III

Soergel bimodules
2020/12/18

(I) Translation functors

(i). For a given $\mathfrak{g}$-module V, we construct an exact functor

$$V \otimes_{\mathbb{C}} - : \mathfrak{g}\text{-mod} \to \mathfrak{g}\text{-mod}$$

to translate modules

- to get some important info. (e.g. compute char.)
- to construct structural modules. (e.g. construct projectives.)
  in a subcate. ⇝

If $V \otimes_{\mathbb{C}} -$ restricts to an endofunctor on $\mathcal{O}$.

Then $V \otimes M(\lambda) \in \mathcal{O}$. But $M(\lambda) \cong \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathrm{Res}\, V \otimes \mathbb{C}_\lambda)$

$\Rightarrow$ $V \otimes M(\lambda) \in \mathcal{O}$ if and only if $\dim V < \infty$

$\Rightarrow$ $V \otimes - : \mathcal{O} \to \mathcal{O} \iff \dim V < \infty$.

(ii). Remark: $V \otimes - \cong (V \otimes U\mathfrak{g}) \otimes_U (-) : \mathcal{O} \to \mathcal{O}$

There are equivalence of (some) blocks in $\mathcal{O}$ and the category of Harish-Chandra bimodules. In the later category the (U,U)-bimodule $V \otimes U(\mathfrak{g})$ is projective. Therefore (direct summands) of $V \otimes -$ are sometimes called "projective functors".

$\mathcal{O}_{\lambda+\Lambda} \cong \mathcal{HC}_\lambda$

Dir. Sum. $V \otimes -$ are "proj. fun.".

Per-Duet

(iii). Now, we define the translation functor T_λ^μ:
For any $\lambda, \mu \in \mathfrak{h}^*$ such that $\mu-\lambda$ is integral,
$T_\lambda^\mu(-) := pr_\mu \circ (L(\nu) \otimes -) \circ in_\lambda : \mathcal{O}_\lambda \to \mathcal{O}_\mu$, where
$\nu \in W(\mu-\lambda)$ is dominant integral, and

$$pr_\mu : \mathcal{O} \twoheadrightarrow \mathcal{O}_\mu \text{ and } in_\lambda : \mathcal{O}_\lambda \hookrightarrow \mathcal{O}$$

is projection and inclusion along central decomposition of $\mathcal{O}$, respectively.

Facts:

(1). $(T_\lambda^\mu, T_\mu^\lambda)$ is an adjoint pair.

(2). T_λ^μ sends projectives to projectives.

(3). If μ is dominant and $\lambda-\mu$ is domi. int.
Then for any w

$$[T_\lambda^\mu \Delta(w\cdot\lambda)] = [\Delta(w\cdot\mu)],$$
$$[T_\mu^\lambda \Delta(w\cdot\mu)] = \sum_{w' \in W_\mu^0} [\Delta(ww'\cdot\lambda)],$$

where W_μ^0 is the stabilizer of μ.
$\subseteq W_{[\mu]}$

(4). If μ is also dom. integral And $\lambda-\mu$: domi. int.
Then $T_\lambda^\mu : \mathcal{O}_\lambda \to \mathcal{O}_\mu$ is an equivalence.

Notation: We define for any $w \in W$
$\Delta_w := \Delta(w\cdot 0)$, $L_w := L(w\cdot 0)$, $P_w := P(w\cdot 0)$.

(iv) Def'n (Wall-crossing functor) θ_s):

- $s \in S$
- $\mu \in \mathfrak{h}^*$ integral with $W_\mu = \{e, s\}$
- $\nu \in \mathfrak{h}^*$ dominant integral s.t. $\nu - \mu$: domi. integ.

