§ Koszul Duality for Hecke Category § Mar. 5 Chiu 1

1. General framework of Koszul algebras
2. Beilinson-Ginzburg-Soergel's discovery for cplx s.s. Lie.
3. Equivalences of derived cats $\begin{cases} \text{BGG} \\ \text{mixed perverse (Ringel} \circ \text{Koszul)} \end{cases}$
4. Hecke cat & Koszul duality for regular reps.
5. ~~Positselski's filtered Koszul duality~~ Examples
6. Monodromic actions & free-monodromic cplx.
7. Monoidal Koszul duality.

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1. Koszul Algebras. $A/k = \bigoplus_{i \geq 0} A_i$ w/ $\begin{cases} A_i \cdot A_j \subset A_{i+j}, \ \dim_k A_i < +\infty \\ 1_A = 1_k \in k = A_0. \end{cases}$

Def. A is "quadratic" if A_1 gen. A and has relations of deg two.
i.e. write $A = TV/R$ $A = \{V, R\}$ w/ $\begin{cases} V = A_1 \rightsquigarrow \text{gen. } A \\ R \subset V \otimes V \rightsquigarrow \text{gen. } \ker(TA_1 \to A) \end{cases}$

Def. The quadratic dual of $A = \{V, R\}$ is the quadratic algebra $A^! := \{V^*, R^\perp\}$ $A^! = TV^*/R^\perp$ where $R^\perp \subset V^* \otimes V^*$ is orth. to $R \subset V \otimes V$

Eg. $TV := \{V, 0\}$ is quadratic, and $(TV)^! = \mathbb{1} \oplus V^* \oplus 0 \oplus \cdots$ (obviously) $(A^!)^! = A$

Eg. $R = (v_i \otimes v_j - v_j \otimes v_i)$ $(S^\bullet V)^! = \Lambda^\bullet V^*$ $\to R = (v_i^* \otimes v_i^*)$ are quadratic Eg. $(k)^! \cong (A^!)$, $(A)^! \cong k$. module Koszul

Obviously $!$ is an involutive equiv: quad $\mathrm{Alg}_k \xrightarrow{\sim} \mathrm{quadAlg}_k$ of $(A^!)^! = A$.

Def. $A \in \mathrm{quadAlg}_k$ is Koszul if $A^!$ is completely determined by cohomological information.
i.e. $A^! \cong \mathrm{Ext}_A^\bullet(k, k)$ i.e. $\mathrm{Ext}_A^{ij}(k, k) = 0 \ \forall i \neq j$.

Another Def. $A = \bigoplus_{j \geq 0} A_j$ is Koszul if ① A_0 s.s. (say $A_0 = k$)
② $\exists$ gr-proj resol of A-mod $P^\bullet \twoheadrightarrow A_0$ s.t. $P^j = A \cdot P^j_j$ (i.e. gen. by its component of deg j)

In this case a model for $A^!$ is $A^! = \mathrm{Ext}_A^\bullet(A_0, A_0)$

Eg. V f.d. vect/k $\Rightarrow$ $S^\bullet V$ is Koszul by $\left(\cdots \to SV\underset{k}{\otimes}\Lambda^2 V \to SV\underset{k}{\otimes} V \to SV \underset{\text{arg}}{\to} k\right)$
and $(SV)^! = \Lambda(V^*)$

Rmk. Let $h_A(z) = \sum \dim A_i z^i$ Hilbert series
Thm A Koszul $\Rightarrow$ $h_A(z)\cdot h_{A^!}(-z) = 1$

2. <u>BGS's discovery</u>. $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ cplx s.s Lie alg. $\mathcal{U} = \mathcal{U}(\mathfrak{g})$
$\mathcal{O}$: cat of f.g. $\mathcal{U}$-mods $\begin{cases} \text{loc. fin. over } \mathfrak{b} \\ \text{s.s. over } \mathfrak{h}. \end{cases}$

$\forall \lambda \in \mathfrak{h}^*$ set $\begin{cases} M(\lambda) := \mathcal{U} \underset{\mathcal{U}(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda \quad \text{Verma} \\ L(\lambda) \text{ its simple quotient} \\ P(\lambda) \text{ its proj (indecomp.) cover of } L(\lambda) \end{cases}$

Let $\mathcal{O} \ni L := \bigoplus_{\lambda \in W}$ simple highest weight $L(\lambda)$ w/ triv. infinitesmal char.
$P := \bigoplus$ such $P(\lambda)$

Thm (BGS). $\mathrm{End}_{\mathcal{O}}(P) \cong \mathrm{Ext}^\bullet_{\mathcal{O}}(L,L)$ is Koszul
$\to$ There should be a categorical intepretation on which Koszul duality acts.

