Intro to Soergel Bimod Chp 16 Perverce Sheaves

Goal: connection btw Soergel bimod and perverse sheaves w/o precise defn

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Defn A stratified space X is a top. sp. with stratification $(X_\lambda \subseteq X)_{\lambda\in\Lambda}$ such that

1. $X = \coprod_{\lambda\in\Lambda} X_\lambda$
2. The stratif Λ is a finite poset
3. $\overline{X_\lambda} = \coprod_{\mu\le\lambda} X_\mu$
4. Each strata X_λ is $\boxed{\qquad}$

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ex(1) Let $X = \mathbb{CP}^1 =$ . The followings are stratifications of X:

(a)

Λ	$X_\bullet$
λ	X

(b)

Λ	$X_\bullet$	
λ $\vee$	$\mathbb{C}$	
μ	pt	

(c)

Λ	$X_\bullet$	
λ:	$\mathbb{C}^\times$	
μ	pt	
ν	pt	

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(2) Let $G = GL_n(\mathbb{C}) \supset B = B_n(\mathbb{C}) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$X = G/B \equiv \{\text{complete flags in } \mathbb{C}^n\}$: flag variety

$gB \mapsto gV_\bullet^{std}$ i.e. $gV_i^{std} = \mathrm{span}\{\text{first } i \text{ columns in } g\}$

Then X is strat'd wrt Σ_n via the Bruhat decomp'n:

$X = \coprod_{w\in\Sigma_n} X_w$ where $X_w = BwB/B \cong \mathbb{C}^{\ell(w)}$ is called $\overset{g}{\vee}$ Schbert cell

$\overline{X_w} = \coprod_{y\le w} X_y$ is called a Schubert variety

$\llcorner$ Bruhat order

For $w\in\Sigma_n$, define a matrix r^w s.t.

$r^w_{ij} = \#\{k\in[1,.,i] \mid w(k)\le j\} = \mathrm{rk}\left(\overset{(1,1)}{\boxed{\quad w \quad}}{}_{(i,j)}\right)$

eg

w	r^w
$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&1\\1&2\end{pmatrix}$
$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&2\end{pmatrix}$

For $V_\bullet \in X$, define a matrix f^V s.t.

$f^V_{ij} = \dim(V_i \cap V^{std}_j)$

Prop $X_w = \{V_\bullet \in G/B \mid f^V = r^w\}$

$\overline{X_w} = \{V_\bullet \in G/B \mid f^V \ge r^w\} \Rightarrow y\le w \Leftrightarrow r^y \ge r^w$

In particular, $\overline{X_{s_i}} = \{V_\bullet \in G/B \mid V_j = V^{std}_j \ \forall j\ne i\} \cong \mathbb{CP}^1$

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e.g. when $n=2$ $V_\bullet = (0 \subset \langle e_1+e_2\rangle \subset \mathbb{C}^2)$ has f^V given by

$$\begin{array}{ccc} 0 & \subset \langle e_1\rangle \subset & \mathbb{C}^2 \\ \cap & & \\ \langle e_1+e_2\rangle & \begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix} & \\ \cap & & \\ \mathbb{C} & & \end{array}$$

hence $V_\bullet \in X_{s_1}$

Actually, $G/B = Gr(1,2) = \mathbb{CP}^1$ recovers (1)(b)

Λ	$X_\bullet$
s_1	$\mathbb{C}$
id	pt

For $s=s_i$, define the parabolic subgrp $P_s = \begin{pmatrix} * & & * \\ & \underset{\overset{\uparrow\ \uparrow}{i\ \ i+1}}{\boxed{GL_2}} & \\ & & * \end{pmatrix}$ of $G=GL_n$

$P_s = \mathrm{Stab}_G$ of the std partial flag $\left(0 \overset{1}{\subset} \cdots \overset{1}{\subset} \langle e_1,\dots,e_{i-1}\rangle \overset{2}{\subset} \langle e_1,\dots,e_{i+1}\rangle \overset{1}{\subset} \cdots \overset{1}{\subset} \mathbb{C}^n\right)$

Defn For $\underline{w} = (s_{i_1},\dots,s_{i_d})$ the corr. Bott–Samelson variety is

