L(y \cdot \lambda)] \mid y \in W\}$ are both bases for the Grothendieck group for $\mathcal{O}_\lambda$.

Kazhdan–Lusztig Conjecture: λ: regular, integral, dominant.

$[\Delta(y \cdot \lambda)] = \sum_{x \geq y} h_{y,x}(1) [L(x \cdot \lambda)]$, where $h_{y,x}$ is given by $b_x = \sum_{x \geq y} h_{y,x}(v) \delta_y$, with b_x: KL-basis, δ_x: std basis for $H(W)$