3. <u>Equivalences of derived cats</u>. Assume $k = A_0 \cong A_0^!$ & some finiteness condition, A Koszul then.
Thm. (BGS) $\exists$ exact functor $K: D^b(A\text{-mod}^{f.g.}) \xrightarrow{\sim} D^b(A^!\text{-mod}^{f.g.})$ such that

① $K(M(n)) = (K(M))[n](-n)$ $\quad M \in D^b$. $\quad M(n)_i := M_{n+i}$ $\quad K\circ(n) \cong [n](-n)\circ K$

② $\forall \mu \in k \Rightarrow$ $\underset{\text{all simples/gr}}{K(k\mu)} = \underset{\text{proj cov.}}{A^!\mu}$ ~~proj cover~~ inj hull
$K(A^*\mu) = k^!\mu$ all simples/gr

Rmk. in BGS they use opp so the order of proj/inj is different.

Rmk ① One can choose $K = R\mathrm{hom}_A(k, -)$
② This recovers BGG's $D^b(SV\text{-mod}^{f.g.}) \xrightarrow{\sim} D^b(\Lambda^\bullet V^*\text{-mod}^{f.g.})$
not so important since later we will move to categorical Koszul

~~BGS~~ Summary (~~BGS~~)

A Koszul

$$D^b(A\text{-mod}^{f.g.}) \xrightarrow[\sim]{K} D^b(A^!\text{-mod}^{fg})$$

simple$_{/gr}$ $\longmapsto$ inj
proj $\longmapsto$ simple$_{/gr}$

Koszul duality (need grading)

G red. alg gp.

mixed geometry, not necessary modules over certain algebras.

G/B

Borel

$$D^b \mathrm{Perv}^{mix}_{(B)}(\mathcal{B}, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} D^b \mathrm{Perv}^{mix}_{(B^\vee)}(\mathcal{B}^\vee, \overline{\mathbb{Q}_\ell})$$

Langlands dual.

simple $\mapsto$ inj
proj $\mapsto$ simple

"Koszul duality" (need grading)

BG & AMRW argue that, since we have.

$$D^b \mathrm{Perv}^{mix}_{(B)}(\mathcal{B}, \overline{\mathbb{Q}_\ell}) \circlearrowleft$$

inj $\mapsto$ tilting
costd $\mapsto$ std
tilting $\mapsto$ proj

Ringel duality (ungraded)

Ringel $\circ$ Koszul gives

$$D^b \mathrm{Perv}^{mix}_{(B)}(\mathcal{B}, \overline{\mathbb{Q}_\ell}) \xrightarrow[\sim]{\text{"}K\text{"}} D^b \mathrm{Perv}^{mix}_{(B^\vee)}(\mathcal{B}^\vee, \overline{\mathbb{Q}_\ell})$$

simple $\mapsto$ tilting
tilting $\mapsto$ simple.

a more natural Koszul duality.

We will see what it means to the Hecke category of Soergel bimodules (under Cartan realization)

Rmk. the Abelian cat Perv^{mix} comes from the heart of perverse t-str. of $K^b\mathcal{H}$ (our triangulated Hecke cat)

4. Hecke category Koszul duality for regular representations.

Given (W, S) coxeter, a Cartan realization. $\mathfrak{h} = (V, \{\alpha_s^\vee\}; V^*, \{\alpha_s\})$

$R = \mathrm{Sym}(V^*)$, $|V^*| = 2$. $\forall s$, $B_s := R \otimes_{R^s} R(1)$

$\mathbb{S}\mathrm{Bim}(\mathfrak{h}, W) := \langle B_s : s \in S \rangle_{\oplus, \otimes_R, (1), \overset{\oplus}{\subset}, \cong} \subset R\text{-mod-}R$ (gr).

ii our candidate of Hecke cat. ($\Bbbk$-linear, add, gr. monoidal.)

