II-3. Shapovalov Bilinear Form

$$\text{PBW} \Rightarrow \mathcal{U}(\mathfrak{g}) = (\mathfrak{n}^- \mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{n}^+) \oplus \underbrace{\mathcal{U}(\mathfrak{h})}_{\mathcal{S}(\mathfrak{h})}$$

Define the v.s. projection $\mathcal{H} : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{S}(\mathfrak{h})$. ($\mathcal{H}$ for HC)

Recall the Cartan involution $\omega : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ $\quad e_i \leftrightarrow -f_i \quad h \mapsto -h$

Let $\tau : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ be the anti-auto given by $\tau(x) = -x, \ \forall x \in \mathfrak{g}$
(extend it to $\mathcal{U}(\mathfrak{g})$)

and set $\sigma := \tau \circ \omega = \omega \circ \tau : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$, an anti-auto of order 2

Note that $\sigma|_{\mathcal{U}(\mathfrak{h})}$ = identity map.

Define the $\overbrace{\text{bilinear}}^{}$ form $\quad S : \mathcal{U}(\mathfrak{g}) \times \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{S}(\mathfrak{h})$
Shapovalov

$$(a, b) \longmapsto \mathcal{H}(\sigma(a) \cdot b)$$

Facts:

(1) $S(a, b) = S(b, a)$ (symmetric) $\quad \underbrace{\sigma(\sigma(a)b)}_{}= \sigma(b)a$ if $\sigma(a)b \in \mathcal{S}(\mathfrak{h})$ $\quad$ ($\sigma(a)b$ = $\sigma(\sigma(a)b)$)

(2) $S(c \cdot a, b) = S(a, \sigma(c) b)$ (contravariant)

(3) $S(\mathcal{U}(\mathfrak{g})_{\gamma_1}, \mathcal{U}(\mathfrak{g})_{\gamma_2}) = 0$ if $\gamma_1 \neq \gamma_2 \in \mathfrak{h}^*$

treat $\mathcal{U}(\mathfrak{g})$ as a $\mathfrak{h}$-module by adjoint, γ_1 is its weight

$\forall \lambda \in \mathfrak{h}^*$, define $P_\lambda : \mathcal{S}(\mathfrak{h}) \to \mathbb{C}$ by $P_\lambda(h_1 h_2 \cdots h_k) = \lambda(h_1) \cdots \lambda(h_k)$ $\quad \forall h_i \in \mathfrak{h}$

$$\rightsquigarrow \quad P_\lambda \circ S : \mathcal{U}(\mathfrak{g}) \times \mathcal{U}(\mathfrak{g}) \longrightarrow \mathbb{C}$$

descends $\rightsquigarrow$ $\quad S_\lambda : M(\lambda) \times M(\lambda) \longrightarrow \mathbb{C}$

$$S_\lambda(a \otimes 1, b \otimes 1) = P_\lambda \circ S(a, b) \qquad \forall a, b \in \mathcal{U}(\mathfrak{n}^-)$$

(~~$\mathcal{U}(\mathfrak{g})$~~)

a sym-bilinear form on $M(\lambda)$.

Recall $M'(\lambda)$: the unique maximal proper submodule of $M(\lambda)$ s.t. $M(\lambda)/M'(\lambda) = L(\lambda)$.

Prop 2.3.2

(1) $S_\lambda(x\cdot v, w) = S_\lambda(v, \sigma(x)\cdot w)$ $\quad \forall v, w \in M(\lambda), \ x \in U(\mathfrak{g})$ [Contravariant]

(2) $S_\lambda(M(\lambda)_\mu, M(\lambda)_\theta) = 0$ if $\mu \neq \theta$

(3) $S_\lambda(M'(\lambda), M(\lambda)) = 0$ $\quad$ ($M'(\lambda)$ is the radical of the form S_λ $\underset{\text{Ker } S_\lambda}{}$)

(4) $\downarrow$ $S_\lambda : L(\lambda) \times L(\lambda) \to \mathbb{C}$ is a non-deg, sym, contravariant bilinear form.

(5) Any contravariant bilinear form on $L(\lambda) = S_\lambda$, up to a scalar, and hence symmetric. (Such a form is unique up to a scalar multiple of $S_\lambda(1\otimes 1, 1\otimes 1)$.)

proof:

(1) follows from Facts (1), (2). (2) follows from (1).

