1. Historical background

Soergel first proved Soergel's conjecture for finite Weyl groups using results the results of Kazhdan and Lusztig that identifies the Kazhdan-Lusztig polynomial as numerical data derived from intersection complex on Schubert varieties. The Kazhdan-Lusztig work in turn is built on the theory on perverse sheaves, in particular the Beilinson-Bernstein-Deligne-Gabber Decomposition Theorem that was established around the same time. The BBDG Decomposition theorem relies on the machinery of Weil II (Deligne's second proof of the Weil Conjecture) which is one of the most important results ever in algebraic and arithmetic geometry. The BBDG proof was done by first doing it over a finite field, then over finitely generated algebra over $\mathbb{Z}$, and then pass to complex geometry if desired. Later Morihiko Saito was able to work out related structure ("mixed Hodge structure") over $\mathbb{C}$ and gave an analytic proof of the Decomposition Theorem.

When Soergel worked on the algebraic approach, he couldn't get around Soergel's conjecture, and at least to Cheng-Chiang's naive eyes it feels like the use of Decomposition Theorem, the technical heart of the theory of perverse sheaves, was sort of an unavoidable difficulty at the time. Yet around the same time, de Cataldo and Migliorini gave a more "elementary" proof[1] of the Decomposition Theorem by extending the classical Hodge theory. It was this simplification of Decomposition Theorem that inspired Elias and Williamson in their proof of the Soergel conjecture, which is our goal this and next week.

2. Start of proof

Fix (W, S) any Coxeter system. The presented result works in the characteristic 0 setting for which we may assume our ground field $k = \mathbb{R}$. We fix a Kac-Moody realization $W \curvearrowright \mathfrak{h}$ over $\mathbb{R}$; when $|W|$ is finite this is the usual one, and for general W this is some construction of Soergel that behaves similarly, see §5.6, 5.7. We write $R = \mathrm{Sym}_{\mathbb{R}}(\mathfrak{h}^*)$ the ring of polynomial functions on $\mathfrak{h}$.

Let $\mathcal{H}$ be the Hecke algebra associated to (W, S) over $\mathbb{Z}[v, v^{-1}]$, $[\mathbb{S}\mathrm{Bim}]_\oplus$ be the split Grothendieck $\mathbb{Z}[v, v^{-1}]$-algebra of the additive graded category $\mathbb{S}\mathrm{Bim}$ of Soergel (R-)bimodules. Recall that in Soergel categorification theorem we have two mutually inverse algebra homomorphisms $c : \mathcal{H} \to [\mathbb{S}\mathrm{Bim}]_\oplus$ and $\mathrm{ch} : [\mathbb{S}\mathrm{Bim}]_\oplus \to \mathcal{H}$. The former is defined by $c(b_s) = B_s$. The latter is defined by the assertion that for any Soergel bimodule $B \in \mathbb{S}\mathrm{Bim}$ we have a R-bimodule filtration

$$0 = B^m \subset B^{m-1} \subset ... \subset B^0 = B$$

such that $B^i/B^{i-1} \cong R_{x_i}^{\oplus h_{x_i}(B)}$ where $x_i \not> x_j$ whenever $i < j$ under Bruhat order. One then puts $\mathrm{ch}(B) = \sum_{i=1}^{m-1} h_{x_i}(B)\delta_{x_i}$. The theorem of Elias and Williamson is

Theorem 1. *(Soergel's conjecture)* $\mathrm{ch}(B_x) = b_x$ *is the Kazhdan-Lusztig basis for any* $x \in W$.

This immediately proves the Kazhdan-Lusztig positivity conjecture. Using Soergel hom formula reviewed in the Appendix, we have the important consequence that

Corollary 2. $\underline{\mathrm{rank}}\,\mathrm{Hom}^*(B_x, B_y) = (b_x, b_y) \in \delta_{xy} + v\mathbb{Z}[v]$ *for* $x, y \in W$.

Inspired by the use of Decomposition Theorem, let us consider the following geometry background: Recall that for a compact complex manifold X we have Poincaré duality $H^{*+n}(X) \cong H^{*-n}(X)^\vee$. In the case when our W is a finite Weyl group and $X = Y(\underline{x})$ is a Bott-Samelson variety, the duality works in an equivariant setting (that $Y(\underline{x})$ is equivariantly formal), resulting in the duality $H_T^{*+\ell(\underline{x})}(Y(\underline{x})) \cong \mathrm{Hom}_R(H_T^{*-\ell(\underline{x})}(Y(\underline{x})), R)$; let us

[1]Technically, de Cataldo and Migliorini proved a more restrictive version of the Decomposition Theorem. That version is nevertheless sufficient for Soergel's use on Schubert varieties.

discuss this in detail. From now on all R-bimodules are assumed to be graded and finite free as a right R-module, and we will omit the adjectives. For any R-bimodule B , denote by

$$\mathbb{D}(B) := \{\phi : B \to R \mid \phi(br) = \phi(b)r,\ \forall b \in B,\ r \in R\}$$

equipped with the natural notion of degree and an R-bimodule structure that $(r_1.\phi.r_2)(b) = \phi(r_1 b r_2)$ (note: the one written in the book has typo). Note that we did not require maps in $\mathbb{D}(B)$ to be left-R-equivariant. It is easy to see that as B is assumed finite free as a right R-module, so is $\mathbb{D}(B)$. We now claim that

Lemma 3. *(Exercise 18.12, 18.13) We have canonical isomorphism $\mathbb{D}(B_s) = B_s$. Moreover, for any R-bimodule B we have $\mathbb{D}(B \otimes_R B_s) = \mathbb{D}(B) \otimes_R B_s$.*