$\mathcal{H}(\mathfrak{h}, W) \xrightarrow[\text{categorifies}]{\text{split Grothendieck}} H(W)$ ($[\mathcal{H}]_{\oplus} \overset{\sim}{\underset{\text{alg}}{=}} H$) Hecke algebra. (over $\mathbb{Z}[v, v^{-1}]$)

$H(W) = \mathbb{Z}[v, v^{-1}][\{\delta_s\}] / \delta_s^2 = (v^{-1} - v)\delta_s + 1,\ \delta_s\delta_t\delta_s \cdots = \delta_t\delta_s\delta_t \cdots$ (KL basis)

For $H(W)$ we have

KL involution: $\delta_s \overset{\bar{(\ )}}{\mapsto} \delta_s^{-1}$ and $v \overset{\bar{(\ )}}{\mapsto} v^{-1}$ making self-dual $b_s := \delta_s + v \overset{\bar{(\ )}}{\mapsto} b_s$

integrated ~~natural for~~ $\mathcal{H}(\mathfrak{h}, W)$ since $\mathcal{H}$ is gen. by B_s ($b_s \rightsquigarrow B_s$)

By categorification we mean ① $\mathcal{H}$ is Krull-Schmidt w/ $\{\text{indecomp.}\}/_{\cong} = \mathbb{Z} \times \{B_w / w \in W\}$ $\overset{1:1}{\updownarrow}$ $\mathbb{Z} \times W$.

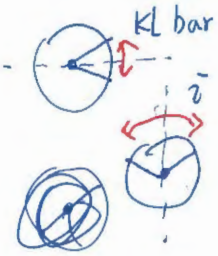
② $b_s \mapsto [B_s]$ determines $H(W) \xrightarrow[\mathbb{Z}[v, v^{-1}]\text{-alg}]{\sim} [\mathcal{H}]_{\oplus}$

$\boxed{\text{A missing duality}}$

Consider $H(W) \xrightarrow{i} H(W)$ by $\boxed{i(v) = -v^{-1}}$, $i(\delta_s) = \delta_s$ $\forall s$. $v \in S^1$

i satisfies $i(b_s) = \delta_s - v^{-1} =: t_s$ (tilting basis)

$i(b_w) =: t_w$.



Question: Who categorifies i (and t_w)?

Assume $\exists$ such monoidal $\mathcal{H}(\mathfrak{h}, W) \xrightarrow[\sim]{\kappa} \mathcal{H}(\mathfrak{h}, W)$.

$\Rightarrow [\kappa(B_{id}(1))] = i(v) = -v^{-1}$ No way!

not a $\mathbb{Z}_{\geq 0}[v, v^{-1}]$ combination of b_w !!

Rmk. To be compatible w/ $\delta_s^2 = (v^{-1} - v)\delta_s + 1$,

$v \overset{\bar{(\ )}}{\mapsto} v^{-1} \Rightarrow \delta_s \overset{\bar{(\ )}}{\mapsto} \delta_s^{-1}$

$v \overset{i}{\mapsto} -v^{-1} \Rightarrow \delta_s \overset{i}{\mapsto} \delta_s$

Instead ② Consider bdd htpy cat $K^b(\mathcal{H}(\mathfrak{h}, W))$.

and its triangulated Grothendieck $[K^b\mathcal{H}]_{\Delta}$ $\circlearrowleft$ $[1] = \cdot(-1)$

we have algebraic isom: $[K^b\mathcal{H}]_{\Delta} \xrightarrow[\sim]{e} [\mathcal{H}]_{\oplus} \xrightarrow[\sim]{ch} H$ negative sign emerges!

