III-1. Category $\mathcal{O}$

$\mathfrak{g}$: a KM alg. For $\lambda, \mu \in \mathfrak{h}^*$, write $\lambda \geq \mu$ if $\lambda - \mu \in Q^+ = \bigoplus_{i=1}^{l} \mathbb{Z}_{\geq 0} \alpha_i$

Set $\mathfrak{h}^*_{\leq \lambda} := \{ \mu \in \mathfrak{h}^* \mid \mu \leq \lambda \}$

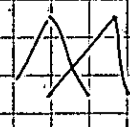
Def. $\mathcal{O}$ for KM alg.

Objects : $\mathfrak{g}$-modules M satisfies

(C1) M is a weight module (w.r.t. $\mathfrak{h}$) and $\underset{\text{mult}_\lambda(M)}{\overset{\parallel}{\dim M_\lambda}} < +\infty$ for any wt.

Set $P(M) = \{ \lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0 \}$ set of wts of M

(C2) $\exists\ \lambda_1, \cdots, \lambda_k$ (finite set!) $\in \mathfrak{h}^*$ s.t. $P(M) \subseteq \bigcup_{i=1}^{k} \mathfrak{h}^*_{\leq \lambda_i}$,



Morphisms : $\mathfrak{g}$-module homomorphisms.

$\mathcal{O}_Y$: parabolic version $Y \subseteq \{1, 2, \cdots, l\}$ $\mathfrak{g} \to \mathfrak{g}_Y$

Facts:

(1) $\mathcal{O}$ is closed under taking submodule and subquotient.
$\oplus$ and $\otimes$ with finitely many objects in $\mathcal{O}$.

ω: Cartan involution.
$e_i \to -f_i$
$f_i \to -e_i$
$h_i \to -h_i$

(2) $M \in \mathcal{O} \Rightarrow M^\sigma \in \mathcal{O}$. $M^\sigma = M^\vee$ with the $\mathfrak{g}$-module structure $x \cdot f(m) := -f(\omega(x) \cdot m)$ $\forall x \in \mathfrak{g},\ f \in M^\vee,\ m \in M$.

and $(M^\sigma)^\sigma = M$. $M = \bigoplus_\lambda M_\lambda$ $M^\vee = \bigoplus_\lambda M_\lambda^*$ (restricted dual).

Verma module

$\lambda \in \mathfrak{h}^* \rightsquigarrow \mathbb{C}_\lambda$: the 1-dim $\mathfrak{h}$-module. $h \cdot z = \lambda(h) z$. $\forall z \in \mathbb{C}_\lambda$

Let $\mathfrak{n}^+$ acts on $\mathbb{C}_\lambda$ as zero $\Rightarrow$ $\mathbb{C}_\lambda$ is a left $U(\mathfrak{b})$-module

$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ a left $U(\mathfrak{g})$-module

$\cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$ a left $U(\mathfrak{b}^-)$-module by PBW.

$\mathfrak{n}^-$ acts as left multiplication.
$\mathfrak{h}$ acts by adjoint.

$M(\lambda)$ is a free $U(\mathfrak{n}^-)$-module with $1 \otimes 1$ a generator.
is a wt module.

Def. A $\mathfrak{g}$-module $L^{\neq 0}$ is called a ht wt module with ht wt λ if L is a quotient of $M(\lambda)$.

Facts:

(1) $\dim L_\lambda = 1$ (the image of $1 \otimes 1$ under the quotient).

(2) $\forall \mu \in P(L)$, $\mu \leq \lambda$

(3) $\mathrm{End}_{\mathfrak{g}} L = \mathbb{C}\, Id_L$

(4) $M(\lambda) \in \mathcal{O}$ (and hence any ht wt module $M \in \mathcal{O}$)

Lemma 2.1.2, 2.1.3 Any Verma $M(\lambda)$ has a unique maximal proper submodule $M'(\lambda)$ ← sum of all proper submodules

and unique irreducible quotient $L(\lambda)$.