$Y(\underline{w}) = P_{s_{i_1}} \times \dots \times P_{s_{i_d}} / B^d$ with B^d-action

$(b_1,\dots,b_d)\cdot(p_1,\dots,p_d) = (p_1 b_1^{-1}, b_1 p_2 b_2^{-1}, \dots, b_{d-1} p_d b_d^{-1})$

$\equiv \{(\underset{\overset{\parallel}{V_\bullet^{std}}}{V_\bullet^0}, V_\bullet^1, \dots, V_\bullet^d) \in (G/B)^{d+1} \mid V_t^j = V_t^{j+1} \ \forall j, t\ne i_j\}$

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eg. $n=3$; $Y(s_1) \equiv \{(V_\bullet^{std}, V_\bullet^1) \mid V_t^{std} = V_t^1 \text{ if } t=0,2,3\}$

$\equiv \{(0\subset V_1^1 \subset \langle e_1,e_2\rangle \subset \mathbb{C}^3) \mid \dim V_1^1 = 1\} \cong \mathbb{CP}^1$

$Y(s_1,s_1) = \{(V_\bullet^{std}, V_\bullet^1, V_\bullet^2) \mid V_t^{std} = V_t^1 = V_t^2 \text{ if } t=0,2,3\} \cong \mathbb{CP}^1\times\mathbb{CP}^1$

Fact (1) $\mathrm{mult}: P_{s_{i_1}}\times\dots\times P_{s_{i_d}} \to G$ induces a map $\mathrm{mult}: Y(\underline{w}) \to G/B$

(2) if $\underline{w}$ is rex, then $\mathrm{mult}: Y(\underline{w}) \to \overline{X_w}$ is a resolution of singularities (called Bott-Samelson res'n)

$\Rightarrow Y(\underline{w})$ is stratified by $\{y\le w\}$, $\begin{cases} \mathrm{mult}^{-1}(X_y) \to X_y \text{ is trivial bundle w/ fiber } F_y \\ \mathrm{mult}^{-1}(X_w) \xrightarrow{\sim} X_w \end{cases}$

Idea: linear dual : (f.d.) vect sp = Verdier dual : (constructible) sheaves

Defn: Let X be a strat'd space.

$\mathcal{F}$ is a sheaf (write $\mathcal{F} \in Sh(X)$) if ____.

$\mathcal{F} \in Sh(X)$ is constructible (write $\mathcal{F} \in Sh_c(X)$) if ____.

Fact Let $\mathcal{F} \in Sh(X)$, then

(a) $\forall x \in X$, one can define a vect sp $\mathcal{F}_x$ (called a stalk)

(b) If $\mathcal{F} \in Sh_c(X)$ then $\mathcal{F}_x$ is f.d. $\forall x \in X$.

Moreover, $\mathcal{F}_x = \mathcal{F}_y$ if $x, y \in X_\lambda$ (hence $\mathcal{F}_\lambda$ is well-defined)

Defn A complex of sheaves $\mathcal{F}$ is constructible if ____.
(Write $\mathcal{F} \in \mathcal{D}_c(X)$)

Fact Let $\mathcal{F} \in \mathcal{D}_c(X)$. Then $\forall i \in \mathbb{Z}$,

cohomology $h^i(\mathcal{F}) \in sh_c(X)$ s.t. $h^i(\mathcal{F})_\lambda \cong h^i(\mathcal{F}_\lambda)$

Rmk

sheaf ⇝ col of stalks

cplx of sheaves ⇝ table of stalks

We write

$$ToS(\mathcal{F}) = \text{incr}\downarrow \begin{array}{c|c} & \xrightarrow{\text{incr}} \\ & \cdots\ i \in \mathbb{Z}\ \cdots \\ \hline \vdots & \vdots \\ \lambda \in \Lambda & \cdots h^i(\mathcal{F})_\lambda \cdots \\ \vdots & \vdots \end{array}$$

In general, table of stalks doesn't determine a cplx of sheaves.