$[B^\bullet] \mapsto \sum (-1)^i [B^i]$

Rmk. $\kappa \rightsquigarrow i : v \mapsto -v^{-1}$

means κ satisfies $\kappa \circ (1) \cong [1](-1) \circ \kappa$. i.e. $\kappa(M(n)) = (\kappa(M))[n](-n)$. $\forall M \in K^b\mathcal{H}$

this is exactly what BGS has for $D^b(A\text{-mod}) \xrightarrow[\sim]{K} D^b(A^!\text{-mod})$

↓ page 5

We ask: $\exists ?$ $K^b\mathcal{H} \xrightarrow[\sim]{K} K^b\mathcal{H}$. s.t. $[K]_\Delta \mapsto i$ under $[K^b\mathcal{H}]_\Delta \xrightarrow[\sim]{e} [\mathcal{H}]_\oplus \xrightarrow[\sim]{ch} H$

Still no way!

This time the problem is we cannot categorify $t_w =: i(b_w)$

Consider the case $[K^b \mathbb{S}Bim(\mathfrak{h}_k^{sl_2}, S_2)]_\Delta \simeq H(S_2)$

$\Rightarrow$ $K(B_s) \rightsquigarrow i(b_s) =: t_s = \delta_s - v^{-1} = b_s - v - v^{-1} \in H(S_2)$

$\Rightarrow$ $K(B_s) =$ "Rouquier cplx" up to [even] $T_s :=$

$$\begin{array}{ccc} R(1) & 1 & fg \\ \uparrow & & \uparrow \\ B_s & 0 & f\Delta_s \quad f\otimes g \\ \uparrow & & \uparrow \\ R(-1) & -1 & f \end{array}$$

However, $\circ = \alpha_s \cdot id \neq 0$. $\Rightarrow$ T_s is not a cplx, $T_s \notin K^b\mathcal{H}$.

Solution (Partial): Find a category where $\alpha_s\, id$ becomes zero!

$\Rightarrow$ Regular representation.

<u>Def</u>. The left quotient $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\mathfrak{h}, W)$ is a category w/ same $\mathcal{H}$ objs whose (graded) homs are

$$\mathrm{Hom}^\bullet_{\overline{\mathcal{H}}}(X,Y) := k \otimes_R \hom^\bullet_{\mathcal{H}}(X,Y).$$

$R \xrightarrow{\varepsilon_R} k$ ~~by~~
$V^* \mapsto 0$.

i.e. $\mathcal{H} \xrightarrow{For} \overline{\mathcal{H}}$: quotient out R^+ from $\hom^\bullet_{\mathcal{H}}$.

◎ We have $[\mathcal{H}]_\oplus \xrightarrow{\sim} [\overline{\mathcal{H}}]_\oplus$ (w/ same indecomp. but no longer monoidal.) as $\mathbb{Z}[v, v^{-1}]$-modules.

the right action $\overline{\mathcal{H}} \circlearrowleft \mathcal{H}$. categorifies right reg rep $H(W) \circlearrowleft H(W)$

◎ geometrically, For is the forgetful functor
(on Kac-Moody flag var.) Borel-equiv parity cplx $\xrightarrow{For}$ Borel-constructible parity cplx.

⊚ Similarly $\underline{\mathcal{H}}$ is the right quotient via $(-)\otimes_R k$.

and $\overline{\mathcal{H}} \xrightarrow{\sim} \underline{\mathcal{H}}$, $B_w \mapsto B_{w^{-1}}$.

Now we can define "tilting cplx" $T_s \in K^b\overline{\mathcal{H}}$ (Similarly in $K^b\underline{\mathcal{H}}$)

Our first Koszul duality relates $\overline{\mathcal{H}}$ and $\underline{\mathcal{H}}$ for Langlands dual realization.

$\boxed{\text{Thm}}$. Let $(\mathfrak{h}, W)$ Cartan realization and $(\mathfrak{h}^*, W)$ its Langlands dual.
$V^* \ni \{\alpha_s\}$ $\quad$ $\{\alpha_s^\vee\} \subset V$. $\quad$ $2 \nmid |k|$.
$|V| = 2$.

Then $\exists$ triangulated equivalence

$$K^b\overline{\mathcal{H}}(\mathfrak{h}, W) \xrightarrow[\sim]{\kappa} K^b\underline{\mathcal{H}}(\mathfrak{h}^*, W)$$

categorifying i; in particular.

① $\kappa \circ (1) \cong \underbrace{[1](-1)}_{\langle 1\rangle \text{ Koszul shift}} \circ \kappa$ ② $\kappa(B_w) = T_w$ ③ $\kappa(T_w) \cong B_w$.