The lemma gives us the general case of equivariant duality that we mentioned:

Corollary 4. *For any Bott-Samelson bimodule $BS(\underline{x})$ we have canonical isomorphism $\mathbb{D}(BS(\underline{x})) = BS(\underline{x})$.*

This consequently implies

Corollary 5. *(Exercise 18.15) For any Soergel bimodule B we have $\mathbb{D}(B) \cong B$.*

Proof. We have seen that any Soergel bimodule B is uniquely characterized by a unique element $x \in W$ in that for any reduced expression $\underline{x}$ of x, B appears in $BS(\underline{x})$ but not in $BS(\underline{x}')$ for reduced expression $\underline{x}'$ with $x' < x$. Since all Soergel bimodules are finite free as right R-modules, we have that $\mathbb{D}(B)$ also appears in $\mathbb{D}(BS(\underline{x}))$ but not in $\mathbb{D}(BS(\underline{x}'))$ with $x' < x$. But $\mathbb{D}(BS(\underline{x})) = BS(\underline{x})$, and thus $\mathbb{D}(B) \cong B$ by its characterizing property. □

For our R-bimodules B we evidently have $\mathbb{D}(\mathbb{D}(B)) = B$ canonically. Thus $\mathbb{D}$ induces a ring involution on $[\mathbb{S}\mathrm{Bim}]_\oplus$ that fixes B_s. Let us also note that we have immediately by definition that $\mathbb{D}(B(1)) = B(-1)$. Hence $\mathbb{D}$ induces a ring involution on $\mathcal{H}$ that fixes b_s but sends v to v^{-1}, i.e. $\mathbb{D}$ exactly induces the Kazhdan-Lusztig involution, and this immediately proves that $\mathrm{ch}(B_x)$ is self-dual.

Remark 6. The contravariant equivalence (duality) of category $\mathbb{D}$ can also be viewed, in the diagrammatic Hecke category, as the "upside-down functor."

In the de Cataldo-Migliorini proof of the Decomposition Theorem, the semi-small case is highlighted. However, the Bott-Samelson resolution in general is not semi-small; see Example 7 below.

Example 7. Suppose $W = S_4 = \langle s_1, s_2, s_3\rangle$. We write $b_1 = b_{s_1}$, etc. We have $b_1b_3b_2b_1b_3 = b_{13213} + (v + v^{-1})b_{13}$. This corresponds to the fact that the resolution from the Bott-Samelson variety $BS(13213)$ is not semismall above the strata $Y(13)$; the fiber is $\mathbb{CP}^1 \times \mathbb{CP}^1$ of dimension 2, while 2×2 is one more than $\ell(13213) - \ell(13) = 3$, implying the degree 1 at the coefficient $(v + v^{-1})$.

This will create some difficulty, and the way around it goes back to something basic in the inductive calculation of the Kazhdan-Lusztig basis:

Lemma 8. *(Theorem 3.27) Let $x \in W$ and $s \in S$ be such that $y := xs > x$. We have*

$$b_x b_s = b_{xs} + \sum_{y<x,\ ys<y} \mu(y, x) b_y,$$

where $\mu(y, x) \in \mathbb{Z}$ (in fact $\in \mathbb{Z}_{\geq 0}$) is the coefficient of v in the Kazhdan-Lusztig polynomial $h_{y,x}$.

So if instead of using the whole Bott-Samelson bimodule, we work with B_yB_s inductively, then philosophically we are in a "semi-small" setup which should be easier to work with. Now we can vaguely describe the Elias-Williamson plan: we will prove Soergel's conjecture $\mathrm{ch}(B_x) = b_x$ inductively in Bruhat order. Denote by $\mathrm{S}(x)$ the assertion $\mathrm{ch}(B_x) = b_x$. To prove $\mathrm{S}(x)$ inductively, it suffices to prove for x with $y = xs > x$ that

$$B_xB_s = B_{xs} \oplus \sum_{y<x,\ ys<y} B_y^{\oplus\mu(y,x)}. \tag{2.1}$$

That is, it suffices to understand how B_y, $y < xs$ can be embedded into B_xB_s. We remark that even if we assume Soergel's conjecture (and the Kazhdan-Lusztig positivity conjecture) it is *a priori* not clear if (2.1) holds, but that was part of the geometric insight[2].

Now naively one will expect that the number of times $B_y(i)$ appears in B_xB_s is the dimension of the space of degree i bimodule morphisms $\mathrm{Hom}^i(B_y, B_xB_s)$. Assuming this, as well as $\mathrm{S}(< xs)$ (i.e. assertion $\mathrm{S}(y)$ for all $y < xs$), we have by the Soergel hom formula that

$$\underline{\mathrm{rank}}\,\mathrm{Hom}^*(B_y, B_xB_s) = (\mathrm{ch}(B_y), \mathrm{ch}(B_x)\,\mathrm{ch}(B_s)) = (b_y, b_xb_s) \in \mu(y,x) + v\mathbb{Z}[v].$$

In particular, for any $y < xs$ we have that B_y appears (without shift) in B_xB_s as many times as $\mu(y,x)$ if $y > ys$ and 0 times if $y < ys$, and $B_y(-i)$ never appears in B_xB_s for $i > 0$. By duality $\mathbb{D}$, $B_y(i)$ for $i > 0$ must also never occur (this is very similar to the decomposition theorem computation). Hence we have proved (2.1) and Soergel's conjecture.

