Chp 19 : Rouquier complexes and homological alg

Goal : $\begin{cases} \text{Use Rouquier complexes to bypass absence of weak Lefschetz} \\ \quad \llcorner \text{ Review related homological algebra} \\ \text{"Diagonal miracle" — key step in the proof of hard Lefschetz} \end{cases}$

Motivation

- Study smooth projective variety $X \subseteq \mathbb{CP}^n$ via hyperplane section $X_H = X \cap H$, where $H \subseteq \mathbb{CP}^n$ is of codim 1.

Thm (Weak Lefschetz) ← topology of X is controlled by topology of X_H

The induced map $H^i(X;\mathbb{Q}) \to H^i(X_H;\mathbb{Q})$ is $\begin{cases} \text{iso} & \text{if } i < \dim X - 1 \\ \text{inj} & \text{if } i = \dim X - 1 \end{cases}$

- SBim analog :
  study Bott-Samelson variety $Y(\underline{w})$ via $Y(\underline{w})_i$ for $1 \le i \le \ell(\underline{w})$, where $Y(\underline{w})_i \subseteq Y(\underline{w})$ is of codim 1

Fact The induced map

$$\Omega : \overline{BS(\underline{w})} \longrightarrow \bigoplus_{i=1}^{\ell(\underline{w})} \overline{BS(\underline{w}_{\hat{i}})}(1) \quad \text{is} \quad \begin{cases} \text{iso} & \text{if "deg} = -\ell(w)\text{"} \\ \text{inj} & \text{if "deg} < \ell(w)\text{"} \end{cases}$$

⚠ Not exactly a weak Lefschetz

⇒ want to understand Ω, which is 1st diff. of a Rouquier complex

Homological algebra

$\mathcal{A}$: additive category

⇝ $C(\mathcal{A})$: cat'y of (cochain) complexes in $\mathcal{A}$, i.e.,

obj : $\cdots \to A^0 \xrightarrow{d} A^1 \xrightarrow{d} \cdots$ s.t. $d^2 = 0$

⇝ $K(\mathcal{A})$: homotopy cat'y of $\mathcal{A}$

obj = obj($C(\mathcal{A})$), mor = homotopy classes in $C(\mathcal{A})$

For complexes A, B, write $A \simeq_h B$ for homotopy equiv., i.e.,

$\exists\ A \underset{b}{\overset{a}{\rightleftarrows}} B$ s.t. $a \circ b \simeq_h \mathrm{id}_B$ and $b \circ a \simeq_h \mathrm{id}_A$

where morphisms $f \simeq_h g : A \to A$ means $f\downarrow^{A^i} - \downarrow g = \angle^{A^i} + {}^{A^i}\!\swarrow$ $\forall i$,

for some homotopy h : $\swarrow\swarrow\swarrow$

Examples

(1) $A \simeq_h 0 \iff \left(A \underset{0}{\overset{0}{\rightleftarrows}} 0 \text{ s.t. } 0 = 0 \text{ and}\right)\ 0_A \simeq_h \mathrm{id}_A$

(A is called contractible in this case)

(2) Let $A = (0 \to X \xrightarrow{\varphi} Y \to 0)$

A is contractible $\iff \exists\ h$ s.t.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}"] \arrow[dl] & Y \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, "h"'] & 0 \arrow[dl] \\ 0 \arrow[r] & X \arrow[r, "\varphi"'] & Y \arrow[r] & 0 \end{tikzcd}$$

$\iff \varphi$ is iso

$\mathrm{id}_X = \angle^{h}_{0} + \angle^{\varphi}_{h} = h\varphi, \quad \mathrm{id}_Y = \varphi h$

Let $C^b(\mathcal{A})$, $K^b(\mathcal{A})$ be full subcat'ies s.t. $A^i = 0$ if $|i| \gg 0$.

