<u>Def</u> Generalized Cartan matrix (GCM) of size l:

$A = (a_{ij}) \in M_{l \times l}(\mathbb{Z})$ s.t. (a) $a_{ii} = 2 \quad \forall i$

(b) $a_{ij} \leq 0 \quad \forall i \neq j$

(c) $a_{ij} = 0$ iff $a_{ji} = 0$

(Minimal) realization of A is a triple $(\mathfrak{h}, \pi, \pi^\vee)$ where

$\mathfrak{h}$: cplx v.s. of dimen $l + \underbrace{\text{corank} A}_{l - \text{rank} A}$

$\pi = \{\alpha_i\}_{i=1}^{l} \subset \mathfrak{h}^*$

$\pi^\vee = \{\alpha_i^\vee\}_{i=1}^{l} \subset \mathfrak{h}$) linearly indep $\qquad \alpha_i^\vee(\alpha_j) = a_{ij}$.

<u>Eg.</u> $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, $\{e_1^*, e_2^*, e_3^*\} \subseteq \mathfrak{h}^*$ $\quad \alpha_i = e_i$

$\{e_1, e_2, e_3\} \subseteq \mathfrak{h}$ $\quad \alpha_1^* = 2e_1 - 2e_2 + k_1 e_3$

$\alpha_2^* = -2e_1 + 2e_2 + k_2 e_3 \quad k_1 \neq k_2$

<u>Fact</u>. Such a triple is unique up to isom.

$\theta : (\mathfrak{h}, \pi, \pi^\vee) \longrightarrow (\mathfrak{h}', \pi', \pi^{\vee\prime})$

$\theta : \mathfrak{h} \longrightarrow \mathfrak{h}'$ s.t. $\theta(\pi) = \pi'$

$\theta^*(\pi^{\vee\prime}) = \pi^\vee$.

<u>Def</u>. Kac-Moody alg $\mathfrak{g} = \mathfrak{g}(A)$ is the Lie alg $/ \mathbb{C}$ generated by $\mathfrak{h}, e_i, f_i, 1 \leq i \leq l$, s.t.

(R1) $[\mathfrak{h}, \mathfrak{h}] = 0$

(R2) $[h, e_i] = \alpha_i(h) \cdot e_i$
$ [h, f_i] = -\alpha_i(h) f_i$

(R3) $[e_i, f_j] = \delta_{ij} \alpha_i^\vee$

(R4) $(\text{ad} e_i)^{1 - a_{ij}} e_j = 0$

(R5) $\phantom{(\text{ad}} f_i \qquad f_j$ $\qquad$ Serre relation.

$\mathfrak{n}^+$: Lie subalg gene by e_i with (R4)

$\mathfrak{n}^-$ $\qquad\qquad\qquad\qquad\quad f_i \qquad$ (R5)

Cartan involution:

ω: $e_i \to -f_i$, $f_i \to -e_i$, $h \to -h$ $\qquad \omega(\mathfrak{n}^\pm) = \mathfrak{n}^\mp$.

A is called decomp if $A = \begin{pmatrix} A_{Y_1} & \\ & A_{Y_2} \end{pmatrix}$ $\rightsquigarrow \mathfrak{g}(A) \simeq$ as Lie alg $\mathfrak{g}(A_{Y_1}) \oplus \mathfrak{g}(A_{Y_2})$

<u>Theorem</u>

(a) (Tri decomp) $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$

(b) (Root space decomp) $\mathfrak{n}^\pm = \bigoplus_{\alpha \in Q^+ \setminus \{0\}} \mathfrak{g}_{\pm\alpha}$

(c) $\underset{\substack{\shortparallel \\ \text{mult } \alpha}}{\dim \mathfrak{g}_\alpha} < \infty$

$Q = \bigoplus_{i=1}^{l} \mathbb{Z}\alpha_i$ (root lattice)

$Q^+ = \bigoplus_i \mathbb{Z}_{\geq 0} \alpha_i$, $Q^- = -Q^+$

$\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid [h, x] = \alpha(h) \cdot x \quad h \in \mathfrak{h} \}$

$\Delta = \{ \alpha \in Q \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0 \}$

For any subset $Y \subseteq \{1, \cdots, l\}$

$\Delta_Y = \Delta \cap \bigoplus_{i \in Y} \mathbb{Z}\alpha_i$.

$\mathfrak{g}_Y := \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_Y} \mathfrak{g}_\alpha \right)$ (Levi comp of $\mathfrak{p}_Y$)

$\mathfrak{u}_Y^\pm = \bigoplus_{\alpha \in \Delta^\pm \setminus \Delta_Y^\pm} \mathfrak{g}_\alpha$ (nil-rad of $\mathfrak{p}_Y$)

$\mathfrak{p}_Y^\pm := \mathfrak{g}_Y \oplus \mathfrak{u}_Y^\pm$ (standard para ...)

<u>Eg</u>. $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$\mathfrak{g} = \mathfrak{sl}_3$, $Y = \{1\}$.