⊚ This duality is just the baby part of a so called "monoidal Koszul duality", working for $\mathcal{H}$

5. Monodromic Actions

First we want to find a good (derived) replacement for our forgetful functor $\mathcal{H} \to \overline{\mathcal{H}}$.

$$\begin{tikzcd} & & LM(\mathfrak{h}, W) \arrow[d, phantom, "\cup\!\!\!\mid"] \\ & & \\ K^b H \arrow[rruu, "\text{For}"] \arrow[rr, "K^b(\text{For})"'] & & K^b\overline{\mathcal{H}} \end{tikzcd}$$

$A \overset{\varepsilon_A}{\twoheadrightarrow} k$. $\quad$ Bia $\quad$ Triv $\quad$ real

$LM(\mathfrak{h}, W)$ = "Cat of left-monodromic cplx"

Obj: ~~$\mathcal{H}$-Sequences~~. $(\mathcal{F}; \delta)$ where $\mathcal{F} = \mathcal{H}$-seq.

$\delta \in \underline{\mathrm{End}}_{LM}(\mathcal{F})$ s.t. $\boxed{\delta \circ \delta + \partial(\delta) = 0}$ $\quad$ MC eqn. flatness eqn.

may have any bi-deg

$\Updownarrow$

$\varepsilon_k(\delta \circ \delta) = 0$

i.e. δ becomes genuine cplx diff after passing to $k \otimes (-)$.

Let's describe $\underline{\mathrm{End}}_{LM}$, $\underline{\mathrm{Hom}}_{LM}$ as follows:

(~~b~~ we are actually describing an dg enhancement of LM)

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① replace k by its (Koszul) resol $(A^\bullet, \overset{\partial}{\cancel{k}}) \xrightarrow{q.i.} k$.

where $A := \Lambda(V^*) \otimes_k R$ $\curvearrowright R := S(V^*)$

$= (\cdots \to \Lambda^2 V^* \otimes_k R \to V^* \otimes_k R \to R) \xrightarrow{\varepsilon_R} k$ as R-grmod

and $\overset{\partial}{\cancel{k}}$ = the (total) differential of the above cplx.

② $K^b \mathcal{H} \xrightarrow{A \otimes_k (-)} LM$ by

(bi-gd) $\underline{\mathrm{Hom}}^{\bullet\bullet}_{LM}(\mathcal{F}, \mathcal{G}) = \Lambda^\bullet \otimes_k \underline{\mathrm{Hom}}^\bullet(\mathcal{F}, \mathcal{G}) = \overset{\bullet}{A} \otimes_R \underline{\mathrm{Hom}}^\bullet(\mathcal{F}, \mathcal{G})$.

gd by resol. gd by polynomial

where $\underline{\mathrm{Hom}}^\bullet(\mathcal{F}, \mathcal{G}) := \prod_{p,q} \mathrm{Hom}_{\mathcal{H}}(\mathcal{F}^p, \mathcal{G}^q(\circ))$ the total cplx.

③ Objs of LM are $(\mathcal{F}, \delta_{\mathcal{F}})$ satisfying $\delta \circ \delta + \partial(\delta) = 0$.

$(\mathcal{F}, \delta_F)$ $(\mathcal{G}, \delta_g)$

~~This is equivalent to~~

To obtain Morphisms of LM, one makes $\underline{\mathrm{Hom}}_{LM}(\mathcal{F}, \mathcal{G}) \ni f$

into a cplx under the diff

$$d_{LM}(f) := \delta_g \circ f - (-1)^{|f|} f \circ \delta_F + \partial(f).$$

Check that

$$d^2_{LM}(f) = \delta_g \circ (\delta_g \circ f - (-1)^{|f|} f \circ \delta_F + \partial(f))$$

$$- (-1)^{|df|} (\delta_G \circ f - (-1)^{|f|} f \circ \delta_F + \partial(f)) \circ \delta_F$$

$$+ \partial(\delta_G \circ f - (-1)^{|f|} f \circ \delta_F + \partial(f))$$

$-\partial(\delta_F)$

$$= \delta_G \circ \delta_G \circ f - (-1)^{|f|} \delta_G \circ f \circ \delta_F + \delta_G \circ \partial(f)$$