They are monoidal if $\mathcal{A}$ is, via

$(A \otimes B)^n = \bigoplus_{i+j} A^i \otimes B^j$, $\quad d : A^i \otimes B^j \to (A \otimes B)^{n+1}$

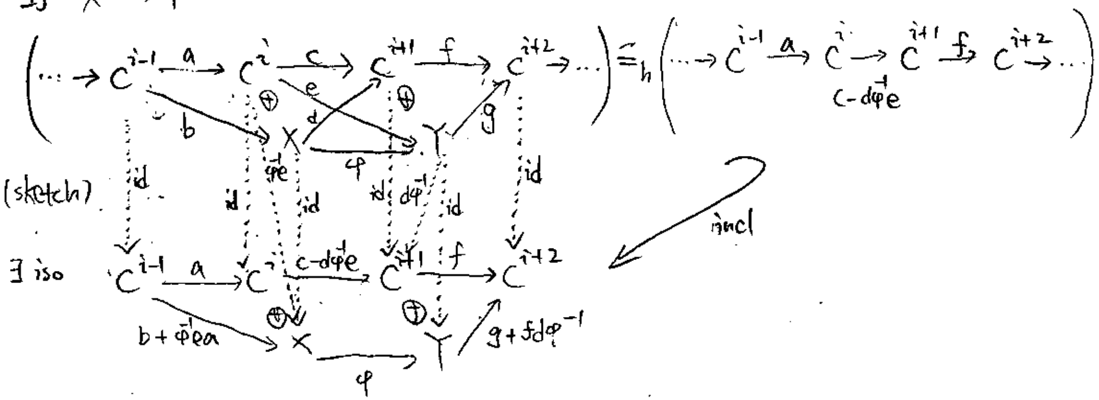
$\qquad a \otimes b \mapsto d(a) \otimes b + (-1)^i a \otimes d(b)$

Example

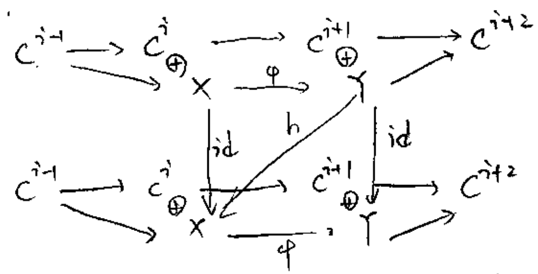
$$(A^0 \xrightarrow{f} A^1) \otimes (B^0 \xrightarrow{g} B^1) = \begin{tikzcd} & A^1 \otimes B^0 \arrow[dr, "-1\otimes g"] & \\ A^0 \otimes B^0 \arrow[ur, "f\otimes 1"] \arrow[dr, "1\otimes g"'] & \oplus & A^1 \otimes B^1 \\ & A^0 \otimes B^1 \arrow[ur, "f\otimes 1"'] & \end{tikzcd}$$

Gaussian Elimination

If $X \xrightarrow{\varphi} Y$ is iso then



The inclusion is indeed a homotopy equiv., e.g., $\mathrm{incl} \circ \mathrm{proj} \simeq_h \mathrm{id}$ since φ is an iso



⚠ It's called GE since the middle matrix $\begin{pmatrix} c & d \\ e & \varphi \end{pmatrix} \sim \begin{pmatrix} c - d\varphi^{-1}e & 0 \\ 0 & \varphi \end{pmatrix}$

Example

$$\left(0 \to X \oplus X \xrightarrow{\begin{pmatrix} id & id \\ id & id \end{pmatrix}} X \oplus X \to 0\right) \simeq_h \left(0 \to X \xrightarrow{0} X \to 0\right)$$

not contractible

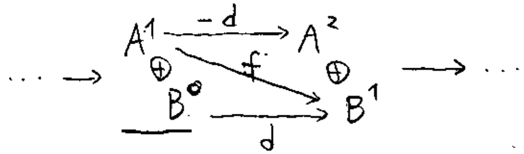
Facts (1) If $\mathcal{A}$ is Karoubian (i.e., every idempotent in $\mathcal{A}$ splits), $C \in C^b(\mathcal{A})$ then C is contractible $\iff$ $C \cong \bigoplus_i (0 \to X_i \xrightarrow[\sim]{\varphi_i} Y_i \to 0)$