$\forall L \in \mathcal{O}$ irreducible, $\exists \lambda \in \mathfrak{h}^*$ s.t. $L \cong L(\lambda)$. ← $\exists$ maximal wt λ of L, $v_\lambda \in L$. Then $\theta: M(\lambda) \to L$, $1 \otimes 1 \to v_\lambda$ is a $\mathfrak{g}$-hom, L: irred. $\Rightarrow \theta$ surj. So $M(\lambda)/\ker\theta \cong L$ $\overset{\text{irred}}{\Rightarrow} L \cong L(\lambda)$

$$\Rightarrow \quad \mathfrak{h}^* \xleftrightarrow{\text{bijection}} \{\text{irred in } \mathcal{O}\}/_{iso}$$
$$\lambda \longleftrightarrow L(\lambda)$$

Formal character

$\lambda \in \mathfrak{h}^* \rightsquigarrow e^\lambda$ a formal symbol.

Let
$$\mathcal{A} = \left\{ a = \sum_{\lambda \in \mathfrak{h}^*} a_\lambda e^\lambda \;\middle|\; (d1) \text{ and } (d2) \right\}.$$

(d1) $a_\lambda \in \mathbb{Z}$ $\forall \lambda$.

(d2) $\exists \lambda_1, \dots, \lambda_k \in \mathfrak{h}^*$ s.t. $a_\lambda = 0$ $\forall \lambda \notin \bigcup_{i=1}^{k} \mathfrak{h}^*_{\leq \lambda_i}$

↖ depending on a

$\mathcal{A}$ is <u>closed</u> under the following <u>well-defined</u> multiplication — Due to (d2)

$$\left(\sum_\lambda a_\lambda e^\lambda\right)\left(\sum_\mu b_\mu e^\mu\right) := \sum_\theta \left(\sum_{\lambda+\mu=\theta} a_\lambda b_\mu\right) e^\theta$$

$\lambda \uparrow, \mu \downarrow$

In particular, $e^\lambda e^\mu = e^{\lambda+\mu}$ and $e^0 = 1$

$$\forall M \in \mathcal{O},\ M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda \rightsquigarrow \mathrm{ch}\, M = \sum_{\lambda \in \mathfrak{h}^*} (\mathrm{mult}_\lambda M) \cdot e^\lambda \in \mathcal{A} \quad \text{by (c1), (c2).}$$

Facts: $M, N \in \mathcal{O}$.

① $ch(M \oplus N) = ch M + ch N$ $\quad ch(M \otimes N) = ch(M) \cdot ch(N)$

③ $ch(M/N) = ch M - ch N$ if N is a submodule of M.

④ $0 \to A \to B \to C \to 0$ SES in $\mathcal{O}$ $\Rightarrow$ $ch B = ch A + ch C$.

⑤ W acts on $\mathcal{A}$ by $w \cdot e^{\lambda} = e^{w(\lambda)}$

Integrable modules

dominant integral wt's

$$D := \{ \lambda \in \mathfrak{h}^* \mid \lambda(\check{\alpha}_i) \in \mathbb{Z}_{\geq 0} \quad \forall i \}.$$

$\forall \lambda \in D$, define $M_1(\lambda)$ = the $\mathfrak{g}$-submodule of $M(\lambda)$ generated by

$$\{ f_i^{\lambda(\alpha_i)+1} \otimes 1 \mid 1 \leq i \leq l \}.$$

← quotient out this makes all f_i nilpotent

Define the maximal integrable ht wt $\mathfrak{g}$-module with ht wt $\lambda \in D$ to be

$$L^{max}(\lambda) := M(\lambda) / M_1(\lambda)$$

integrable: e_i and f_i act locally nilpotently $\forall i$.

- Verma $M(\lambda)$ is NOT integrable

Lemma 2.1.6.
$\forall \lambda \in D$, $\forall 1 \leq i \leq l$, $\quad e_j (f_i^{\lambda(\check{\alpha}_i)+1} \otimes 1) = 0$ in $M(\lambda)$ $\quad \ulcorner M_1(\lambda) \subsetneq M(\lambda) \lrcorner$

pf: Follows from $e_i f_i^n = f_i^n e_i + n f_i^{n-1} (\check{\alpha}_i - n + 1)$. ← induction on n.

Lemma 2.1.7.
(1) $\forall \lambda \in D$, $L^{max}(\lambda)$ is an integrable $\mathfrak{g}$-module and hence its subquotients are integrable as well.