$$- (-1)^{|df|} \delta_G \circ f \circ \delta_F + \boxed{(-1)^{|df| + |f| + 1}} f \circ \delta_F \circ \delta_F - (-1)^{|df|} \partial(f) \circ \delta_F$$

$$+ \partial(\delta_G \circ f) - (-1)^{|f|} \partial(f \circ \delta_F) + \partial(\cancel{\partial(f)}) \to 0$$

$$= [\partial(\delta_G \circ f) - \partial(\delta_G) \circ f + \delta_G \circ \partial(f)]$$

$$- (-1)^{|f|} [\partial(f \circ \delta_F) - \partial(f) \circ \delta_F + (-1)^{|df|} f \circ \partial(\delta_F)]$$

Leibniz.

$$= 0 + 0$$

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Therefore one defines $\mathrm{Hom}_{LM}(F,G) := \mathrm{Ker}(d_{LM}^{0,0}) / \mathrm{Im}(d_{LM})$.

② the pairing $(-) \frown (-) : V \otimes \Lambda^i V^* \to \Lambda^{\cdot} V^*$. induces derivation

Koszul shift $\langle r_i \rangle$, dg shift $[|r_i|]$, dg shifts $(-|r_i|)$

$$x \frown (r_1 \wedge \cdots \wedge r_k) = \sum_{i=1}^{k} (-1)^{i+1} (r_1 \wedge \cdots \wedge \hat{r}_i \wedge \cdots r_k)\, r_i(x)$$

$\overset{1:1}{\Longrightarrow}$ interior derivation by V on $\underline{\mathrm{End}}_{LM}$

$\forall F$, $\frown : V \otimes \underline{\mathrm{End}}_{LM}(F) \to \underline{\mathrm{End}}_{LM}(F)$.

Now for $(F, \underset{\mathrm{End}_{LM}}{\delta}) \in LM$, $x \in V \Rightarrow x \frown \delta$ gives functorially $\boxed{\mu(x)_F : F \to F\langle 2 \rangle}$

alg of int. der. $= S(V)$ $\overset{\shortparallel}{}$

$\Rightarrow \mu : R^\vee \xrightarrow[\text{gr-alg}]{} \bigoplus_{d \in \mathbb{Z}} \mathrm{Hom}_{K^b H(h)}(\mathrm{Id}, \langle d \rangle)$.

forgetting δ; i.e. taking cohomology of Koszul resol.

"left monodromic action"

$\Updownarrow$ Koszul to the usual.

$m : R \longrightarrow \bigoplus_{d \in \mathbb{Z}} \mathrm{Hom}_{K^b H(h^*)}(\mathrm{Id}, (d))$.

6. Examples.

$\boxed{\mathrm{Ex}}$. $h_k^{SL_2} = V$

$R = S(V^*)$.

$\mathcal{H} = \mathbb{S}\mathrm{Bim}$.

(T_s, δ_{T_s})

$LM \xrightarrow{\sim} K^b \mathbb{S}\mathrm{Bim}$

$$\begin{array}{ccc} R(1) & & R(1) \\ \uparrow \scriptstyle{\bullet} & & \uparrow \scriptstyle{\bullet} \\ B_s & \mapsto & B_s \\ \uparrow \scriptstyle{\bullet} & & \uparrow \scriptstyle{\bullet} \\ R(-1) & & R(-1) \end{array}$$

$-\alpha_s \otimes \mathrm{id}$

$\delta \circ \delta + \partial(\delta) = 0$

$\Rightarrow \partial(\delta) = -\delta \circ \delta$.

lifts to deg 2 components of δ

Koszul resol $\quad k\langle -\alpha_s \rangle \otimes R \xrightarrow{\partial} R \xrightarrow{\varepsilon_R} k$.