(3) $a\otimes 1 \in M'(\lambda)$, $v \in M(\lambda)$.

$$S_\lambda(v, a\otimes 1) = S_\lambda(\underbrace{\sigma(a)v}_{\substack{M'(\lambda) \\ \lambda \text{ is not its wt}}}, \underbrace{1\otimes 1}_{\text{wt } \lambda}) = 0 \quad \text{by (2)}$$

By (1), Ker S_λ is $\underbrace{\text{a}}_{\text{proper}}$ $\mathfrak{g}$-submodule of $M(\lambda)$ containing $M'(\lambda)$ $\Rightarrow$ Ker $S_\lambda = M'(\lambda)$.

(4) follows from (3).

(5) Any such a form B is determined by the value $B(1\otimes 1, 1\otimes 1)$. ※

Definition: for KM.

$\forall \lambda \in \mathfrak{h}^*$, $M(\lambda)$ admits a Jantzen filtration:

Choose $\rho \in \mathfrak{h}^*$ $\rho(\alpha_i^\vee) = 1$ and define a family of sym bilinear forms $\hat{S}_\lambda$ on $M(\lambda)$

$$\hat{S}_\lambda(a\otimes 1, b\otimes 1) = P_{\lambda + t\rho}(S(a,b)) \qquad \forall a, b \in U(\mathfrak{n}^-)$$

here $P_{\lambda+t\rho} : S(\mathfrak{h}) \to \mathbb{C}[t]$ is given by

$$P_{\lambda+t\rho}(h_1 \cdots h_k) = (\lambda(h_1) + t\rho(h_1)) \cdots (\lambda(h_k) + t\rho(h_k))$$

By Ex 2.3.E (5).

By Ex 2.3.E (5), $\exists$ a filtration of $\mathfrak{g}$-submodules

Need the bilinear form to define them.

$$M(\lambda) = M^0(\lambda) \supset M^1(\lambda) \supset \cdots\cdots$$

where $M^k(\lambda) := \left\{ v_0 \in M(\lambda) \;\middle|\; \begin{array}{l} \exists\, v_1, v_2, \ldots, v_{k-1} \in M(\lambda) \text{ such that} \\ \hat{S}_\lambda(v_0, w) + t\,\hat{S}_\lambda(v_1, w) + \cdots + t^{k-1}\hat{S}_\lambda(v_{k-1}, w) \\ \text{is divisible by } t^k, \quad \forall w \in M(\lambda) \end{array} \right\}$

each $M^k(\lambda)$ is a submodule of $M(\lambda)$.

Note that $M^1(\lambda) = \operatorname{Ker} S_{\lambda+t\rho} = M'(\lambda)$.

$v_0 \in M^1(\lambda) \Leftrightarrow \hat{S}_\lambda(v_0, w)\big|_{t=0} = 0 \quad \forall w \in M(\lambda)$

$\Leftrightarrow P_\lambda \circ S(v_0, w) = 0 \quad \forall w \in M(\lambda)$

For $\theta \in Q^+$, let $\mathcal{U}(\mathfrak{n}^-)_\theta = \{ x \in \mathcal{U}(\mathfrak{n}^-) \mid h \cdot x = -\theta(h)\, x, \ \forall h \in \mathfrak{h} \}$

Let $S_\theta^- : \mathcal{U}(\mathfrak{n}^-)_\theta \times \mathcal{U}(\mathfrak{n}^-)_\theta \to S(\mathfrak{h})$ be the restriction of $S : \mathcal{U}(\mathfrak{g}) \times \mathcal{U}(\mathfrak{g}) \to S(\mathfrak{h})$

Define the Kostant partition function $P(\theta) \in \mathbb{Z}_{\geq 0}$ by

$$\prod_{\alpha \in \Delta^+} \left(1 + e^{-\alpha} + e^{-2\alpha} + \cdots\right)^{\operatorname{mult}\alpha} = R^{-1} = \operatorname{ch} M(0) = \sum_{\theta \in Q^+} P(\theta)\, e^{-\theta} \qquad (\text{set } P(\theta) = 0 \text{ if } \theta \notin Q^+)$$

Choose a basis $F = \{F_I\}$ of $\mathcal{U}(\mathfrak{n}^-)_\theta$

a matrix with entries in

$$\text{Define} \quad \det_F(S_\theta^-) := \det\left[S_\theta^-(F_I, F_{I'})\right]_{I, I'} \in S(\mathfrak{h})$$

[Shapovalov det]

Simply write $\det(S_\theta^-)$, well-defined up to a nonzero scalar multiple

Theorem 2.3.4

$\mathfrak{g}$ symmetrizable. Then $\forall \theta \in Q^+$,

θ appears here

$$\det(S_\theta^-) = \prod_{(\alpha, n)} \Big(\underbrace{\nu^{-1}(\alpha)}_{\in \mathfrak{h}} + \underbrace{\langle \rho - \tfrac{n\alpha}{2}, \alpha \rangle}_{\text{scalar}} \Big)^{P(\theta - n\alpha)\,(\operatorname{mult}\alpha)}$$

$\longleftarrow$ a product of linear terms.