We define the wall-crossing functor
$$\theta_s := T_\nu^0 T_\mu^\nu T_\nu^\mu T_0^\nu : \mathcal{O}_0 \to \mathcal{O}_0.$$

It can be shown that θ_s is independent of μ and ν (We shall show it later on)

Example: Consider $\mathfrak{g} = \mathfrak{sl}(2)$. Then

- $\theta_s \Delta_e$ is projective.
- $\theta_s \Delta_e$ has Verma subquotients Δ_s and Δ_e

$\Rightarrow 0 \to \Delta_e \to \theta_s \Delta_e \to \Delta_s \to 0$ exact $(\star)$

and $\theta_s \Delta_e = P_s \Rightarrow$ and so $(\star)$ is nonsplit.

Lemma: For any $w \in W$ and $s \in S$ we have a non-split short exact sequence:
$$0 \to \Delta_w \to \theta_s \Delta_w \to \Delta_{ws} \to 0 \quad \text{if } ws > w$$
$$0 \to \Delta_{ws} \to \theta_s \Delta_w \to \Delta_w \to 0 \quad \text{if } ws < w$$

- $\Delta_w \xrightarrow{T_0^\nu} \Delta_{w \cdot \nu} \xrightarrow{T_\nu^\mu} \Delta_{w \cdot \mu}$.
- We have exact sequence
$$0 \to \Delta_{w \cdot \mu} \to T_\mu^\nu \Delta_{w \cdot \mu} \to \Delta_{ws \cdot \mu} \to 0, \quad \text{if } ws > w$$
$$0 \to \Delta_{ws \cdot \mu} \to \quad " \quad \to \Delta_{w \cdot \mu} \to 0, \quad \text{if } ws < w.$$

This induces $\mathbb{Z}$-linear functions θ_s on $[\mathcal{O}_0]$ such that $[\theta_s \Delta_w] = [\Delta_w] + [\Delta_{ws}]$.

Corollary: Let $\varphi : [\mathcal{O}_0] \to \mathbb{Z}[W]$ be the isomorphism sending $[\Delta_w]$ to w. Then for any simple reflection s the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}[W] & \xrightarrow{\cdot(e+s)} & \mathbb{Z}[W] \\ \varphi \downarrow & & \downarrow \varphi \\ [\mathcal{O}_0] & \xrightarrow{[\theta_s]\cdot} & [\mathcal{O}_0] \end{array}$$

Example: Consider $\mathfrak{g} = \mathfrak{sl}(2)$. We have

- $\varphi(e) = \Delta_e = P_e$;
  $\varphi(s) = \Delta_s = L_s$.
- SES. $0 \to \Delta_e \to \theta_s \Delta_e \to \Delta_s \to 0 \Rightarrow \theta_s \Delta_e = P_s \Rightarrow \varphi(e+s) = P_s$.
- $\varphi^{-1}(L_e) = e - s \Rightarrow \varphi^{-1}(\theta_s L_e) = (e-s)(e+s) = e - e = 0$.
  $\Rightarrow \theta_s L_e = 0$.
- $\varphi^{-1}(\theta_s P_s) = \varphi^{-1}(P_s)(e+s) = \varphi^{-1}(\theta_s P_e)(e+s) = (e+s)^2$
  $= 2(e+s) = P_s \oplus P_s$

Remark: $\theta_s^2 = \theta_s \oplus \theta_s : \mathcal{O}_0 \to \mathcal{O}_0$. (We will check this later on). Therefore θ_s^2 can be decomposable.
Note $\theta_s^2 = \theta_s \oplus \theta_s \leftrightsquigarrow b_s^2 = (v+v^{-1}) b_s$.

For any expression $(s_1, \cdots, s_m)$, we define

$$P_{\underline{x}} := \theta_{s_m} \cdots \theta_{s_1} (P_e)$$

Then by repeatly using " $[\theta_s \Delta_w] = [\Delta_w] + [\Delta_{ws}]$ "
we have

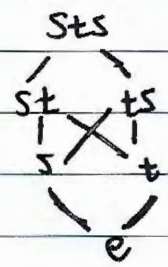
$$P_{\underline{x}} = P_x \oplus \bigoplus_{y<x} P_y^{m_y},$$

for any reduced expression $\underline{x}$ for x..