$\alpha_s \otimes 1 \mapsto \alpha_s \mapsto 0$

$\boxed{\text{Eg}}$ $W = S_3,\ S = \{s,t\}$.
$V^* = \mathfrak{h}_k^{SL_3} = k\langle \alpha_s, \alpha_t \rangle$

$B_s \to R(1)$, $f \otimes_s g \mapsto fg$ $\qquad R(-1) \to B_s$, $f \mapsto f\Delta_s$

$B_t \to R(1)$ $\qquad R(-1) \to B_t$

$$LM(\mathfrak{h}_k^{SL_3}, S_3) \xrightarrow{\sim} K^b \overline{\mathbb{S}Bim}(\mathfrak{h}_k^{SL_3}, S_3).$$

$$F := \begin{tikzcd} R(3) \\ B_t(2) \arrow[u] \\ B_s \arrow[u] \arrow[uu, bend right, "(-\alpha_t)\otimes"'] \\ R(-1) \arrow[u] \arrow[uu, bend right, "(-\alpha_s)\otimes"'] \arrow[uuu, bend left, "(\alpha_s \wedge \alpha_t)\otimes id"] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} R(3) \\ B_t(2) \arrow[u] \\ B_s \arrow[u] \\ R(-1) \arrow[u] \end{tikzcd}$$

δ_F has several components.
the eqn $\delta_F \circ \delta_F = -\partial \delta_F$ describes their lifts (being diff up to higher htpy)

deg 2 component: $R(-1) \xrightarrow{\alpha_s \otimes} B_t(2)$, $B_s \xrightarrow{\alpha_t \cdot} R(3)$.
which corresponds to $V^* \otimes R$ term in the resolution.

deg 3 components $R(-1) \xrightarrow[\ \circ(-\alpha_s \otimes\ ) + (-\alpha_t \otimes\ )\circ\ ]{} R(3)$

$$= \alpha_s \otimes (\alpha_t \cdot id) - \alpha_t \otimes (\alpha_s \cdot id).$$

which lifts to $(\alpha_s \wedge \alpha_t) \otimes 1$ in $\wedge^2 V^* \otimes R$ term.

$\boxed{\text{Eg}}$ (Morphism of LM cplx whose cone gives F).

$$\begin{tikzcd} & R(3) \\ B_s \arrow[ur, "-\alpha_t \otimes"] \arrow[r] & B_t(2) \arrow[u] \\ R(-1) \arrow[u] \arrow[ur, "-\alpha_s \otimes"'] \arrow[uur, bend right, "(\alpha_s \wedge \alpha_t)\otimes id"'] & \end{tikzcd}$$

complete $f \otimes_s g \mapsto fg\Delta_t$

resolve

records the failure of to be a ~~cplx map~~ morphism of cplx.

w/ we can describe F as the "cone"

Recall $S(V) \xrightarrow{\mu} \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{K^b\overline{H}}(\mathrm{Id}, \langle d\rangle)$, $\langle d\rangle = [d](-d)$

$\forall x \in V$, $\mu(x)_{T_s} : T_s \to T_s\langle 2\rangle$ is given by chain map

$$\begin{tikzcd}
R(1) & & R(-1) \\
B_s \arrow[u] & & B_s(-2) \arrow[u] \\
R(-1) \arrow[u] \arrow[uurr, "-\alpha_s(x)"] \arrow[uu, bend left, "-\alpha_s\otimes id"] & & R(-3) \arrow[u] \arrow[uu, bend right]
\end{tikzcd}$$

$\forall x \in V$, $\mu(x)_{F} : F \to F\langle 2\rangle$ is given by chain map.

$$\begin{tikzcd}
R(3) & & R(1) \\
B_t(2) \arrow[u] & & B_t \arrow[u] \\
B_s \arrow[u] \arrow[uurr, "(\alpha_t\otimes\alpha_s(x) id_R"] & & B_s(-2) \arrow[u] \\
R(-1) \arrow[u] \arrow[uurr, "-\alpha_s\otimes\alpha_t(x) id_R"] \arrow[rr, phantom, "-\alpha_s(x)"] & & R(-3)
\end{tikzcd}$$

$-\alpha_t(x)$

monoidal product.

$$\begin{tikzcd}
\boxed{???} \arrow[r, leftrightarrow, "\overset{?}{\sim}"] \arrow[d, "For"] & (\mathcal{H}(h^*), \star) \arrow[d, "For"] \\
K^b\overline{H}(h) \arrow[r, leftrightarrow, "\sim", "K"'] & K^b\underline{H}(h^*)
\end{tikzcd}$$

7. Monoidal Koszul Duality.

(Main Result of AMRW).