(2) Any integrable ht wt $\mathfrak{g}$-module L must have ht wt $\lambda \in D$ and L is a quotient of $L^{max}(\lambda)$.

proof:
(1) by def
(2) L is a quotient of $M(\lambda)$ for some $\lambda \in \mathfrak{h}^*$ (ht wt). $\quad L_\lambda$: ht wt space of L.

L is integrable $\mathfrak{g}$-module and hence as $\mathfrak{g}_i$-module $\forall 1 \leq i \leq l$ $\quad \mathfrak{g}_i \cong \mathfrak{sl}_2$.

L_λ generates a f.d $\mathfrak{sl}_2$-module $\Rightarrow$ $\lambda(\check{\alpha}_i) \in \mathbb{Z}_{\geq 0}$ $\forall i$ and $f_i^{\lambda(\check{\alpha}_i)+1} \cdot v_\lambda = 0 \quad \forall v_\lambda \in L_\lambda$.

$\Downarrow$

$\lambda \in D$

Crollary 2.1.8 $\quad D \xleftrightarrow{\text{bijection}} \left\{ \begin{array}{l} \text{irred.} \\ \text{integrable} \end{array} \text{in } \mathcal{O} \right\} \Big/ \text{iso}$

$\lambda \longleftrightarrow L(\lambda)$

For $V \in \mathcal{O}$, $\mu \in \mathfrak{h}^*$, define $[V : L(\mu)]$

Remark: Do NOT have composition series in KM in general!

Lemma 2.1.9 (replacement)

$\forall V \in \mathcal{O}$, $\lambda \in \mathfrak{h}^*$, $\exists$ a filtration of submodules

$0 = V_0 \subset V_1 \subset \cdots\cdots \subset V_p = V$ and a subset $J \subseteq \{1, 2, \cdots\cdots p\}$

such that ① if $j \in J$ then $W_j := V_j / V_{j-1} \cong L(\lambda_j)$ for some $\lambda_j \geq \lambda$

② if $j \notin J$ then $(W_j)_\mu = 0 \quad \forall \mu \geq \lambda$

proof: Induction on $a(V, \lambda) = \sum_{\mu \geq \lambda} \dim V_\mu$ ※

Let F_λ be a filtration in the lemma, where λ is choosen with $\lambda \leq \mu$.

Define $[V : L(\mu)]_{F_\lambda} := \left| \{ j \mid W_j \cong L(\mu) \} \right|$

$=$ counting the number of appearances of μ in $\{ \lambda_j \mid j \in J \}$

Then for any $V \in \mathcal{O}$, take ch to the filtration F_λ.

$$(☆) \quad \operatorname{ch} V = \sum_{\nu \geq \lambda} [V : L(\nu)]_{F_\lambda} \cdot \operatorname{ch} L(\nu) + R_\lambda$$

① ↓ finite sum by Lemma 2.1.9

② $R_\lambda = \sum_{\theta \in \mathfrak{h}^*} a_\theta(\lambda) e^\theta$ such that $\theta \geq \lambda \Rightarrow a_\theta(\lambda) = 0$

If we take some $\lambda' \leq \mu$, then for any $\nu \geq \lambda$ and $\nu \geq \lambda'$,

we have $\quad [V : L(\nu)]_{F_\lambda} = [V : L(\nu)]_{F_{\lambda'}}$

$\Rightarrow$ May drop F_λ so $[V : L(\nu)]$ is well-defined

$\forall V \in \mathcal{O}$ and $\mu \in \mathfrak{h}^*$, take $\lambda \in \mathfrak{h}^*$ with $\lambda \leq \mu$. $\rightarrow F_\lambda$

Define the multiplicity of $L(\mu)$ in V by

$$[V : L(\mu)] := \left| \{ j \mid W_j \cong L(\mu) \} \right|$$

in F_λ

This is well-defined

We say $L(\mu)$ is a component of V if $[V : L(\mu)] \neq 0$.

($\Leftrightarrow$ $L(\mu)$ is a subquotient of V).

Let $\{ a_i = \sum_\lambda a_i(\lambda) e^\lambda \}_{i \in I}$ be a family of elements in $\mathcal{A}$. I: index set.

We call it locally finite if $\forall \mu \in \mathfrak{h}^*$, the set $\{ i \in I \mid a_i(\mu) \neq 0 \}$ is finite.