In particular, P_x is the unique indecomposable direct summand of $P_{\underline{x}}$ that does NOT appear as a direct summand for any $\underline{w}$ with $\ell(\underline{w}) < \ell(\underline{x})$.

$P_{\underline{s}} = P_s$,

☆ Example: Consider $\mathfrak{g} = \mathfrak{sl}(3)$. $P_{\underline{t}} = P_t$, $P_{\underline{e}} = P_e$.



$P_{\underline{ts}} = \theta_t P_s = \theta_t \Delta_s + \theta_t \Delta_e = P_{st}$

$P_{\underline{sts}} = \theta_s P_{st} = \theta_s(\Delta_{st} + \Delta_s + \Delta_t + \Delta_e)$

$= \Delta_{sts} + \Delta_{st} + \Delta_s + \Delta_e + \Delta_{ts} + \Delta_t + \Delta_s + \Delta_e$

$= P_{sts} + P_s$

Similarly, $P_{\underline{st}} = P_{ts}$

We compare $[\mathrm{Proj}(\mathcal{O}_0)]_\oplus$ with $\mathcal{H}$ as follows:

$P_w \longleftrightarrow b_w$

$P_{\underline{s}} \longleftrightarrow b_{\underline{s}}$

$P_{\underline{st}} \longleftrightarrow b_{\underline{st}} = b_s b_t$

$P_{\underline{sts}} \longleftrightarrow b_{\underline{sts}} := b_s b_t b_s = b_{sts} + b_s$

i.e. This is an analog to KL basis element

$P_w \leftrightarrow b_w$

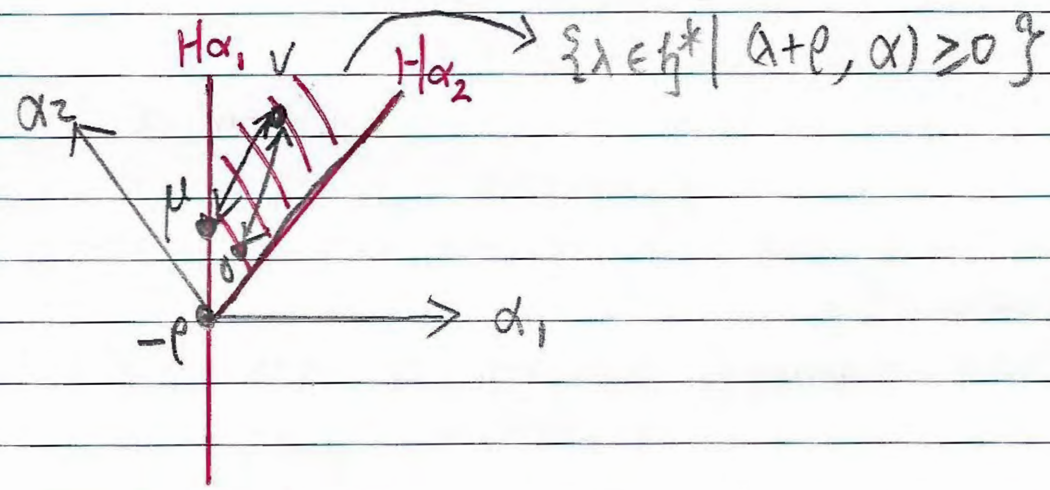
$P_{\underline{w}} \longleftrightarrow b_{\underline{w}}$

Walls

$\mathfrak{g} = sl(3)$.

$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3$

$H_\alpha := \{\lambda \in \mathfrak{h}^* \mid (\lambda+\rho, \alpha) = 0\}$



$T_\nu^0 T_\mu^\nu : \mathcal{O}_\mu \to \mathcal{O}_\nu \to \mathcal{O}_0$

$T_\nu^\mu T_0^\nu : \mathcal{O}_0 \to \mathcal{O}_\nu \to \mathcal{O}_\mu$

§ Soergel's $\mathbb{V}$ functor

$R := \mathrm{Sym}(\mathfrak{h}) = \mathbb{C}[\alpha_s \mid s \in S]$.