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ex Let $X = X_\lambda \sqcup X_\mu \sqcup X_\nu$ with $\nu < \mu < \lambda$. We have

(a) Constant sheaf on X (b) shifted const sheaf (for $i \in \mathbb{Z}$) (c) const. sheaf on $\overline{X_\mu}$

$$ToS(\underline{\mathbb{C}}_X) = \begin{array}{c|c} & 0 \\ \hline \lambda & \mathbb{C} \\ \mu & \mathbb{C} \\ \nu & \mathbb{C} \end{array} \qquad ToS(\underline{\mathbb{C}}_X[i]) = \begin{array}{c|c} & -i \\ \hline \lambda & \mathbb{C} \\ \mu & \mathbb{C} \\ \nu & \mathbb{C} \end{array} \qquad ToS(\underline{\mathbb{C}}_{\overline{X_\mu}}) = \begin{array}{c|c} & 0 \\ \hline \lambda & 0 \\ \mu & \mathbb{C} \\ \nu & \mathbb{C} \end{array}$$

Recall that $mult^{-1}(X_y) \cong X_y \times F_y \xrightarrow{Pr_1} X_y$ is a trivial bundle w/ fiber F_y

Note : $mult : Y(\underline{w}) \to \overline{X}_w$ ⇝ pushforward $mult_* : \mathcal{D}_c(Y(\underline{w})) \to \mathcal{D}_c(\overline{X}_w)$

Prop (proper base change thm : special case) If $f : Y \to X$ is proper, strat'd, (eg res'n of singularities) then

$$ToS(f_* \underline{\mathbb{C}}_Y) = \begin{array}{c|c} \lambda & i \in \mathbb{Z} \\ \hline \vdots & \vdots \\ y & \mathcal{H}^i(F_y) \\ \vdots & \vdots \end{array}$$ ie. det by coh(fiber)

ex $n = 3$, $s = s_1$, $t = s_2$, $\underline{w} = (s, t, s)$

$$mult^{-1}(X_y) \rightsquigarrow \begin{array}{c|c} & F_y \\ \hline sts & pt \\ ts & pt \\ st & pt \\ t & pt \\ s & \mathbb{C}P^1 \\ id & \mathbb{C}P^1 \end{array} \Rightarrow ToS(mult_* \underline{\mathbb{C}}_{Y(\underline{w})}) = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline sts & \mathbb{C} & & \\ ts & \mathbb{C} & & \\ st & \mathbb{C} & & \\ t & \mathbb{C} & & \\ \hline s & \mathbb{C} & 0 & \mathbb{C} \\ id & \mathbb{C} & 0 & \mathbb{C} \end{array}$$

Defn The Verdier duality is a contravariant autoequiv $\mathbb{D} : \mathcal{D}_c(X) \to \mathcal{D}_c(X)$ given by ____.

Fact (a) $\mathbb{D}(\mathcal{F}[1]) = (\mathbb{D}\mathcal{F})[-1]$

(b) If X is smooth then $\mathbb{D}\underline{\mathbb{C}}_X \simeq \underline{\mathbb{C}}_X[2\dim_{\mathbb{C}} X]$

$\overset{(a)}{\Rightarrow}$ $\underline{\mathbb{C}}_X[\dim_{\mathbb{C}} X]$ is self-dual

(c) If f is proper then $f_* \mathbb{D} = \mathbb{D} f_*$

$\Rightarrow$ f_* preserves self-duality

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ex $Y = Y(s, t, s)$ is smooth of dim 3

$\overset{(b)}{\Longrightarrow}$ $\underline{\mathbb{C}}_Y[3]$ is self-dual $\overset{(c)}{\Longrightarrow}$ So is $mult_* \underline{\mathbb{C}}_Y[3]$

Defn Let $d_\lambda = \dim_{\mathbb{C}} X_\lambda$. Call
$$\begin{array}{c|cc} & -d_\lambda & -d_\mu \\ \hline \vdots & \ddots & \\ \lambda & 0 & \\ \mu & & 0 \\ \vdots & & \ddots \end{array}$$
the twisted diagnol

A perverse sheaf is a complex of constructible sheaves $\mathcal{F} \in \mathcal{D}_c(X)$ s.t.

(i) table of stalks of $\mathcal{F}$ is supported on or below the twisted diag

(ii) ____ $\mathbb{D}\mathcal{F}$ ____

Thm (a) The cat'y Perv(X) of perverse sheaves is abelian.