Unfortunately, it is not for granted that the number of times $B_y(i)$ appears in B_xB_s is equal to $\dim_{\mathbb{R}} \mathrm{Hom}^i(B_y, B_xB_s)$. Let us think about this question abstractly: suppose we have an $\mathbb{R}$-linear additive category with objects X and Y (typically X is indecomposable). To say that X appears in Y m times is to find morphisms $i_1, ..., i_m : X \to Y$ and $p_1, ..., p_m : Y \to X$ such that $p_j \circ i_k = \delta_{jk}\mathrm{id}_X \in \mathrm{End}(X)$. Assume for the moment that $\mathrm{End}(X) \cong \mathbb{R}$. Then this is saying that the pairing

$$\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,X) \to \mathrm{End}(X) \cong \mathbb{R},\ (i,p) \mapsto p \circ i$$

has rank m. Now in our case $X = B_y$, and what is $\mathrm{End}(B_y)$? We have seen the trick that assuming $\mathrm{S}(y)$, we have $\mathrm{End}^*(B_y) = (\mathrm{ch}(B_y), \mathrm{ch}(B_y)) \in 1 + v\mathbb{Z}[v]$ and thus $\mathrm{End}^0(B_y) \cong \mathbb{R}$. For the non-zero degree case we have less control, but at least we know $\mathrm{End}^*(B_y)$ is a graded algebra whose "completion" is a local algebra with maximal ideal $\mathrm{End}^{>0}(B_y)$. One may work out some non-commutative ring theory and get to that

Proposition 9. *(Prop. 11.69, Cor. 11.71) Assume X and Y are two objects in a graded additive category so that $\mathrm{End}(X)$ is graded local with graded maximal (Jacobson) ideal $\mathfrak{m}_X$, so that $\mathrm{End}(X)/\mathfrak{m}_X$ is a division algebra. Then X appears in Y as many times as the graded rank of the pairing*

$$\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,X) \to \mathrm{End}(X) \twoheadrightarrow \mathrm{End}(X)/\mathfrak{m}_X.$$

Thanks to Prop. 9, to inductively prove Soergel's conjecture it suffices to prove that the pairing

$$\mathrm{Hom}^0(B_y, B_xB_s) \times \mathrm{Hom}^0(B_xB_s, B_y) \to \mathrm{End}^0(B_y) \cong \mathbb{R} \tag{2.2}$$

has rank equal to $\dim_{\mathbb{R}} \mathrm{Hom}^0(B_y, B_xB_s)$, i.e. (2.2) is non-degenerate. This was realized by Soergel, but he couldn't prove it. Yet miraculously, similar non-degeneracy also appears at the technical heart of the de Cataldo-Migliorini work via the **local intersection form**.

To turn a pairing into a form, we note that by duality we have $\mathrm{Hom}^0(B_xB_s, B_y) = \mathrm{Hom}^0(\mathbb{D}(B_y), \mathbb{D}(B_xB_s)) \cong \mathrm{Hom}^0(B_y, B_xB_s)$. So (2.2) becomes a form on $\mathrm{Hom}^0(B_z, B_xB_s)$

[2] Actually Cheng-Chiang hasn't worked out how to rigorously prove this geometrically when W is a finite Weyl group, but I believe it's not hard.

- not yet! The isomorphism $B_y \cong \mathbb{D}(B_y)$ is only unique up to (non-zero) scalar. So the form we get is also only unique up to scalar. At this moment, it feels fine because the non-degeneracy of the form is insensitive to the scalar. The question is: how do we proceed? It turns out that the strategy of Elias-Williamson, inspired by that of de Cataldo-Migliorini, is to prove that the form is $(-1)^{\ell(xs)-\ell(y)}$-definite via Hodge theory.

A bit more preparation: while our Soergel bimodules originate from equivariant (hyper-)cohomology, classical Hodge theory is applied to non-equivariant cohomology. That is, instead of Soergel bimodules B we want to look at Soergel modules $\bar{B} := B \otimes_R \mathbb{R}$ where recall that the map $R \twoheadrightarrow \mathbb{R}$ is given by $R/R_+ = \mathbb{R}$. Define for any left R-module $\bar{B}$ (always assumed to be finite over $\mathbb{R}$) that $\mathbb{D}(\bar{B}) := \mathrm{Hom}_{\mathbb{R}}(\bar{B}, \mathbb{R})$ as a left R-module. For any R-bimodule B we have $\mathbb{D}(B \otimes_R \mathbb{R}) = \mathbb{D}(B) \otimes_R \mathbb{R}$. In particular, we have $\mathbb{D}(\bar{B}) \cong \bar{B}$ for any Soergel module $\bar{B}$. We also note that there is yet another way to view the duality functor; a R-bimodule morphism $B \to \mathbb{D}(B)$ is a degree 0 bilinear form $(\cdot,\cdot) : B \times B \to \mathbb{R}$ such that $(rb_1, b_2) = (b_1, rb_2)$ and $(b_1 r, b_2) = (b_1, b_2 r) = (b_1, b_2)r$; let us call such bilinear forms **invariant forms**. The morphism is an isomorphisms iff the invariant form is non-degenerate. An invariant form also induces an invariant form from $\bar{B} \times \bar{B}$ to $\mathbb{R}$.