$\Rightarrow$ may throw away every contractible $\oplus$-summands (as long as there're finitely many of them) and obtain a minimal subcomplex C^{min}

(2) If $\mathcal{A}$ is Krull-Schmidt then C^{min} is well-defined $\forall\, C \in K^b(\mathcal{A})$

Recall that for $A \xrightarrow{f} B$ we define a complex $\mathrm{Cone}(f) \in C(\mathcal{A})$ by

$\mathrm{Cone}(f) = A[1] \oplus B$, or



(In other words, $X \xrightarrow{f} Y \to \mathrm{Cone}(f)$ is a std Δ in $K^b(\mathcal{A})$ Δ-cat'y)

Recall Euler characteristic $\chi : [C^b(\mathcal{A})]_\oplus \to [\mathcal{A}]_\oplus$, $[A] \mapsto \sum_{i \in \mathbb{Z}} (-1)^i [A^i]$

$\Rightarrow \chi(\mathrm{cone}(f)) = \sum_i (-1)^i [A^{i+1} \oplus B^i] = \chi(B) - \chi(A)$

Def'n Triangulated Grothendieck group of $K^b(\mathcal{A})$, is

$$[K^b(\mathcal{A})]_\Delta = [K^b(\mathcal{A})]_\oplus / ([\mathrm{Cone}(f)] = [B] - [A])$$

Facts (Let $\mathcal{A}$ be Karoubian.

(1) $\chi(A) = 0 \quad \forall$ contractible A,

$\Rightarrow \chi(C) = \chi(C^{min}) \quad \forall\, C \in K^b(\mathcal{A})$

(2) If $\mathcal{A}$ is essentially small, then χ induces a grp iso

$$\chi : [K^b(\mathcal{A})]_\Delta \xrightarrow{\sim} [\mathcal{A}]_\oplus$$

(3) This upgrades to a $\mathbb{Z}[v, v^{-1}]$-alg iso when $\mathcal{A}$ is monoidal and equips w/ a grading shift functor (1)

Want: Categorify $\delta_s = b_s - v$ using cplx of SBim

Two choices:

δ_s ~~$\leadsto$~~ $(0 \to R(1) \to \underline{B_s} \to 0)$ ← not invertible, bad

$\delta_s \leadsto (0 \to \underline{B_s} \xrightarrow{\bullet} R(1) \to 0) =: F_s$

$f \otimes g \mapsto fg$

↑ underline indicates deg 0 terms

Thm (Rouquier): $\{F_s\}$ satisfy the braid relations

If $m_{st} < \infty$ then $F_s F_t F_s \cdots \simeq_h F_t F_s F_t \cdots$

Defn

$\Rightarrow$ F_w, up to homotopy equiv., is well-defined, and it categorifies $\delta_w \in H$ since $[K^b(\mathbb{S}Bim)]_\Delta \cong [\mathbb{S}Bim]_\oplus \cong H$

Q: Explicit description of F_w

Example

Recall $p_{+1} =$, $i_{+1} =$, $p_{-1} =$, $i_{-1} =$

$$F_s F_s = \underline{B_s \otimes B_s} \to \begin{matrix} R\otimes B_s(1) \\ \oplus \\ B_s \otimes R(1)\end{matrix} \to R\otimes R(2)$$

$\cong$ recall $\begin{matrix} B_s(-1) \\ \oplus \\ B_s(1)\end{matrix} \to B_s B_s$ is an iso

$$\begin{tikzcd} 0 \arrow[r] & B_s(-1) \arrow[r, "c"] \arrow[dr, "e"] & B_s(1) \arrow[r] & R(2) \\ & \oplus & \oplus & \\ & B_s(1) \arrow[r, "\cong"] \arrow[ur, "d"] & B_s(1) \arrow[uur] & \end{tikzcd}$$

with $c =$, $d =$, $e =$, $\varphi =$

$\simeq_h$ (by Gaussian elim'n)