Then $\sum_{i \in I} a_i \in \mathcal{A}$ 「$\notin A$ in general!」

Lemma 2.1.12. $\forall V \in \mathcal{O}$, $\{ [V : L(\mu)] \, ch\, L(\mu) \}_{\mu \in \mathfrak{h}^*}$ is locally finite.

Also, $$ch\, V = \sum_{\mu \in \mathfrak{h}^*} [V : L(\mu)] \, ch\, L(\mu)$$

proof: Choose $\lambda \leq \mu$. $\rightarrow$ Take $(*)$ on both sides, and compare coefficients of e^λ. #

Lemma 2.1.13. ← new for KM.

$\forall \lambda \in \mathfrak{h}^*$, $$ch\, M(\lambda) = e^\lambda \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{-\text{mult}\, \alpha}$$

$$= e^\lambda \times \Big(\underbrace{\prod_{\alpha \in \Delta^+} (1 + e^{-\alpha} + e^{-2\alpha} + \cdots)^{\text{mult}\, \alpha}}_{\text{denoted by } R^{-1}} \Big)$$

proof: PBW $\Rightarrow$ $M(\lambda)$ has a basis in terms of monomials in a basis for $\mathfrak{n}^-$. #

1

II-2 Weyl-Kac character formula

Assume $\mathfrak{g}$: symmetrizable. $\langle\ ,\ \rangle$: a normalized inv. form on $\mathfrak{h}^*$

Theorem 2.2.1

L : an integrable hт wt $\mathfrak{g}$-module with hт wt λ (Then $\lambda \in D$ by lemma 2.1.7)

$$\Rightarrow \operatorname{ch}(L) = \frac{\sum_{w \in W} \varepsilon(w)\, e^{w*\lambda}}{R} \in \mathcal{A}$$

「determined by λ」

where $R = \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\operatorname{mult}\alpha}$

$*$: shift action : $w * \lambda := w(\lambda+\rho) - \rho \quad \forall \lambda \in \mathfrak{h}^*$ $\quad \rho \in \mathfrak{h}^*$ s.t. $\rho(\alpha_i^\vee) = 1 \ \forall i$

does not depend on ρ.

$\varepsilon : W \to \pm 1$ sign repn.

Corollary : $R = \sum_{w \in W} \varepsilon(w)\, e^{w(\rho)-\rho}$ since $\operatorname{ch} L(0) = e^0 = 1$

Same version for $\bar{\mathfrak{g}}$, $\bar{\Delta}^+$, $\overline{\operatorname{mult}}\,\alpha$ where $\bar{\mathfrak{g}} = \mathfrak{g}/\tau$ ← max ideal of $\mathfrak{g}$ s.t. $\tau \cap \mathfrak{h} = 0$

Recall that $w \cdot e^\lambda = e^{w(\lambda)}$ so $w(e^\lambda e^\mu) = w(e^{\lambda+\mu}) = e^{w(\lambda)} e^{w(\mu)}$

Its proof is based on several lemmas.

Lemma 2.2.2

By Lemma 2.1.12 enough to prove when $V = L(\nu)$. since V_ν is 1-dim so $\operatorname{ch} V = 1 \cdot \operatorname{ch} L(\nu) + \sum_{\lambda<\nu} [V : L(\lambda)] \operatorname{ch} L(\lambda)$

V : a hт wt module of hт wt ν

$$\operatorname{ch} V = \sum_{\mu \le \nu} c_\mu \operatorname{ch} M(\mu) \qquad \text{where } c_\mu \in \mathbb{Z} \text{ and } c_\nu = 1.$$

$\langle \mu+\rho, \mu+\rho \rangle = \langle \nu+\rho, \nu+\rho \rangle$

Proved by Ω. $\quad \Omega \equiv \langle \nu, \nu+2\rho \rangle$ on $(1 \otimes 1_\nu) \in M(\nu)$

$\equiv \langle \mu, \mu+2\rho \rangle$ on $L(\mu)$ appearing as a subquotient of $M(\nu)$.