$\mathfrak{g}_Y = \begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix}$

$\mathfrak{u}_Y^+ = \left(\begin{array}{cc|c} & & * \\ & & * \\ \hline & & \end{array}\right)$

$\mathfrak{p}_Y^+ = \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}$

Weyl group $W \subseteq \mathrm{Aut}(\mathfrak{h}^*)$ generated by s_i

$\chi \in \mathfrak{h}^*$ $\quad s_i(\chi) := \chi - \langle \chi, \alpha_i^\vee \rangle \alpha_i$.

<u>Def</u>. $T: V \to V$ linear map of cplx v.s.

T: locally finite if for any $v \in V$, $\exists_{v \in} W \subseteq V$ f.d s.t. $T(W) \subseteq W$.

locally nilpotent

$T|_W$ is nilpotent

If T is locally finite, $\exp T := I + \sum_{n\geq 1} \frac{T^n}{n!}$.

$T, S : V \to V$ two linear maps s.t.

T is locally finite & $\{(\mathrm{ad}\,T)^n S\}_{n\in\mathbb{N}}$ spans a f.d subspace $\subseteq \mathrm{End}(V)$

$$(\exp T)\cdot S\, \exp(-T) = \sum_{n\geq 0} \frac{(\mathrm{ad}\,T)^n}{n!}(S) = \exp(\mathrm{ad}(T))\cdot S.$$

$\mathfrak{g}$: K.-M. alg.

$(V\ (= \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda), \pi)$ is called an integrable repre if all e_i and f_i act locally nilpotently on V.

$$s_i(\pi) := \exp \pi(f_i)\, \exp(-\pi(e_i))\, \exp \pi(f_i) \in \mathrm{Aut}\, V.$$

(V,π) integrable repre

<u>Lemma</u>

- ad is an integrable rep of $\mathfrak{g}$.
- $s_i(\pi)(V_\lambda) = V_{s_i\lambda}$ ($\Rightarrow$ $\mathrm{mult}_V \lambda = \mathrm{mult}_V w\lambda$ $w\in W$; Δ is stable under the action of W; Δ the set of roots)
- $s_i(\pi)(x\cdot v) = (s_i(\mathrm{ad})x)(s_i(\pi)\cdot v)$, $x\in\mathfrak{g}$, $v\in V$ ($\Rightarrow$ $\pi = \mathrm{ad}$, $s_i(\mathrm{ad})$ is a Lie alg auto)
- Let m_{ij} be the order of $s_i s_j \in W$, then
$$(s_i(\pi)\, s_j(\pi))^{m_{ij}} = (s_j(\pi)\, s_i(\pi))^{m_{ij}}$$

<u>Coxeter group</u>. (W, S), S: a fixed subset of ele of order 2

$$W := \hat{W}/\sim \qquad \hat{W} = \coprod S_i$$

$s^2 = 1$, $\forall s\in S$

$s\neq t$, $(st)^{m_{s,t}} = 1$ with $m_{s,t}\geq 2$ or ∞

$\varepsilon : W \longrightarrow \{\pm 1\}$

$l(w) = l$, $w = s_1\cdots s_l \longrightarrow (-1)^l$.

Prop. Let W be the Weyl group of $\mathfrak{g}$.
Then W is a crystallographic Coxeter group with finite S.
$m_{ij} \in \{2, 3, 4, 6, \infty\}$

More precisely, $S = \{s_i\}_{1 \le i \le l}$. The order m_{ij} of $s_i s_j$ is given as follows

	m_{ij}	a
(1)	$=2$	$=0$
(2)	3	$=1$
(3)	4	2
(4)	6	3
(5)	∞	≥ 4.

$a = a_{ij} \cdot a_{ji}$

Eg. $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, $\quad W = \langle s, t \mid s^2 = t^2 = 1 \rangle$

$m_{12} = \infty \qquad\qquad = D_\infty$

GCM

Def. A is called symmetrizable if $\exists\ D = \mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_l)$, $\varepsilon_i \in \mathbb{Q}$
s.t. $D^{-1}A$ is symmetric.

$\exists!\ D = \mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_l)$ ← minimal
s.t. (a) $D^{-1}A$ is sym
(b) $\varepsilon_i \in \mathbb{N}$
(c) If $D' = \mathrm{diag}(\varepsilon_1', \cdots, \varepsilon_l')$ also satisfies (a) and (b)
then $\varepsilon_i \le \varepsilon_i' \ \forall i$

Prop. $\mathfrak{g}$ (sym K-M. alg)
$\mathfrak{h}$ carries a non deg symm W-inv bilinear form, $\langle\ ,\ \rangle$.

(pf) $\mathfrak{h}' := \bigoplus_{i=1}^{\ell} \mathbb{C}\alpha_i^\vee \subset \mathfrak{h}$

$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$ ← vector space complement.