We remark that the canonical isomorphism $\mathbb{D}(BS(\underline{x})) = BS(\underline{x})$ gives us a canonical form on $BS(\underline{x})$ and on $\overline{BS(\underline{x})}$. In fact, we know such a form! It is the form $(a, b) = \mathrm{Tr}(ab) \in R$, the coefficient of c_{top} when ab is expressed in terms of 01-basis (with right R-multiplication), defined in Definition 12.13. The form was called **global intersection form** for a reason that was unclear, and the reason is that it matches with what de Cataldo-Migliorini called global intersection form in their work! The trick of Elias-Williamson inspired by that of de Cataldo-Migliorini is to embed the form on $\mathrm{Hom}^0(B_y, B_x B_s)$ to the global intersection form on a subspace of $\overline{B_x B_s}$, and use that the latter is definite. In the next section we will describe the Hodge-theory we want (thanks to de Cataldo-Migliorini) for $\overline{B_x B_s}$.

3. Hodge theory: preparation and motivation

Fix V a finite-dimensional graded vector space over $\mathbb{R}$.

Lemma 10. *(Exercise 17.7, 17.11) Let $e : H \to H$ be a degree 2 operator. TFAE*

1. *Its power $e^i : H^{-i} \to H^i$ is an isomorphism for any $i \geq 0$.*
2. *It can be completed into an action of $\mathfrak{sl}_2(k) = \langle e, h, f \rangle$ such that $h.v = nv$ for any $v \in H^n$.*

In this case, the space $P^{-i} := H^{-i} \cap \ker(e^{i+1}|_{H^{-i}} : H^{-i} \to H^{i+2})$ is the lowest weight subspace in H^{-i} for any $i \geq 0$.

The space P^{-i} is called the **primitive subspace**. In Hodge theory, H should be the shifted cohomology group of a compact Kähler manifold and e the cupping with a strictly positive Kähler class (this is the so-called Hard Lefschetz Theorem) or its generalization. The cohomology group has a Poincaré pairing, which motivates:

Definition 11. *Suppose H is a finite-dimensional graded vector space over $\mathbb{R}$ equipped with $(\cdot,\cdot) : H \times H \to \mathbb{R}$ a symmetric non-degenerate graded bilinear form. We say $L : H \to H$ of degree 2 satisfies* **hard Lefschetz** *if it is self-adjoint and satisfies the condition for e in Lemma 10.*

The definition is motivated that L satisfies hard Lefschetz iff $(\cdot,\cdot)_L^{-i} : H^{-i} \times H^{-i} \to \mathbb{R}$ defined by $(a, b)_L^{-i} := (a, L^i b)$ is non-degenerate. Now suppose furthermore $m \in \mathbb{Z}$ is the minimal with $H^m \neq 0$ such that $H^{m+2j+1} = 0$ for all $j \geq 0$. (This is motivated when H is the shifted cohomology of a dimension m Bott-Samelson variety or the shifted intersection cohomology of a dimension m Schubert variety.) In this case we put

Definition 12. *The triple* $(H, (\cdot,\cdot), L)$ *is said to satisfies the* **Hodge-Riemann bilinear relation** *at degree* i *if* $(\cdot,\cdot)^i_L|_{P^i}$ *is* $(-1)^{\frac{i-m}{2}}$*-definite. It is said to satisfies the Hodge-Riemann bilinear relation if it does so in all degrees.*

Remark 13. Suppose $(H, (\cdot,\cdot), L)$ satisfies Hodge-Riemann, and V is the standard representation of $\mathfrak{sl}_2$ with the standard invariant form on V; in fact $V \cong \overline{B_s}$. Then $(H \otimes_{\mathbb{R}} V, (\cdot,\cdot), L+e)$ also satisfies Hodge-Riemann; it's elementary to show so, but is not totally obvious and is a good exercise. (As Shun-Jen remarked, this is also true if V is anything that satisfies Hodge-Riemann.) In our case $H = \overline{B_x}$, while we **DON't** have that $\overline{B_xB_s} \ncong \overline{B_x} \otimes_{\mathbb{R}} \overline{B_s}$ as left R-module. However, the key will be that a limit certain family of operators on $\overline{B_xB_s}$ will behave as $L_0 \curvearrowright \overline{B_x} \otimes \overline{B_s}$, thus giving the Hodge-Riemann bilinear relation on $\overline{B_xB_s}$ that we will desire in Proposition 22.

If the triple satisfies the Hodge-Riemann bilinear relations, then L apparently satisfies hard Lefschetz. This relation (or inequality) is the same as the Hodge-Riemann bilinear relation for the cohomology of compact Kähler manifolds in the case when $h^{p,q} = 0$ for $p \neq q$; examples of such manifolds include flag varieties and Bott-Samelson varieties.

In our case, we want to look at the situation when $H = \overline{B_xB_s}$ and next $H = \overline{B_{xs}}$, and that $(\cdot,\cdot)$ is the invariant form coming from a choice of $H \cong \mathbb{D}(H)$; we'll later claim that there is a natural choice of the isomorphism up to positive scalars, for sign is definitely important here! We have yet to explain L. In the theory of Kac-Moody realization, we have linearly independent coroots $\{\alpha_s^\vee \in \mathfrak{h}\}_{s\in S}$ (but not necessary independent roots) so that there always exist $\rho \in \mathfrak{h}^*$ such that $\langle \alpha_s^\vee, \rho\rangle > 0$ for all $s \in S$. We fix such a ρ once and for all; this is the same as fixing an ample line bundle or a positive Kähler class on G/B if W is a finite Weyl group for (G, B). Anyhow, denote by L_0 the left multiplication by ρ on any Soergel module. It is indeed self-adjoint as our form is an invariant form. This L_0 will be our Lefschetz operator L.