(b) $\{\text{Simple objects in } Perv(X)\} = \{IC_\lambda \mid \lambda \in \Lambda\}$, (called <u>intersection cohomology sheaves</u>)

(c) Each IC_λ is uniquely (up to isom) specified by

(i) IC_λ is self-dual

(ii) $ToS(IC_\lambda) =$

Λ	$\cdots$	$-d_\lambda$	$\cdots$	$-d_\mu$	$\cdots$
$\nu \not< \lambda$	0	0	0	0	0
λ	0	$\mathbb{C}$	0	0	0
$\mu < \lambda$	0	$*$	$*$	0	0

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ex. Take parabolic $P = \begin{pmatrix} GL_2 & * \\ 0 & GL_2 \end{pmatrix} \subseteq G = GL_4(\mathbb{C}) \curvearrowright X = Gr(2,4)$
$\equiv \{W \subset \mathbb{C}^4 \mid \dim W = 2\}$

$X = \coprod P\text{-orbits} = X_0 \sqcup X_1 \sqcup X_2$

where $X_i = \{W \in X \mid \dim(W \cap \langle e_1, e_2\rangle) = i\}$

$\overline{X_i} = \{W \in X \mid \dim(W \cap \langle e_1, e_2\rangle) \geq i\}$

$\Rightarrow \overline{X_0} = X$, $\overline{X_1} = X_1 \sqcup X_2$ and $\overline{X_2} = X_2 = pt$

We construct a res'n of singularities $f : Y \to \overline{X_1}$ by setting

$Y = \{(0 \overset{1}{\subset} L \overset{1}{\subset} W \overset{2}{\subset} \mathbb{C}^4) \mid L \subset W \cap \langle e_1, e_2\rangle\} \overset{f}{\mapsto} W$

$f^{-1}(X_i) \cong X_i \times F_i$ has fiber

i	F_i
0	$\emptyset$
1	pt
2	$\mathbb{CP}^1$

$\Rightarrow ToS(f_* \underline{\mathbb{C}}_Y) =$

i	0	1	2
0	0	0	0
1	$\mathbb{C}$	0	0
2	$\mathbb{C}$	0	$\mathbb{C}$

One can show that Y is smooth of $\dim_{\mathbb{C}} 3 \Rightarrow \underline{\mathbb{C}}_Y[3]$ is self-dual

$\Rightarrow f_* \underline{\mathbb{C}}_Y[3]$ is self-dual $\rightsquigarrow$ $ToS =$

i	-3	-2	-1	0
0	0	0	0	0
1	$\boxed{\mathbb{C}}$	0	0	0
2	$\mathbb{C}$	0	$\mathbb{C}$	$\boxed{0}$

where $d_1 = 3$, $d_2 = 0$

$\Rightarrow f_* \underline{\mathbb{C}}_Y[3]$ is perverse, in fact $f_* \underline{\mathbb{C}}_Y[3] = IC_1$ by Thm(c)

<u>ex</u> $mult_* \underline{\mathbb{C}}_{Y(sts)}[3]$ is perverse but not IC $\rightsquigarrow$ $ToS =$

	-3	-2	-1	0
sts	$\boxed{\mathbb{C}}$			
ts	$\mathbb{C}$	$\boxed{0}$		
st	$\mathbb{C}$	$\boxed{0}$		
t	$\mathbb{C}$		$\boxed{0}$	
s	$\mathbb{C}$		$\boxed{\mathbb{C}}$	
id	$\mathbb{C}$		$\mathbb{C}$	$\boxed{0}$

<u>ex</u> If $\overline{X_\lambda}$ is smooth, $IC_\lambda = \underline{\mathbb{C}}_{\overline{X_\lambda}}[d_\lambda]$, e.g.