Lemma 2.2.3

$$\forall w \in W, \quad w(e^\rho R) = \varepsilon(w) \cdot e^\rho R$$

Lemma 1.3.11 (b4) $\quad s_i(\Delta^+ \setminus \{\alpha_i\}) = \Delta^+ \setminus \{\alpha_i\}$

pf: Enough for $w = s_i$.

$$s_i(e^\rho R) = e^{\rho-\alpha_i} \prod_{\alpha \in \Delta^+} (1 - s_i e^{-\alpha})^{\operatorname{mult}\alpha}$$

invariant

$$= e^{\rho-\alpha_i} (1 - e^{\alpha_i}) \left[\prod_{\alpha \in \Delta^+ \setminus \{\alpha_i\}} (1 - \cancel{s_i}\, e^{-\alpha})^{\operatorname{mult}\alpha} \right]$$

$$= e^\rho (e^{-\alpha_i} - 1) \cdot [\text{same}] = (-1) \cdot e^\rho \cdot R$$

#

Lemma 2.2.4. Let $\lambda \in \mathfrak{h}^*$ with $\lambda(\alpha_i^\vee) \geq 0$ $\forall i$.

If $\nu \in \mathfrak{h}^*$ satisfies ① $\nu \leq \lambda+\rho$, ② $\langle \nu, \nu \rangle = \langle \lambda+\rho, \lambda+\rho \rangle$ ③ $\nu(\alpha_i^\vee) \geq 0$ $\forall i$

then $\nu = \lambda+\rho$.

proof: Write $\lambda+\rho-\nu = \sum_i a_i \alpha_i$, $a_i \in \mathbb{Z}_{\geq 0}$

$$\langle \lambda+\rho, \lambda+\rho \rangle = \langle \nu, \nu \rangle = \langle \lambda+\rho, \lambda+\rho \rangle - \langle \lambda+\rho, \sum a_i \alpha_i \rangle - \langle \nu, \sum a_i \alpha_i \rangle$$

$$\Rightarrow \quad 0 < \underbrace{\langle \lambda+\rho, \sum a_i \alpha_i \rangle}_{\sum a_i (\langle \lambda, \alpha_i \rangle + \underbrace{\langle \rho, \alpha_i \rangle}_{\frac{1}{2}\langle \alpha_i, \alpha_i \rangle > 0})} = -\underbrace{\langle \nu, \sum a_i \alpha_i \rangle}_{= \sum a_i \langle \nu, \alpha_i \rangle} \leq 0 \quad \Rightarrow \text{all } a_i = 0$$

proof of the Thm:

By Lemma 2.2.2
$$\mathrm{ch}\, L = \sum_{\mu \in S(\lambda)} d_\mu \, \mathrm{ch}\, M(\mu) \quad \text{with } d_\lambda = 1 \quad c_\mu \in \mathbb{Z}$$

$$S(\lambda) = \{ \mu \in \mathfrak{h}^* \mid \mu \leq \lambda, \ \langle \mu+\rho, \mu+\rho \rangle = \langle \lambda+\rho, \lambda+\rho \rangle \}$$

$\times e^\rho R \Rightarrow$
$$e^\rho R \, \mathrm{ch}\, L = \sum_{\mu \in S(\lambda)} d_\mu \, e^{\mu+\rho} \qquad \text{by Lemma 2.1.13}$$

L is integrable $\Rightarrow$ $\mathrm{ch}\, L$ is W-invariant since $\dim L_\mu = \dim L_{w(\mu)}$ $\boxed{\text{Lemma 1.3.5 (a)}}$

$\Leftarrow$ Ex 2.2.E.

By Lemma 2.2.3

$$w(e^\rho R \, \mathrm{ch}\, L) = \varepsilon(w)\, e^\rho R \, \mathrm{ch}\, L = \sum_{\mu \in S(\lambda)} \varepsilon(w)\, d_\mu \, e^{\mu+\rho}$$

$\mu = w(\mu+\rho) - \rho$

$$\| $$

$$w\Big(\sum_{\mu \in S(\lambda)} d_\mu e^{\mu+\rho} \Big) = \sum_{\mu \in S(\lambda)} d_\mu \, e^{w(\mu+\rho)}$$

Thus
$$\varepsilon(w)\, d_{w*\mu} = d_\mu \quad \forall w \in W \qquad (\ast\ast)$$

Fix $\mu \in S(\lambda)$ with $d_\mu \neq 0$. $\Rightarrow d_{w*\mu} \neq 0$ $\forall w \in W$ $\Rightarrow w*\mu \in S(\lambda) \Rightarrow w*\mu \leq \lambda$ $\forall w \in W$.