Define $\langle , \rangle$

$\langle \mathfrak{h}'', \mathfrak{h}'' \rangle = 0$, $\langle h, \alpha_i^\vee \rangle = \langle \alpha_i^\vee, h \rangle = \alpha_i(h)\cdot \varepsilon_i$

Def $\langle , \rangle$: normalized invariant form on $\mathfrak{h}$ (depends on the choice of $\mathfrak{h}''$)

$\nu : \mathfrak{h} \to \mathfrak{h}^*$, $\nu(h)(h') = \langle h, h' \rangle$ for $h, h' \in \mathfrak{h}$

Theorem $\mathfrak{g}$: sym K.-M. alg

$\exists$ a bilinear form $\langle , \rangle$ on $\mathfrak{g}$ satisfying

(a1) $\langle , \rangle$ is invariant, i.e. $\langle [x,y], z \rangle + \langle y, [x,z] \rangle = 0$ $\forall x, y, z \in \mathfrak{g}$

(a2) $\langle , \rangle|_{\mathfrak{h}}$ is a normalized inv form.

Such a form is unique (if we fix $\langle , \rangle|_{\mathfrak{h}}$) and satisfies

(b1) $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0$ if $\alpha + \beta \neq 0$

(b2) $[x, y] = \langle x, y \rangle \nu^{-1}(\alpha)$ for $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, $\alpha \in \Delta$

(pf). $k \in \mathbb{Z}$, $\mathfrak{g}_k := \bigoplus_{\substack{\alpha \in \Delta \cup \{0\} \\ |\alpha| = k}} \mathfrak{g}_\alpha$, $\mathfrak{g}(N) := \bigoplus_{k=-N}^{N} \mathfrak{g}_k$

$\alpha = \sum_i n_i \alpha_i$

$|\alpha| = \sum_i n_i$

Extend the bilinear form on $\mathfrak{g}(1)$ by — $\langle f_j, e_i \rangle = \langle e_i, f_j \rangle = \delta_{ij} \varepsilon_i$

— $\langle \mathfrak{g}_{k_1}, \mathfrak{g}_{k_2} \rangle = 0$ if $k_1 + k_2 \neq 0$ & $|k_1|, |k_2| \leq 1$

Assume $\langle , \rangle$ is defined on $\mathfrak{g}(N-1)$,

Want to define $\langle x, y \rangle$ for $x \in \mathfrak{g}_N$, $y \in \mathfrak{g}_{-N}$

write $y = \sum_i [u_i, v_i]$, $u_i, v_i \in \mathfrak{g}(N-1)$ homogeneous, $\langle x, y \rangle := \sum_i \langle [x, u_i], v_i \rangle$

$\mathfrak{g}$: K.-M. alg (not necess sym).

Let $\tau \subset \mathfrak{g}$ be the maximal ideal s.t. $\tau \cap \mathfrak{h} = \{0\}$

$$\bar{\mathfrak{g}}(A) := \frac{\mathfrak{g}(A)}{\tau}, \quad \text{define } \bar{\Delta}^{\pm}, \bar{\mathfrak{n}}^{\pm}, \dots \text{ similarly}$$

<u>Def.</u> $\hat{U}(\bar{\mathfrak{g}}) := \prod_{d \geq 0} U(\bar{\mathfrak{b}}^{-}) \otimes_{\mathbb{C}} \underbrace{U_d(\bar{\mathfrak{n}}^{+})}_{\text{homoge of degree } d \,\subseteq\, U(\bar{\mathfrak{n}}^{+})}$ a certain completion of $U(\bar{\mathfrak{g}})$

for $x_d, y_d \in U(\bar{\mathfrak{b}}^{-}) \otimes U_d(\bar{\mathfrak{n}}^{+})$

$$\Big(\sum_{d \geq 0} x_d\Big) \cdot \Big(\sum_{m \geq 0} y_m\Big) = \sum_{k} \sum_{d, m \geq 0} (x_d \cdot y_m)_k .$$

Back to symmetrized case.

$\alpha \in \bar{\Delta}^{+}$, $\{e_\alpha^1, \dots, e_\alpha^p\}$ be a basis of $\mathfrak{g}_\alpha$

$\{e_{-\alpha}^1, \dots, e_{-\alpha}^p\}$ dual basis of $\mathfrak{g}_{-\alpha}$

$$\Omega_\alpha := \sum_{i=1}^{p} e_{-\alpha}^i e_\alpha^i$$

$$\Omega_0 = \sum_k u_k u^k \quad \text{on } \mathfrak{h}$$

Casimir-Kac element ($\rho \in \mathfrak{h}^*$, $\langle \rho, \alpha_i^\vee \rangle = 1$)

$$\Omega^Y := 2\nu^{-1}(\rho) + \Omega_0 + 2 \sum_{\alpha \in \bar{\Delta}_Y^{+}} \Omega_\alpha \in \hat{U}(\bar{\mathfrak{g}}_Y)$$

indep of the choice of ρ.

<u>Theorem</u> $\Omega^Y \in Z(\hat{U}(\bar{\mathfrak{g}}_Y))$ $\quad Y \subset \{1, \dots, \ell\}$