$\underline{\mathbb{C}}_{\overline{X_{sts}}}[3] = IC_{sts} \rightsquigarrow$

	-3	-2	-1	0
sts	$\boxed{\mathbb{C}}$			
ts	$\mathbb{C}$	$\boxed{0}$		
st	$\mathbb{C}$	$\boxed{0}$		
t	$\mathbb{C}$		$\boxed{0}$	
s	$\mathbb{C}$		$\boxed{0}$	
id	$\mathbb{C}$			$\boxed{0}$

$\underline{\mathbb{C}}_{\overline{X_s}}[1] = IC_s \rightsquigarrow$

	-3	-2	-1	0
sts				
ts				
st				
t				
s			$\boxed{\mathbb{C}}$	
id			$\mathbb{C}$	$\boxed{0}$

⚠ This would imply:
<u>$mult_* \underline{\mathbb{C}}_{Y(sts)}[3] \overset{?}{\cong} IC_{sts} \oplus IC_s$</u> if we can show it's semisimple

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Thm (Decomposition theorem, special case)

If $f : Y \to X$ is proper, Y is smooth then $f_* \underline{\mathbb{C}}_Y$ is semisimple

Cor If $f : Y \to \overline{X_\lambda}$ is a res'n of singularities, then

$$f_* \underline{\mathbb{C}}_Y[\dim_{\mathbb{C}} Y] = IC_\lambda \oplus \left(\bigoplus_{\mu < \lambda} IC_\mu^{\oplus m_\mu}\right)$$

$\Rightarrow$ an algorithm computing table of stalks of IC_λ recursively.

For $\mathcal{F} \in \mathcal{D}_c(X)$, define <u>character</u> $ch(\mathcal{F}) = \sum_{\substack{w \in W \\ k \in \mathbb{Z}}} (\dim h^{-\ell(w)-k}(\mathcal{F})_w) v^k \delta_w$

If $\mathcal{F} \in Semis(X) = \{\bigoplus_y IC_y^{\oplus m_y}\}$ then

Fact (a) $ch(\mathbb{D}\mathcal{F}) = \overline{ch(\mathcal{F})}$

(b) $\overline{ch(IC_w)} = ch(IC_w) = \delta_w + \sum h_{y,w} \delta_y$ for $h_{y,w} \in v\mathbb{Z}[v]$

$\xRightarrow{\text{uniqueness}}$ $ch(IC_w) = b_w \Rightarrow$ KL positivity conj when $W =$ Weyl grp

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Rmk Geometric cat'n of Hecke alg is available if we work with B-equivariant complexes of sheaves.

Fact (a) $\exists$ convolution functor $\star$ s.t. $(\mathrm{Semis}_B(X), \star)$ is monoidal,

$$IC_{id} \star \mathcal{F} = \mathcal{F} \star IC_{id} = \mathcal{F}$$

$$\mathbb{D}(\mathcal{F}) \star \mathbb{D}(\mathcal{G}) \cong \mathbb{D}(\mathcal{F} \star \mathcal{G})$$

$$\underbrace{IC_s \star IC_t \star \cdots \star IC_u}_{d \text{ terms}} \cong \mathrm{mult}_* \underline{\mathbb{C}}_{Y(s,t,\dots,u)}[d] \cong IC_{st\dots u} \oplus \Big(\bigoplus_{y \in st\dots u} IC_y^{\oplus m_y} \Big)$$

(b) $\mathrm{ch} : K_\oplus(\mathrm{Semis}_B(X)) \xrightarrow{\sim} \mathcal{H}$ is an isom of alg

$$[\mathcal{F}] \mapsto \mathrm{ch}(\mathcal{F})$$

<u>Summary</u>

W = Weyl

W = Coxeter

$$\begin{tikzcd}
\mathrm{Mod}(\mathfrak{g}) & & & \\
\cup & \mathcal{D}_c(G/B) & & \\
\text{Cat'y } \mathcal{O} & \cup & & \\
\cup & \mathrm{Perv}(G/B) \arrow[rr, "\text{hypercoh. functor}"] & & R\text{-gmod} \\
\text{principal block} & \cup & \text{Hecke alg} & \cup \\
\mathcal{O}_0 \arrow[r, Leftrightarrow] & \mathrm{Semis}_B(G/B) \arrow[r, Leftrightarrow] & \mathcal{H}(W) \arrow[r, Leftrightarrow] & \mathrm{SBim}(W)
\end{tikzcd}$$

$$\begin{tikzcd}
\text{simple} & \text{IC sheaf} & \text{dual KL basis} & \text{indecom.} \\
L_w \arrow[r, leftrightarrow] & IC_w \arrow[r, leftrightarrow] & b_w \arrow[r, leftrightarrow] & B_w \\
\text{Verma} & & \text{std basis} & \\
\Delta_w \arrow[rr, leftrightarrow] & & \delta_w & \\
 & \mathrm{mult}_* \underline{\mathbb{C}}_{Y(st..u)} \arrow[r, leftrightarrow] & b_s b_t \cdots b_u \arrow[r, leftrightarrow] & BS(s,t,\dots,u)
\end{tikzcd}$$