Remark 14. Let us remark in advance that the choice $H = \overline{BS(\underline{x})}$ will not satisfy the Hodge-Riemann bilinear relation; in fact, our result will eventually show that for Bott-Samelson modules, our L satisfies hard-Lefschetz iff "The Bott-Samelson resolution is semismall" in the sense of Example 7 even if W is not a finite Weyl group. This gives another reason to work with B_xB_s.

Let us inspect the minimal degree m for $\overline{B_x}$:

Lemma 15. *(Exercise 18.16) For* $\underline{x}$ *reduced and any decomposition of* $BS(\underline{x})$*, the element* $c_{bot} \in BS(\underline{x})$ *lives in the indecomposable summand* B_x*. Consequently, the minimal degree in* B_x *and* $\overline{B_x}$ *is* $-\ell(x)$*.*

Proof. Since the element c_{bot} is up to constant the unique element of degree $-\ell(x)$ in $BS(\underline{x})$, it must lives inside some single indecomposable summand. On the other hand, c_{bot} has non-trivial projection to the twisted Q-bimodule Q_x (this can be proved inductively on length). Since all other summands comes from $BS(\underline{y})$ with $\underline{y} < \underline{x}$ and have trivial image in Q_x, c_{bot} cannot live in any summand other than B_x, hence the result. We remark that once we know in some decomposition c_{bot} lives in B_x, the same result must hold for any embedding $B_x \hookrightarrow BS(\underline{x})$ (not necessarily as a summand). □

Proposition 16. *(Proposition 12.29) For* $\underline{x}$ *reduced and any embedding* $B_x \hookrightarrow BS(\underline{x})$*, denote by* $(\cdot,\cdot)_x$ *the restriction to* $\overline{B_x}$ *of the global intersection form on* $\overline{BS(\underline{x})}$*. Then* $(\overline{B_x}, (\cdot,\cdot)_x, L_0)$ *satisfies the Hodge-Riemann bilinear relation at the minimal degree* $-\ell(x)$*.*

Proof. The assertion is that the coefficient of the c_{top}-term in $\rho^{\ell(x)}c_{bot}$ under the 01-basis is > 0. That in turn is an application of an inductive calculation using polynomial forcing, which we refer to our textbook. $\square$

If we assume $\mathrm{S}(x)$, then $\underline{\mathrm{rank}}\,\mathrm{Hom}^*(B_x, \mathbb{D}(B_x)) = (b_x, b_x) \in 1+v\mathbb{Z}[v]$ and $\dim_{\mathbb{R}} \mathrm{Hom}^0(B_x, \mathbb{D}(B_x)) = 1$, i.e. the dimension of invariant forms is 1-dimensional. Proposition 16 then tells us that the restriction of the global intersection form to B_x all lie in a positive ray. Let us highlight this result:

Proposition 17. *Assuming* $\mathrm{S}(x)$*, the restriction of the global intersection form on* $BS(\underline{x})$ *to* B_x *is well-defined up to a strictly positive scalar, independent of the choice of* $\underline{x}$ *and the choice of embedding* $B_x \hookrightarrow BS(\underline{x})$*.*

We shall continue to denote this important invariant form by $(\cdot,\cdot)_x : B_x \times B_x \to R$, understood to be well-defined up to a strictly positive scalar. This form gives an isomorphism $B_x \xrightarrow{\sim} \mathbb{D}(B_x)$ unique up to positive scalar, and thus also $B_xB_s \xrightarrow{\sim} \mathbb{D}(B_xB_s)$. The last isomorphism can also be read as an invariant form on B_xB_s which we denote by $(\cdot,\cdot)_{x,s}$. This allows us to revisit (2.2): adjointness gives us an isomorphism

$$\mathrm{Hom}^0(B_xB_s, B_y) = \mathrm{Hom}^0(\mathbb{D}(B_y), \mathbb{D}(B_xB_s)) \xrightarrow{\sim} \mathrm{Hom}^0(B_y, B_xB_s)$$

where for the latter arrow we used the isomorphisms given respectively by $(\cdot,\cdot)_y$ and $(\cdot,\cdot)_{x,s}$, assuming both $\mathrm{S}(y)$ and $\mathrm{S}(x)$. That is, we may turn (2.2) into a form

$$(\cdot,\cdot)_y^{x,s} = \mathrm{Hom}^0(B_y, B_xB_s) \times \mathrm{Hom}^0(B_y, B_xB_s) \to \mathrm{End}^0(B_y) = \mathbb{R}$$

that is well-defined up to positive scalar. This is the **local intersection form**. We have explained in (2.2) that the non-degeneracy of this form implies $\mathrm{S}(xs)$ and thus is good for the inductive proof. The plan is to prove that $(\cdot,\cdot)_y^{x,s}$ is $(-1)^{\ell(xs)-\ell(y)}$-definite and thus automatically non-degenerate. Assume $\mathrm{S}(y)$, $\mathrm{S}(x)$ and $x < xs$. Consider a map $\iota : \mathrm{Hom}^0(B_y, B_xB_s) \to \overline{B_xB_s}^{-\ell(y)}$ given by $\iota(\phi) = \overline{\phi(c_{bot})}$. Equipping $\overline{B_xB_s}$ with the form $(\cdot,\cdot)_{x,s}$, we have

Proposition 18. *The image of* ι *lies in* $\ker L_0^{\ell(y)+1}$*, the map* ι *is injective, and*

$$(\iota(\phi_1), \iota(\phi_2))_{L_0}^{-\ell(y)} = (\phi_1, \phi_2)_y^{x,s} \cdot (\rho^{\ell(y)}c_{bot}, c_{bot})_y.$$