Take $w_0 \in W$ s.t. $\underbrace{|\lambda - w_0 * \mu|}_{= \sum a_i}$ is minimal

$\lambda - w*\mu = \sum a_i \alpha_i$, $a_i \in \mathbb{Z}_{\geq 0}$

Let $\nu = w_0 * \mu + \rho$. Then $\underline{\nu \leq \lambda+\rho}$ since $w_0 * \mu \leq \lambda$.

$$\langle \nu, \nu \rangle = \langle w_0*\mu+\rho, w_0*\mu+\rho \rangle = \langle w_0(\mu+\rho), w_0(\mu+\rho) \rangle = \langle \mu+\rho, \mu+\rho \rangle = \langle \lambda+\rho, \lambda+\rho \rangle.$$

Also $|\lambda - w_0 * \mu| \underset{\text{minimal}}{\leq} |\lambda - s_i w_0 * \mu| = |\lambda - w_0 * \mu + \langle \nu, \check{\alpha}_i \rangle \alpha_i| \quad \forall i$

$\therefore \quad \langle \nu, \check{\alpha}_i \rangle \geq 0 \quad \forall i.$

Lemma 2.2.4 $\Rightarrow$ $\nu = \lambda + \rho$, or $w_0 * \mu = \lambda$ or $\mu = w_0^{-1} * \lambda$.

Hence $\forall \mu \in S(\lambda)$ with $d_\mu \neq 0$, we have $\mu = w * \lambda$ for some unique $w \in W$

and $\quad d_\mu \underset{(\ast\ast)}{=} \varepsilon(w^{-1})\, d_{w^{-1} * \mu} = \varepsilon(w^{-1})\, d_\lambda = \varepsilon(w)$.

Then

$$e^\rho R \operatorname{ch} L = \sum_{\mu \in S(\lambda)} d_\mu e^{\mu+\rho} = \sum_{w \in W} \varepsilon(w)\, e^{w(\lambda+\rho)}$$

$$\Rightarrow \quad R \operatorname{ch} L = \sum_{w \in W} \varepsilon(w)\, e^{w * \lambda}$$

$\#$

Corollary: Any integrable ht wt $\mathfrak{g}$-module is irreducible

proof: Let L be such a module with L' its irreducible quotient.

They have the same ht wt $\Rightarrow$ same character $\Rightarrow$ $L = L'$ $\#$

[Hence if $\lambda \in D$, then $L^{max}(\lambda) = L(\lambda)$ and $M_1(\lambda) = M'(\lambda)$.]

Corollary: Any integrable $M \in \mathcal{O}$ is completely reducible:

$$M \cong \bigoplus_{\lambda \in D} L(\lambda)^{\oplus [M : L(\lambda)]}$$

[In particular, if $\lambda, \mu \in D$, then $L(\lambda) \otimes L(\mu)$ is integrable $\Rightarrow$ $L(\lambda) \otimes L(\mu) \cong \bigoplus_{\nu \in D} C^{\nu}_{\lambda\mu} L(\nu)$]

very long proof

Ex. 2.2.E

$M \in \mathcal{O}$ and $\operatorname{ch} M$ is W-invariant $\Rightarrow$ M is integrable

Motivation:

$\lambda \in \mathfrak{h}^*$ $M(\lambda) \in \mathcal{O}$ $\Rightarrow \mathrm{ch}\, M(\lambda) = \sum_{\substack{\mu \in \mathfrak{h}^* \\ \mu \leq \lambda}} [M(\lambda) : L(\mu)]\, \mathrm{ch}\, L(\mu)$.

Question: $[M(\lambda) : L(\mu)] \neq 0 \iff$??

For s.s. Lie alg. (linkage principal)

$[M(\lambda) : L(\mu)] \neq 0 \iff \mu$ is strongly linked to λ.

$(\Leftarrow)$ Verma's embedding theorem. $M(\mu) \hookrightarrow M(\lambda)$.

$(\Rightarrow)$ [BGG Thm] ① Jantzen filtration of $M(\lambda)$.

② Jantzen sum formula

To establish ① and ②, one needs Shapovalov bilinear form / determinant.

Want to have KM analogues of these things.