Detailed computation of F_y for $n=3$, $\underline{w} = (s,t,s)$

- <u>$y = sts$</u>

res'n of sing. $\Rightarrow$ $\underset{\overset{\parallel}{X_y \times F_y}}{\mathrm{mult}^{-1}(X_y)} \xrightarrow{\sim} X_y$ hence $F_y = pt$

- <u>$y = t$</u>

$$X_t = \left\{ V_\bullet \;\middle|\; f^V = r^{\begin{pmatrix}1&&\\&0&1\\&1&0\end{pmatrix}} = \begin{pmatrix}1&1&1\\1&1&2\\1&2&3\end{pmatrix} \right\} \quad \text{i.e.} \quad \begin{array}{ccc} 0 \subset & \langle e_1 \rangle \subset & \langle e_1, e_2 \rangle \\ \cap & & \\ V_1 & 1 & 1 \\ \cap & & \\ V_2 & 1 & 1 \end{array}$$

$$= \{ (0 \subset \langle e_1 \rangle \subset \langle e_1, \lambda e_2 + \mu e_3 \rangle \subset \mathbb{C}^3 \mid \mu \neq 0 \}$$

$\leftarrow$ otw $\dim(V_2 \cap \langle e_1, e_2 \rangle) \neq 1$

$$\mathrm{mult}^{-1}(X_t) = \left\{ \left(V_\bullet^{std}, \begin{matrix} 0 \\ \cap \\ V_1^1 \\ \cap \\ \langle e_1, e_2 \rangle \\ \cap \\ \mathbb{C}^3 \end{matrix}, \begin{matrix} 0 \\ \cap \\ V_1^1 = V_1^2 \\ \cap \\ V_2^2 \\ \cap \\ \mathbb{C}^3 \end{matrix}, \begin{matrix} 0 \\ \cap \\ V_1^3 \\ \cap \\ V_2^2 = V_2^3 \\ \cap \\ \mathbb{C}^3 \end{matrix} \right) \;\middle|\; \begin{matrix} V_2^2 = V_2^3 = \langle e_1, \lambda e_2 + e_3 \rangle \text{ for some } \lambda \in \mathbb{C} \\ \cup \\ V_1^1 = \langle a e_1 + b e_2 \rangle = \langle e_1 \rangle \text{ is fixed} \\ V_1^3 \text{ is fixed} \end{matrix} \right\}$$

$$\cong X_t \times F_t \quad \text{with} \quad F_t = pt$$

- <u>$y = id$</u>

$$X_{id} = \left\{ V_\bullet \;\middle|\; f^V = \begin{pmatrix}1&1&1\\1&2&2\\1&2&3\end{pmatrix} \right\} \text{ i.e. } \begin{array}{ccc} 0 \subset & \langle e_1 \rangle \subset & \langle e_1, e_2 \rangle \\ \cap & & \\ V_1 & 1 & 1 \\ \cap & & \\ V_2 & 1 & 2 \end{array}$$

$$= \{ V_\bullet^{std} \}$$

$$\mathrm{mult}^{-1}(X_{id}) = \left\{ \left(V_\bullet^{std}, \begin{matrix} 0 \\ \cap \\ V_1^1 \\ \cap \\ \langle e_1, e_2 \rangle \\ \cap \\ \mathbb{C}^3 \end{matrix}, \begin{matrix} 0 \\ \cap \\ V_1^1 = V_1^2 \\ \cap \\ \langle e_1, e_2 \rangle \\ \cap \\ \mathbb{C}^3 \end{matrix}, \begin{matrix} 0 \\ \cap \\ V_1^3 \\ \cap \\ \langle e_1, e_2 \rangle \\ \cap \\ \mathbb{C}^3 \end{matrix} \right) \;\middle|\; V_1^3 \text{ is fixed} \right\} \cong \mathbb{C}P^1$$

$$\cong X_{id} \times F_{id} \quad \text{with} \quad F_{id} = \mathbb{C}P^1$$