In particular, $(\cdot,\cdot)_y^{x,s}$ *is* $(-1)^{\ell(xs)-\ell(y)}$*-definite if* $(\overline{B_xB_s}, (\cdot,\cdot)_{x,s}, L_0)$ *satisfies the Hodge-Riemann bilinear relation.*

Proof. The image of ι lies in $\ker L_0^{\ell(y)+1}$ because $\rho^{\ell(y)+1}\overline{\phi} = 0$ for any $\phi \in \mathrm{Hom}^0(B_y, B_xB_s)$ as $\rho^{\ell(y)+1}$ annihilates $\overline{B_y}$. To prove the displayed identity, we argue as

$$(\iota(\phi_1), \iota(\phi_2))_{L_0}^{-\ell(y))} = (\rho^{\ell(y)}\overline{\phi_1(c_{bot})}, \overline{\phi_2(c_{bot})})_{x,s} = (\rho^{\ell(y)}\phi_2^*\phi_1(c_{bot}), c_{bot})_y$$

where $\phi_2^* \in \mathrm{Hom}^0(B_xB_s, B_y)$ is the adjoint, and the $\overline{(\cdot)}$ can be dropped for degree reason. But then by definition of $(\cdot,\cdot)_y^{x,s}$, the last item is equal to

$$(\rho^{\ell(y)}\phi_2^*\phi_1(c_{bot}), c_{bot})_y = (\phi_1, \phi_2)_y^{x,s}(\rho^{\ell(y)}c_{bot}, c_{bot})_y.$$

Hence we have proved the displayed identity. The last sentence follows from Proposition 16 and the injectivity of ι. To prove the injectivity of ι, assume that $\phi \in \mathrm{Hom}^0(B_y, B_xB_s)$ is such that $\overline{\phi(c_{bot})} = 0$. We may choose reduced expressions $\underline{y}$ and $\underline{x}$, and view ϕ as a composition $\phi : B_y \overset{i}{\hookrightarrow} BS(\underline{y}) \twoheadrightarrow B_y \xrightarrow{\phi} B_xB_s \hookrightarrow BS(\underline{x}s) \overset{p}{\twoheadrightarrow} B_xB_s$. That is, we make it factoring through a morphism $\tilde{\phi} : BS(\underline{y}) \to BS(\underline{x}s)$ so that $\phi = p \circ \tilde{\phi} \circ i$ and we have the tool from diagrammatic Hecke category to study $\tilde{\phi}$. We know $\tilde{\phi}$ is an R-linear combination of double leaves, of degree 0. Each of the double leaf factors through some $\underline{z}$ (i.e. $BS(\underline{z})$) for some $\underline{z} \leq \underline{y}$. We claim that

Lemma 19. *If a double leaf* $\varphi \in \mathrm{Hom}^*(BS(\underline{y}), BS(\underline{xs}))$ *of degree* ≤ 0 *(resp. degree* < 0*) factors through* $BS(\underline{z})$ *with* $\underline{z} < \underline{y}$ *(resp.* $\underline{z} = \underline{y}$*), then* $p \circ \varphi \circ i = 0$.

Proof. The composition $p \circ \varphi \circ i$ is itself a sum of compositions $B_y \to B_{z'} \to B_x B_s$ of two graded morphisms for various $z' \leq z$. By Soergel hom formula and S($< xs$), $\mathrm{Hom}^i(B_y, B_{z'}) = 0$ if $z' < y$ and $i \leq 0$ and $z' < y$, or if $z' = y$ and $i = 0$. Similarly $\mathrm{Hom}^i(B_{z'}, B_x B_s) = 0$ if $i < 0$. This proves the lemma. □

With the lemma we may drop those double leaves that factor through $\underline{z} \neq \underline{y}$ and we are left with only double leaves of the form $\varphi_f r_f$ where $\varphi_f = \overline{LL}_{\bar{x}s,f} \circ LL_{\bar{y},\overline{11\ldots1}} = \overline{LL}_{\bar{x}s,f}$ and $r_f \in R$, i.e. $\tilde{\phi} = \sum \varphi_f r_f$ where $f \subset \bar{x}s$ runs over subexpressions such that $(\bar{x}s)^f = y$. Theorem 12.15 asserts that $\varphi_f(c_{bot})$ for various $f \subset \bar{x}s$ is a subset of some right R-basis for $BS(\underline{x}s)$. Thus $\overline{\phi(c_{bot})} = 0$ iff all r_f lives in R^+. But if $\deg(r_f) > 0$ then Lemma 19 asserts that $p \circ \varphi_f \circ i = 0$, so $\phi = 0$ and this proves the injectivity of ι. □

4. Hodge theory: more to do

So far, we have reduced the inductive proof of Soergel's conjecture to the assertion that $(\overline{B_x B_s}, (\cdot,\cdot)_{x,s}, L_0)$ satisfies the Hodge-Riemann bilinear relation, with the inductive hypothesis heavily used at every step. The new assertion will again be proved inductively, but we still need the Soergel's conjecture at the same time! That is, the induction must be done in a package and, in fact, this is also how de Cataldo and Migliorini did it - they put Decomposition Theorem and Hodge theoretic properties into an inductive package. Let us name the assertions:

(1) S(x): this is Soergel's conjecture that $\mathrm{ch}(B_x) = b_x$.
(2) HR(x): this is the assertion that $(\overline{B_x}, (\cdot,\cdot)_x, L_0)$ satisfies Hodge-Riemann bilinear relation.
(3) HR(x, s): this is the assertion that when $xs > x$, $(\overline{B_x B_s}, (\cdot,\cdot)_{x,s}, L_0)$ satisfies Hodge-Riemann bilinear relation.

