Kac-Moody Groups: An Overview

I. What are Kac-Moody groups/algebras?

- A KM algebra is a Lie alg / k: field (For now let $k = \mathbb{C}$)

$$\mathfrak{g} = \mathfrak{g}(A) = \langle e_i, f_i, \mathfrak{h} \mid i \in I \rangle / \sim$$

for a generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$,

a "Cartan subalg" $\mathfrak{h}$,

subject to relations coming from A.

Facts

(1) (classification)

If $A \neq \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ and is symmetrizable, then exactly one of the following holds:

(i) $\mathfrak{g}$ is of finite type, i.e., (f.d.) simple Lie alg

(ii) $\mathfrak{g}$ is of affine type, i.e., (∞-dim) Lie alg that affords a loop alg realization

(iii) $\mathfrak{g}$ is of indefinite type, i.e., (∞-dim) Lie alg we know very little

(2) (Cartan decomp'n)

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

⚠ root space $\mathfrak{g}_\alpha$ is NOT always of dim 1.

For affine Lie alg, $\dim \mathfrak{g}_\alpha = \begin{cases} 1 & \text{if } \alpha \in \Phi \text{ is "real"} \\ |I| - 1 & \text{if } \alpha \in \Phi \text{ is "imaginary"} \end{cases}$

(3) (Weyl group)

$W \subset \mathrm{Aut}(\mathfrak{h}^*)$ is a (poss. ∞) Coxeter group generated by s_i $(i \in I)$

⚠ In general, any argument involving the longest elt fails

(4) (Repn theory)

BGG cat'y $\mathcal{O}$ can be defined (w/o finite generation)

$\cup$

{Verma modules $M(\lambda)$, irred. modules $L(\lambda)$, integrable modules}

$\Rightarrow$ Weyl-Kac character formula for integrable modules

e.g. for affine $\hat{A}_1 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, it translates to the

Jacobi triple product identity:

$$\prod_{m=1}^{\infty} (1 - x^{2m})(1 - x^{2m-1} y)(1 - x^{2m-1} y^{-1}) = \sum_{n=-\infty}^{\infty} (-1)^n x^{n^2} y^n$$

$\Rightarrow$ appl'ns to "$\Pi = \Sigma$" identities i.e. $\mathcal{M}$ identities

(5) (Kazhdan-Lusztig theory) generalized by Casian, Kashiwara-Tanisaki for sym'ble KM alg:

$$[M(y \cdot 0) : L(x \cdot 0)] = h_{y,x}(1)$$

This makes sense since Weyl groups are still Coxeter groups

- Kac-Moody groups : Kac-Moody alg = algebraic grp : simple Lie alg

II. Why Kac-Moody/algebraic grps?

- study repn in both char 0 & $p > 0$ from alg geom / cohomology viewpts.
- Let G : reductive algebraic grp / $k = \bar{k}$ e.g. $GL_n(k)$

$\cup$ $\qquad$ $\cup$

B : Borel subgrp $\qquad \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$

$\cup$ $\qquad$ $\cup$

T : maximal torus $\qquad \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$

Facts (Classification/constr'n of irreducibles)

(1) $\mathrm{Irr}\,G = \{ L(\lambda) \mid \lambda \in X(T)_+ \}$, where

$X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$: characters of T

$X(T)_+ = \{$dominant weights in $X(T)\}$

$L(\lambda) = $ socle of $H^0(\lambda) := \mathrm{ind}_B^G \lambda$ ← char computed by Weyl's char formula

⊂ Σ simple submod.

(2) One defines $H^i(\lambda) := R^i \mathrm{ind}_B^G \lambda$

≅ ith cohomology grp of line bundle on flag variety G/B

(3) [Borel-Bott-Weil thm]

If char $k = 0$ then $H^i(w\cdot\lambda) = \begin{cases} H^0(\lambda) & \text{if } i = \ell(w) \\ 0 & \text{otw} \end{cases}$, $\lambda \in X(T)_+$

⚠ If char $k > 0$ then for a fixed $\mu \in X(T)$, there can be more than one i s.t. $H^i(\mu) \neq 0$

(4) [Kempf vanishing thm]