Def'n: Let $\underline{w} := (s_1, \dots, s_m)$ be an expression. The Bott-Samelson module $\overline{BS}(\underline{w})$ is defined as

$$\overline{BS}(\underline{w}) = \mathbb{C} \otimes_R BS(\underline{w})$$

$$= \mathbb{C} \otimes_R B_{s_1} \otimes_R B_{s_2} \otimes_R \cdots \otimes_R B_{s_m}$$

$$= \mathbb{C} \otimes_{R^{s_1}} R \otimes_{R^{s_2}} R \otimes_{R^{s_3}} \cdots \otimes_{R^{s_m}} R(\ell(m))$$

A Soergel module is any graded right R-module that is isomorphic to a finite direct sum of shifts of direct summands of BS modules.

Proposition:
(Classification of indecomposable Soergel modules)
There is a bijection between W and the set of indecomposable Soergel modules up to shifts and isom.

$$W \overset{1:1}{\longleftrightarrow} \{\text{indec. Soergel modules}\}/\simeq$$
$$w \longleftrightarrow \mathbb{C} \otimes_R B_w =: \overline{B}_w$$

Recall the coinvariant algebra

$C := R/I_W$, where I_W is the homogenous ideal in R generated by the set of W-invariant polynomials in R of strictly positive degree.

$\overline{BS}(\underline{w})\, I_W = 0 \Rightarrow \overline{BS}(\underline{w}) \in \text{mod-}C$, for any $w \in W$.

$C^s := \{ s\text{-invariants in } C \}$.

Fact : $\overline{BS}(s_1, \dots, s_m) \cong \mathbb{C} \otimes_{C^{s_1}} C \otimes_{C^{s_2}} C \otimes \cdots \otimes_{C^{s_m}} C$

Example : Consider $\mathfrak{g} = sl(2)$. $R = U(\mathfrak{h}) \cong \mathbb{C}[h]$

Then $- \otimes_{R^s} R = - \otimes_{\mathbb{C}[x^2]} \mathbb{C}[x] \cong - \otimes_{\mathbb{C}} \mathbb{C}[x]/(x^2) = - \otimes_{C^s} C$

as functors on mod-C ($\subseteq$ mod-R)

(Recall that $R = \mathbb{C}[h]$, $R^s = R^W = \mathbb{C}[h^2]$, $R^W_+ = \mathbb{C}[h^2]h^2$
$I_W = \mathbb{C}[h]h^2$, $C = R/I_W = \mathbb{C}[h]/(h^2)$, $C^s = \mathbb{C} \subseteq C$.)

Fact: Denote by $\overline{SBim}$ the cate of Soergel modules.
- $[SBim]_\oplus \cong \mathcal{H}$ as graded alg.
- $[\overline{SBim}]_\oplus$ is a right $[SBim]$-module.

$\Rightarrow [\overline{SBim}]_\oplus$ can be identified with the right regular repn. of $\mathcal{H}$.

$$Z(U\mathfrak{g}) \xrightarrow{\gamma} U(\mathfrak{h}) \xrightarrow{q} C$$

$$Z(U\mathfrak{g}) \xrightarrow{\gamma'} \mathrm{End}_{\mathcal{O}_0}(P_{w_0})$$

Thm (Soergel's endomorphism theorem):

γ' is surjective and $\ker(q\gamma) = \ker(\gamma')$.
In particular, $C \cong \mathrm{End}_{\mathcal{O}_0}(P_{w_0})$.

Def'n: Soergel's $\mathbb{V}$ functor is defined as

$$\mathbb{V}(-) := \mathrm{Hom}_{\mathcal{O}_0}(P_{w_0}, -) : \mathcal{O}_0 \to \text{mod-}C$$

we shall show $C \cong \mathrm{End}_{\mathcal{O}_0}(P_{w_0})$.