We emphasize that $(\cdot,\cdot)_x$ and $(\cdot,\cdot)_{x,s}$ are well-defined up to positive scalars (so Hodge-Riemann makes sense) only thanks to S(x)! Now our inductive package is:

(A) Assuming S($< xs$) and HR($< xs$) , prove HR(x, s).
(B) With S($< xs$) and HR(x, s) we can prove S(xs); this is what we have worked out in the last few pages.
(C) Lastly, prove HR(xs), so that the induction can continue.

We almost had (C): we want to prove that L_0 on $\overline{B_{xs}}$ satisfies Hodge-Riemann, that is to show certain forms are positive/negative definite. In HR(x, s) we already know that L_0 on $\overline{B_x B_s}$ satisfies Hodge-Riemann. Since a restriction of a positive/negative definite form to a subspace is still positive/negative definite, HR(xs) is a consequence of the next lemma (plus some verification of degree).

Lemma 20. B_{xs} *is direct summand of* $B_x B_s$.

It remains to prove (A). Mimicking de Cataldo-Migliorini, the proof of (A) consists of three steps: (A1) Establish a "weak Lefschetz" via the technique of Rouquier complexes, (A2) Prove some general case of hard Lefschetz, and (A3) Prove HR(x, s). (A1) is to be done in Chun-Ju's talk and (A2) is to be done in Ziqing's ending talk. We will do the simplest (A3) in the rest of our time.

Recall that we our operator L_0 is left multiplication by ρ, for example on $B_x B_s$. Consider another operator $M : B_x B_s \to B_x B_s$ given by $a \otimes_R b \mapsto a \otimes_R (\rho b) = (a\rho) \otimes_R b$. We also write $L_\zeta = L_0 + \zeta M$ for any real $\zeta \geq 0$. Now the result to be proved (sketched?) in two weeks is

Theorem 21. *(Theorem 20.15) (Hard Lefschetz) For any $\zeta \geq 0$, $(\overline{B_xB_s}, (\cdot,\cdot)_{x,s}, L_\zeta)$ satisfies hard Lefschetz.*

To say L_0 satisfies the Hodge-Riemann relation is to assert that a non-degenerate form has a certain signature. When ζ increases, the form varies continuously but remain non-degenerate and thus cannot have its signature. Hence if any of the form $(\cdot,\cdot)^i_{L_\zeta}|_{P^i}$ induced by L_ζ for some ζ satisfies Hodge-Riemann, then it does for all $\zeta \geq 0$. It thus suffices to prove it for $\zeta \gg 0$.

Proposition 22. *Suppose $x \in W$, $s \in S$ and $xs > x$. If the triple $(\overline{B_x}, (\cdot,\cdot)_x, L_0)$ satisfies the Hodge-Riemann bilinear relation, then so does $(\overline{B_xB_s}, (\cdot,\cdot)_{x,s}, L_\zeta)$ for $\zeta \gg 0$.*

Proof. Let us recall some setup. On B_s we have $c_{bot} = 1 = 1 \otimes 1 \in R \otimes_{R^s} R$ and $c_{top} = c_s$. Left multiplication works like $\rho \cdot 1 = 1 \cdot (s.\rho) + c_{top} \cdot (\partial_s \rho)$. This shows that left multiplication by ρ annihilates $\overline{c_s} \in \overline{B_s}$, left multiplication by ρ^2 is trivial on $\overline{B_s}$ and consequently M^2 is trivial on $\overline{B_xB_s}$. The form on B_s is the one given by $(1,1)_s = 0$, $(1, c_s) = (c_s, 1) = 1$, $(c_s, c_s) = \alpha_s$. Consequently, the form $(\cdot,\cdot)_{x,s}$ is built from $(\cdot,\cdot)_x$ by $(b_1 \otimes 1, b_2 \otimes 1) = 0$, $(b_1 \otimes 1, b_2 \otimes c_s)_{x,s} = (b_1 \otimes c_s, b_2 \otimes 1)_{x,s} = (b_1, b_2)_x$ and $(b_1 \otimes c_s, b_2 \otimes c_s)_{x,s} = (b_1, b_2)\alpha_s$.

Fix $i \geq 0$. Let $u_1, \ldots$ be part of a basis of B_x that lift a basis of $\overline{B_x}^{-i-1}$, and let $v_1, \ldots$ be part of a basis of B_x that lift a basis of the primitive subspace of $\overline{B_x}^{-i+1}$ under L^0. Then we have that

$$\overline{\rho u_1 \otimes 1}, \ldots, \overline{u_1 \otimes c_s}, \ldots, \overline{v_1 \otimes 1}, \ldots$$

is a basis for $\overline{B_xB_s}^{-i}$. Because $\alpha_s \in R$ maps to $0 \in \mathbb{R}$, we have

$$(L^i_\zeta \overline{u_j \otimes c_s}, \overline{u_k \otimes c_s})_{x,s} = 0.$$

On the other hand, as $\rho^i v_j$ has trivial projection to $\overline{B_x}$ we have

$$(\overline{u_j \otimes c_s}, L^i_\zeta \overline{v_k \otimes 1})_{x,s} = (L^i_\zeta \overline{v_k \otimes 1}, \overline{u_j \otimes c_s})_{x,s} = 0.$$