$$0 \to \underline{B_s(-1)} \to B_s(1) \to R(2)$$

Next we categorify $\delta_s^{-1} = b_s - v^{-1}$ by

$$F_s^{-1} = (\ 0 \to R(-1) \to \underline{B_s} \to 0\ )$$

$$1 \mapsto c_s := \tfrac{1}{2}(1\otimes \alpha_s + \alpha_s \otimes 1)$$

Indeed, $F_s^{-1} F_s \simeq_h \underline{R} \cong F_s F_s^{-1}$

Example

$$F_s F_s^{-1} \cong B_s(-1) \to \underline{\begin{matrix} R \\ \oplus \\ B_sB_s\end{matrix}} \to B_s(1)$$

$\cong$

$$\left(B_s(-1) \to \underline{\begin{matrix} R\oplus B_s(1) \\ \oplus \\ B_s(-1)\end{matrix}} \to B_s(1) \right)$$

$\simeq_h$

$$\left(0 \to \underline{\begin{matrix} R \\ \oplus \\ B_s(1)\end{matrix}} \to B_s(1) \right) \cong (0 \to \underline{R} \to 0)$$

Defn $F_{s_1}^{e_1} \cdots F_{s_m}^{e_m}$ is called a Rouquier cplx ($e_i \in \{\pm 1\}$) for braid $\beta = (\delta_{s_1}^{e_1} \cdots \delta_{s_m}^{e_m})$

Recall SES (Δ): $0 \to R_s(-1) \to B_s \to R(1) \to 0$

$$\Rightarrow H^i(F_s) = \begin{cases} R_s(-1) & \text{if } i=0 \\ 0 & \text{otw}\end{cases}$$

Recall SES (∇): $0 \to R(-1) \to B_s \to R_s(1) \to 0$

$$\Rightarrow H^i(F_s^{-1}) = \begin{cases} R_s(1) & \text{if } i=0 \\ 0 & \text{otw}\end{cases}$$

$$\Rightarrow H^i(F_{s_1}^{e_1} \cdots F_{s_m}^{e_m}) = \begin{cases} R_{s_1}(-e_1)\cdots R_{s_m}(-e_m) & \text{if } i=0 \\ 0 & \text{otw}\end{cases}$$

For $\underline{w}$ we define positive Rouquier cplx $F_{\underline{w}} = F_{s_1}\cdots F_{s_m}$.

Cor

$$H^i(F_{\underline{w}}) = \begin{cases} R_{\underline{w}}(-\ell(\underline{w})) & \text{if } i=0 \\ 0 & \text{otw}\end{cases}$$

In particular, 1st differential is inj in all deg $< \ell(\underline{w})$

Thm (Diagonal miracle)

Assume the Soergel conjecture. If $\underline{w}$ is a rex, then

$$F_{\underline{w}}^{min} = \left(B_w \longrightarrow \bigoplus_{x<w} B_x^{\oplus n_x}(1) \longrightarrow \bigoplus_{y<w} B_y^{\oplus n_y}(2) \to \cdots \right)$$

This is "diagonal" in the sense: if we arrange décomposition of $X = (\cdots \to X^0 \to X^1 \to \cdots) = \left(\to \bigoplus_j X_j^0 \to \bigoplus_j X_j^1 \to \cdots \right)$

where $X_j^i \cong \bigoplus_w B_w(j)$ by

$$\begin{array}{ccc} X_0^0 & X_0^1 & X_0^2 \\ X_1^0 & X_1^1 & X_1^2 \\ X_2^0 & X_2^1 & X_2^2 \end{array}$$

That means $F_{\underline{w}}^{min}$ lies entirely on the diagonal.

Examples

$$F_s = \begin{array}{cc} \underline{B_s} & \\ & \searrow \\ & R(1) \end{array} \qquad \text{while} \quad F_sF_s = \begin{array}{ccc} & B_s(-1) & \\ \underline{0} & \downarrow & \\ & B_s(1) & \\ & & \searrow \\ & & R(2) \end{array}$$