Example: Consider $\mathfrak{g} = \mathfrak{sl}(2)$. Set $J := \mathrm{Ann}_{Z(U\mathfrak{g})} P_s$.
$\Omega := 4yx + h^2 + 2h \in Z(U\mathfrak{g})$. the Casimir element.
Recall that $0 \to \Delta_0 \to P_s \xrightarrow{f} \Delta_s \to 0$
Let $v \in P_s$ be the preimage of the highest weight vector of Δ_s. We claim that $xv \neq 0$.
Suppose on the contrary that $xv = 0$. Then $U\mathfrak{g}.v = L_s$, which implies that $\mathrm{soc}\, P_s = L_s \oplus L_s$.
But $P_s = I_s$, a contradiction. Thus $xv \neq 0$ and so $xv \in \Delta_0$. Note $yxv \neq 0$ in Δ_0 since Δ_0 is y-free. Consequently,

$$\Omega \cdot v = 4yxv + (h^2 + 2h)v = 4yxv \neq 0.$$

We know that $\Omega^2 P_s = 0$ but $\Omega P_s \neq 0$.

Also, we note $\dim Z(U\mathfrak{g})/\Omega^2 Z(U\mathfrak{g}) = \dim \mathrm{End}_{\mathcal{O}_0} P_s = 2$.

There is a natural homo. by

$$\psi: z \mapsto z \overset{\circ}{\circ} x \to zx,\ \forall z \in Z(\mathfrak{g}),\ \forall x \in P_s.$$

Note $\Omega P_s \neq 0$ (in fact, $\Omega P_s \cong L_s$) and Ω is NOT an automorphism of P_s (otherwise, $\Omega^2 P_s \neq 0$, contradiction). Thus, the natural homo.

$$\psi: UZ(\mathfrak{g}) \to \mathrm{End}_{\mathcal{O}_0}(P_s)$$

has image of dimension ≥ 2. Thus $\mathrm{Ker}(\psi) = J$ is equal to $\Omega^2 Z(U\mathfrak{g}) \Rightarrow UZ\mathfrak{g}/\Omega^2 UZ\mathfrak{g} \cong \mathrm{End}_{\mathcal{O}_0}(P_s)$ ⁂

Thm (Soergel's structure theorem):

(1).

$$\mathbb{V}: \mathrm{Hom}_{\mathcal{O}_0}(M, Q) \xrightarrow{\cong} \mathrm{Hom}_{mod\text{-}C}(\mathbb{V}M, \mathbb{V}Q),$$

for any $M, Q \in \mathcal{O}_0$ with Q projective.

(2). For any s

$$\mathbb{V} \circ \theta_s(-) \cong (- \otimes_{C^s} C) \circ \mathbb{V}(-).$$

Coro.

(1). $\mathbb{V}: \mathrm{Proj}(\mathcal{O}_0) \xrightarrow{\cong}$ Cate. of (ungraded) Soergel modules

(2). For any $s \in S$, $x \in W$ and expression $\underline{y}$

$$\mathbb{V}(P_{\underline{y}}) \cong \mathbb{C} \otimes_R BS(\underline{y}) = \overline{BS(\underline{y})}$$

$$\mathbb{V}(P_x) \cong \mathbb{C} \otimes_R B_x =: \overline{B}_x.$$

Casimir element

Example: Consider $\mathfrak{g} = sl(2)$. Then

$\mathrm{End}_{\mathcal{O}_0}(P_s) \cong \mathbb{C}[x]/(x^2)$, where $x \equiv C$ and so $xP_s = L_s$.

$\mathbb{V}(P_0) = \mathbb{V}(\Delta_0) \cong \mathbb{C}$ trivial C-module.

$\mathbb{V}(P_s) = C$

Note: S.E.S. $0 \to \Delta_0 \xrightarrow{\hookrightarrow} P_s \xrightarrow{\Omega(\cdot)} \Delta_s \to 0$

$\Rightarrow$ S.E.S. $0 \to \mathbb{V}\Delta_0 \xrightarrow{\mathbb{V}(\hookrightarrow)} \mathbb{V}P_s \xrightarrow{\mathbb{V}(\Omega(\cdot))} \mathbb{V}\Delta_s \to 0$

is equivalent to $0 \to \mathbb{C} \xrightarrow{i} C \xrightarrow{p} \mathbb{C} \to 0$ in $\mathrm{Ext}^1_C(\mathbb{C}, \mathbb{C})$.