This says that the matrix for $(L^i_\zeta \cdot, \cdot)_{x,s}|_{\overline{B_xB_s}^{-i}}$ is symmetric of the shape

$$\begin{bmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & Q_\zeta \end{bmatrix}$$

That is, this symmetric form has signature $(\dim \overline{B_x}^{-i+1}, \dim \overline{B_x}^{-i+1})$ plus whatever given by the Q_ζ-part, namely the pairing

$$(L^i_\zeta \overline{v_j \otimes 1}, \overline{v_k \otimes 1})_{x,s} = (L^i_0 \overline{v_j \otimes 1}, \overline{v_k \otimes 1})_{x,s} + i\zeta(L^{i-1}_0 M \overline{v_j \otimes 1}, \overline{v_k \otimes 1})_{x,s} \tag{4.1}$$

where we used that M^2 acts trivially. Now thanks to that $(1,1)_s = 0$, the second term in (4.1) is equal to

$$i\zeta(L^{i-1}_0 M \overline{v_j \otimes 1}, \overline{v_k \otimes 1})_{x,s} = i\zeta(\partial_s \rho) \cdot (L^{i-1}_0 \overline{v_j \otimes c_s}, \overline{v_k \otimes 1})_{x,s}$$

which is up to a positive scalar the definite form $(L^{i-1}_0 \cdot, \cdot)_x|_{P^{-i+1}}$. Hence if we have $\zeta \gg 0$, then the signature of $(L^i_\zeta \cdot, \cdot)_{x,s}|_{\overline{B_xB_s}^{-i}}$ is determined. It is then a routine to check that this signature indeed satisfies Hodge-Riemann; see Remark 13 for a heuristic behind such routine check. ∎

With the proposition, we have finished our inductive proof of Soergel's conjecture, together with a Hodge-Riemann statement, modulo Theorem 21.

Appendix A. Soergel hom formula

Recall that a Δ-filtration is for a R-bimodule B (again, all such are assumed to be finite free as a right R-module) is an R-bimodule filtration

$$0 = B^m \subset B^{m-1} \subset ... \subset B^0 = B$$

such that $B^i/B^{i+1} \cong R_{x_i}^{\oplus h_{x_i}(B)}$ for some $x_i \in W$ such that $x_i \not> x_j$ under Bruhat order whenever $i < j$. In this case we define $\mathrm{ch}^\Delta(B) = \sum_{i=0}^{m-1} v^{\ell(x_i)} h_{x_i} \delta_{x_i}$ Likewise, a ∇-filtration is

$$0 = B^0 \subset B^1 \subset ... \subset B^m = B$$

subject to the similar condition $B^i/B^{i-1} \cong R_{x_i}^{\oplus h_{x_i}(B)}$ where $x_i \not> x_j$ whenever $i < j$. And we put $\mathrm{ch}^\nabla(B) = \sum_{i=1}^{m} v^{-\ell(x_i)} \overline{h_{x_i}} \delta_{x_i}$. Lastly, for a finite free right R-module M, denote by $\underline{\mathrm{rank}} M$ the graded dimension of $M \otimes_R \mathbb{R}$. With a slight abuse of language whenever we write $\underline{\mathrm{rank}} M = f$ for some $f \in \mathbb{Z}[v^{\pm 1}]$ we mean that M is finite free as a right R-module with $\dim^*_{\mathbb{R}}(M \otimes_R \mathbb{R}) = f$. Now the so-called Soergel hom formula is

Theorem 23. *(Theorem 5.15 in [Soe07]) Let B, B' be R-bimodules. Suppose that either (i) B affords a Δ-filtration and B' is a Soergel bimodule or (ii) B is a Soergel bimodule and B' affords a ∇-filtration. Then we have $\underline{\mathrm{rank}}\, \mathrm{Hom}^*(B, B') = (\overline{\mathrm{ch}^\Delta(B)}, \mathrm{ch}^\nabla(B'))$.*

Some comments on the proof. We will only discuss case (i); case (ii) follows from case (i) via $\mathbb{D}$. In some sense, the statement has two parts: the assertion in the special case when $B = R_x$, and that this special case implies the general case. Since both sides of the identity is linear in B' and the class of Bott-Samelson bimodules generate the split Grothendieck group of the category of Soergel bimodules, we may assume $B' = BS(\underline{y})$ for some reduced $\underline{y} = s_1...s_k$. Now, the formula is also linear in B in the following sense: for $x \in W$ denote by $\Gamma_{>x}B$ the submodule of B within the filtration given by those $x_i > x$, by $\Gamma_{\not> x}B$ the quotient, so that we have $0 \to \Gamma_{>x}B \to B \to \Gamma_{\not> x}B \to 0$. The Soergel hom formula suggests that

$$0 \to \mathrm{Hom}(\Gamma^\Delta_{\not> x}B, B') \to \mathrm{Hom}(B, B') \to \mathrm{Hom}(\Gamma^\Delta_{>x}B, B') \to 0 \tag{A.1}$$

is exact, or essentially that $\mathrm{Hom}(B, B') \to \mathrm{Hom}(\Gamma^\Delta_{>x}B, B')$ is surjective. On the other hand, if we can prove this exactness for $B' = BS(\underline{y})$, then we can reduce to the case when B is also a Bott-Samelson bimodule, or when $B = R_x$. We note that the case when B is a Bott-Samelson bimodule is equivalent to the big claim in diagrammatic that the functor from the diagrammatic Hecke category to the Soergel bimodule category is full. ☐