First we construct.

$$[FM(h,W) \xrightarrow[\delta \mapsto \delta_{R^\vee}(\delta)]{For} LM(h,W)] \overset{\text{Koszul}}{\rightsquigarrow} [K^b\mathcal{H}(h^*) \xrightarrow{K^b(For)} K^b\underline{H}(h^*)].$$

free-monodromic cplx.

by relaxing the MC eqn $\delta\circ\delta + \partial(\delta) \overset{\text{flat}}{=} 0$ in $\underline{End}_{LM}$.

to $\delta\circ\delta+\partial(\delta) = \Theta$ in. $\underline{End}_{FM}$. curved.

where $\begin{cases} \underline{End}_{FM}(F) := \Lambda^{\bullet} \otimes_k \underline{End}(F) \otimes_k R^\vee \left(= A \otimes_R \underline{End}(F) \otimes_k R^\vee\right), \\ \Theta := \sum (\mathrm{Id}_F \star \overset{V^*}{e_i}) \otimes \overset{V}{e_i^\vee} \in \underline{End}_{FM}(F) \end{cases}$

↳ multiplication on $K^b\mathcal{H}(h)$

$\boxed{\begin{array}{l} A = (\Lambda^{\bullet} \otimes_k R, \partial) \\ \Lambda^{\bullet} = \Lambda^{\bullet}(V^*) \\ V^* = \bigoplus k\langle e_i \end{array}}$

Rmk. the curv. Θ becomes zero after passing to $\underline{\mathrm{End}}_{LM} := \Lambda \otimes_{\mathbb{k}} \underline{\mathrm{End}}(\mathcal{F})$.

Eg Let $\theta = \sum e_i \otimes \mathrm{id} \otimes e_i^\vee$, $\theta_s = \sum s(e_i) \otimes \mathrm{id} \otimes e_i^\vee$

Eg the obj $\widetilde{T_{id}} := R \circlearrowleft \theta$ is the FM lift of T_{id}.

Eg $\mathfrak{h}_{\mathbb{k}}^{SL_2}$.

$$\widetilde{T_s} := \begin{array}{ll} R(1) \circlearrowleft \theta_s & \\ \uparrow \quad \curvearrowleft 1 \otimes d \otimes \alpha_s^\vee & \\ B_s \circlearrowleft \theta_s & \\ \uparrow d & \\ R(-1) \circlearrowleft \theta & \end{array} \quad -\alpha_s \otimes \mathrm{id} \otimes 1 \text{ from } R(-1) \text{ to } R(1)$$

is the FM lift of T_s.

Thm (ARMRW) (JAMS 2019) ("Koszul duality for Kac-Moody & chars of tilting modules")

There exists a monoidal additive category

$$\begin{array}{ccc} (\mathrm{Tilt}\,\mathrm{FM}(\mathfrak{h},W), \hat{\star}) & \subset & \mathrm{FM}(\mathfrak{h},W) \\ K^{mon} \ \wr\!\uparrow & & \updownarrow K \\ (\mathcal{H}(\mathfrak{h}^*,W), \star) & \subset & K^b\mathcal{H}(\mathfrak{h}^*,W) \end{array}$$

① K^{mon} is a monoidal equiv.

② $K^{mon} \circ (1) \cong \langle 1 \rangle \circ K^{mon}$

③ $K^{mon}(B_w) \cong \widetilde{T_w}$

④ $\mathrm{TiltFM} = \boxed{???}$ is the full additive consists of $\oplus$ of $\widetilde{T_w}\langle n \rangle$, $\widetilde{T_w} = \widetilde{T_{s_1}} \hat{\star} \cdots \hat{\star} \widetilde{T_{s_r}}$

Rmk To prove that $\hat{\star}$ needs geometry (why we restrict ourself to Cartan realization of crystallographic coxeter) satisfying $(g \circ f) \hat{\star} (g' \circ f') = (g \hat{\star} g') \circ (f \hat{\star} f')$ (bifunctor property).

$$\begin{array}{l} (F \xrightarrow{f} G \xrightarrow{g} H) \\ F' \xrightarrow{f'} G' \xrightarrow{g'} H' \end{array}$$