$\forall (\alpha, n) \in \Delta^+ \times \mathbb{N}$ such that $\theta - n\alpha \in Q^+$ and any normalized invariant form on $\mathfrak{h}^*$

θ control the sum as well

Also, $\det(S_\theta^-) \neq 0 \quad \forall \theta \in Q^+$.

[Step 2]

Cor 2.3.5. [Jantzen sum formula for KM.]

$\mathfrak{g}$ symmetrizable. $\forall \lambda, \mu \in \mathfrak{h}^*$, $M^k(\lambda)_\mu = 0$ for k large enough.

in Jantzen filtration of $M(\lambda)$.

depends on λ and μ.

Also, $$\sum_{k\geq 1} \mathrm{ch}\, M^k(\lambda) = \sum_{(\alpha,n)\in D_\lambda} \mathrm{ch}\, M(\lambda - n\alpha)$$

where $$D_\lambda = \{ (\alpha, n) \in \underbrace{\tilde{\Delta}^+}_{} \times \mathbb{N} \mid \langle \lambda + \rho - \tfrac{n}{2}\alpha, \alpha \rangle = 0 \}$$

$= \Delta^+$ but counting $\mathrm{mult}\,\alpha$ for each $\alpha \in \Delta^+$

e.g. $\mathrm{mult}\,\alpha = 3$, (α^1, n), (α^2, n), (α^3, n) then set $\alpha^1 = \alpha^2 = \alpha^3 = \alpha$

Cor 2.3.6

For $\underset{\sim\sim}{\mu \leq \lambda} \in \mathfrak{h}^*$,

$[M(\lambda) : L(\mu)] > 0 \iff \exists\, \beta_1, \cdots, \beta_p \in \Delta^+$ and $k_1, \cdots, k_p \in \mathbb{N}$ s.t.

(1) $\lambda - \mu = \sum_{i=1}^{p} k_i \beta_i \quad (\in Q^+)$

and

(2) $\forall\, 1 \leq j \leq p$, $\quad 2\langle \lambda + \rho - \sum_{i=1}^{j-1} k_i\beta_i, \beta_j \rangle = k_j \langle \beta_j, \beta_j \rangle$

Proof by induction on $|\lambda - \mu|$.

Prop 2.3.8. [Linkage principal for KM.]

Let $\lambda \in K^{W, g}$

a certain subgroup of W.

$[M(\lambda) : L(\mu)] > 0 \iff \exists\, \beta_1, \cdots, \beta_p \in \Delta^+_{re}$ ($\Delta^+_{re} \subset \Delta^+$) and $s_{\beta_1}, \cdots, s_{\beta_p} \in W(\lambda)$ (reflections)

such that

$$\mu = (s_{\beta_1} \cdots s_{\beta_p}) * \lambda < (s_{\beta_2} \cdots s_{\beta_p}) * \lambda < \cdots < (s_{\beta_{p-1}} s_{\beta_p}) * \lambda < s_{\beta_p} * \lambda < \lambda$$

$\Delta = \Delta_{re} \sqcup \Delta_{im}$.

set of roots

A root β is called real if $\beta = w \cdot \alpha_i$ for some $w \in W$, simple root α_i.

not real $\Leftrightarrow$ imaginary

For $\alpha \in \Delta^+_{im} := \Delta^+ \cap \Delta_{im}$. $\quad (\Delta^+_{re} := \Delta^+ \cap \Delta_{re})$

define $C_\alpha := \left\{ \lambda \in \mathfrak{h}^* \;\middle|\; \langle \lambda+\rho, \alpha \rangle = \frac{\langle \alpha, \alpha \rangle}{2} \right\}$

and $C := \bigcup_{\alpha \in \Delta^+_{im}} C_\alpha$

Define $K^{w.g} := \mathfrak{h}^* \setminus C$

C and $K^{w.g.}$ are stable under the $*$ action of W

$W(\lambda)$: the subgroup of W gen by $\{ S_\beta \mid \beta \in \Delta^+_{re},\ (\lambda+\rho)(\check{\beta}) \in \mathbb{Z} \}$.