If $\lambda \in X(T)_+$ then $H^i(\lambda) = 0 \quad \forall\, i > 0$

(5) [Andersen'88] (strong linkage principle)

If $[H^0(\lambda) : L(\mu)] \neq 0$ then $\mu = \mu_0 \uparrow \mu_1 \uparrow \dots \uparrow \mu_r = \lambda$, where

$x \uparrow y \iff x = s_{\beta, mp} \cdot y \leq y$ for some $\beta \in \Phi^+$, $m \in \mathbb{Z}_{>0}$

$= s_\beta \cdot y + mp\beta$

$= y - (\langle y+\rho, \beta^\vee \rangle - mp)\beta$

$\Rightarrow \langle s_{\beta,mp} \rangle =$ affine Weyl group ("p-dilated") $W_p = p\mathbb{Z}\Phi \rtimes W$

⚠ SLP is first proved by Jantzen (under restriction on p) from Lie alg

(6) (SLP ⇒ Linkage principle)

If $L(\lambda)$, $L(\mu)$ are composition factors of an indecomposable, then $\lambda \in W_p \cdot \mu$

(7) (Kazhdan-Lusztig theory)

When char $k = p > 0$, the role of Verma is replaced by Weyl modules $V(\lambda) := H^0(\lambda)^\vee$

- [Lusztig's conj '90] $\mathrm{ch}\, L(y\cdot 0) = \sum_{y \leq x \in W_p} (-1)^{\ell(x)-\ell(y)} h_{w_0 x, w_0 y}(1)\, \mathrm{ch}\, V(x\cdot 0)$

  if $p \geq 2h-3$ ← Coxeter #

  ⇒ It's then widely believed that one only needs $p \geq h$ due to Kato
- [Andersen-Jantzen-Soergel '93] via QG at $\sqrt[p]{1}$, 300+ pp.

  Lusztig's conjecture holds except for finitely many p's
- [Fiebig '11] via moment graphs/parity sheaves

  ——— if $p > U(h) =$!!! ← insanely large
- [Williamson '13]

  ∃ counter examples to Lusztig's conj (when $p = 89 > h = 35$)

  Moreover, ∄ polyn $P(h)$ s.t. ——— holds when $p > P(h)$
- [Riche-Williamson '18 (type A), Achar-Makisumi-R-W '19]

  LC holds if one replaces KL poly by p-KL polynomials.

  ⚠ This is NOT the end - p-KL polyn are difficult to compute - Very little is known beyond (affine) SL_2

  ⇒ Lusztig-Williamson's billiards conjecture for (affine) SL_3

(8) (KL-Vogan polyn).

replace $h_{x,y}$ by $\sum_{i \geq 0} v^{2i} \dim \mathrm{Ext}_G^{\ell(x)-\ell(y)-2i}(H^0(x\cdot 0), L(y\cdot 0))$

⇒ data can be obtained by computing all $\mathrm{Ext}_G^n(H^0(\mu), L(\lambda))$ using Kostant's thm in Lie alg (co)homology and spectral sequence

Algebraic grp	$\rightsquigarrow$	KM grps
class'n of irred.	$\rightsquigarrow$	✓
const'n via		
$H^i(\lambda) \cong H^i(G/B, \mathcal{L}(\lambda))$ $\uparrow$ flag var, $\uparrow$ line bundle	$\rightsquigarrow$	gen. flag var $\mathcal{G}/\mathcal{B}$ ✓ line bundles on $\mathcal{G}/\mathcal{B}$
Weyl char formula	$\rightsquigarrow$	Weyl-Kac char formula
Borel-Bott-Weil	$\rightsquigarrow$	✓ for $p=0$
SLP	$\rightsquigarrow$	conj [Lai-Wang]
$\Downarrow$		
LP		unknown
$\Downarrow$		
KL theory		not even formulated.
Kostant's thm & spectral seq	$\rightsquigarrow$	✓

III. Organization

Next week: Kozual duality by Chiu (cont'n from last term)

Chp I. KM alg

II. Repn theory of KM alg

III. Lie alg (co)homology

IV. Intro to ind-var/pro-grp

V. Tits sys

VI. KM grps