(1). $\mathbb{V}: \mathrm{Hom}_{\mathcal{O}}(P_0, P_0) \cong \mathrm{Hom}_C(\mathbb{C}, \mathbb{C})$ dim=1

Pf: This is clear.

(2). $\mathbb{V}: \mathrm{Hom}_{\mathcal{O}}(P_s, P_s) \cong \mathrm{Hom}_C(C, C)$ dim=2

pf: because their dimensions are 2, and $\mathbb{V}(1_{P_s}) = 1_C$ and $\mathbb{V}(\Omega: P_s \to P_s)$ = multiplication by x.

(3). $\mathbb{V}: \mathrm{Hom}_{\mathcal{O}}(P_s, P_0) \cong \mathrm{Hom}_C(C, \mathbb{C})$ dim=1

Pf: Note $\mathrm{Hom}_{\mathcal{O}}(P_s, P_0) = \mathbb{C}\cdot\{f: P_s \twoheadrightarrow \mathrm{soc}\, P_0 \subseteq P_0\}$

$= \mathbb{C}\{\Omega: P_s \to P_s \text{ restricts to } \Omega': P_s \to P_0\}$

$\Rightarrow \mathbb{V}(\Omega') = p$

(4). $\mathbb{V}: \mathrm{Hom}_{\mathcal{O}}(P_0, P_s) \cong \mathrm{Hom}_C(\mathbb{C}, C)$ dim=1

Pf:

$\mathbb{V}(\hookrightarrow)$ is injective $\Rightarrow$ $\mathbb{V}$ is an isom.

Per-Duet

(5) $\mathbb{V}\,\theta_S P_0 = \mathbb{V}\,P_S = C$

$$(- \otimes_{C^S} C) \circ (\mathbb{V} P_0) = \mathbb{C} \otimes_{C^S} C \cong C$$

$$\mathbb{V}\,\theta_S P_S = \mathbb{V}\,(P_S \oplus P_S) = C \oplus C$$

$$(- \otimes_{C^S} C) \circ (\mathbb{V} P_S) = C \otimes_{C^S} C = \mathbb{C}[x]/(x^2) \otimes_{\mathbb{C}} C \cong C \oplus C.$$

Proposition: Soergel's conjecture implies the KL Conjecture. That is,

$$(P_x : \Delta_y) = h_{y,x}(1).$$

Proof: Using the identification $[\mathcal{O}_0] \cong \mathbb{Z}[W]$, $w \leftrightarrow \delta_w$, our goal is to show $[P_x] = b_x|_{v=1}$, for each x.

We shall proceed with induction on the Bruhat order.

- $[P_e] = b_e|_{v=1}$
- Let $x \neq e$. Choose $s \in S$ such that $xs < x$ and write $x = ws$ with $w < x$. Then

$$\theta_s P_w \cong P_x \oplus \bigoplus_{z<x} P_z^{m_z}$$

$$\Rightarrow [P_x] = [P_w](1+s) - \sum_{z<x} m_z [P_z]$$

$$= (b_w b_s - \sum_{z<x} m_z b_z)|_{v=1}$$

Apply $\mathbb{V}$ $\Rightarrow$ $\overline{B_w} B_x \cong \overline{B_x} \oplus \bigoplus_{z<x} B_z^{\oplus m_z}$

$$\Rightarrow B_w B_x \cong B_x \oplus \bigoplus_{z<x} B_z^{m_z} \text{ since } [\overline{\mathrm{BSim}}]_{\oplus}$$

is the right regular $\mathcal{H}$-module.

Use Soergel's conjecture we have

$$b_x = b_w b_x - \sum_{z<x} m_z b_z \Rightarrow [P_x] = b_x|_{v=1}$$