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Lie algebra (co)homology

(I). Introduction

(II). Applications:

- Weyl-Kac character formulas.
- Blocks of $\mathcal{O}$.
- $\mathfrak{g} = \bar{\mathfrak{g}}$, for $\mathfrak{g}$: symmerizable.

Definitions:

(1). (Lie algebra homology):

s: Lie algebra

V: s-module.

Consider $\Lambda_* = \Lambda_*(s, V) := \quad \cdots \to \Lambda_p \xrightarrow{\partial_p} \Lambda_{p-1} \to \cdots \xrightarrow{\partial_1} \Lambda_0 \xrightarrow{\partial_0} 0$

$\Lambda_p := \Lambda^p(s) \otimes V$, $\Lambda_0 = V$ with differentials ∂_p defined by $\boxed{\partial_{p+1}(x \wedge f) = -x \cdot f - x \wedge \partial_p(f)}$, $\forall x \in s, \forall f \in \Lambda_p$. $\forall p \geq 0$.

∂_{p+1}: well-defined on Λ_{p+1} by induction on p.

$\Rightarrow \partial_1(x \otimes v) = -xv - 0 = -xv$, $\forall x \in s$, $v \in V$

and, in general,

$$\partial_p(x_1 \wedge \cdots \wedge x_p \otimes v) := \sum_{i<j} (-1)^{i+j} [x_i\, x_j]\, x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_p \otimes v$$

$$+ \sum_i (-1)^i x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_p \otimes x_i v$$

and $\boxed{\partial_p(x \cdot f) = x \cdot \partial_p(f)}$, $\forall x \in \mathfrak{g}$, $f \in \Lambda_p$.

"$\boxed{\Lambda_* \text{ is a complex}}$":

$\forall f \in \Lambda_{p-1}$, $\forall x \in s$ we have

$$\partial_{p-1}\partial_p(x \wedge f) = -\partial_{p-1}(x \cdot f) - \partial_{p-1}(x \wedge \partial_{p-1}(f))$$

$$= -x \cdot \partial_{p-1}(f) + x \cdot \partial_{p-1}(f) + x \wedge \partial_{p-2}\partial_{p-1}(f)$$

$$= x \wedge \partial_{p-2}\partial_{p-1}(f) = 0 \quad \text{by induction.}$$

<u>Define</u> the Lie algebra homology of s with coefficients in V

$$H_p(s,V) := \mathrm{Ker}(\partial_p)/\mathrm{Im}(\partial_{p+1}), \quad \forall p \geq 0.$$

" $\boxed{H_p(s,V) \text{ is a trivial } s\text{-module}}$ ":

$$x \cdot f = -\partial_{p+1}(x \wedge f) - x \wedge \partial_p(f) = 0,$$
for any $f \in \mathrm{Ker}(\partial_p)$.

<u>e.g.</u> Consider $s = sl_2(\mathbb{C})$ and $V := \mathbb{C}$.

$$\Lambda_*: \quad 0 \to \underset{\Lambda_3}{\Lambda^3 s \otimes \mathbb{C}} \xrightarrow{\partial_3} \underset{\Lambda_2}{\Lambda^2 s \otimes \mathbb{C}} \xrightarrow{\partial_2} \underset{\Lambda_1}{\Lambda^1 s \otimes \mathbb{C}} \xrightarrow{\partial_1} \underset{\Lambda_0}{\Lambda^0 s \otimes \mathbb{C}} \to 0$$

Note: $\partial_3(f \wedge h \wedge e \otimes c) = -2 f \wedge e \otimes c + 0 - 2 e \wedge f \otimes c = 0$
$\Rightarrow \partial_3 = 0$

Note: $\partial_2(f \wedge h \otimes c) = -2 f \otimes c$
$\partial_2(f \wedge e \otimes c) = h \otimes c \quad \Rightarrow \Lambda_2 \overset{\partial_2}{\cong} \Lambda_1$
$\partial_2(h \wedge e \otimes c) = -2 e \otimes c$

Note: $\partial_1(x \otimes v) = -x v = 0 \Rightarrow \partial_1 = 0$

$$H_p(s,V) = \begin{cases} \mathbb{C}, & \text{if } p = 0 \text{ or } 3 \\ 0, & \text{otherwise} \end{cases}$$

(2). (Lie algebra cohomology):

$\mathfrak{s}$: Lie algebra

V: $\mathfrak{s}$-module.

Consider $C^* := C^*(\mathfrak{s},V) := \quad 0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \cdots$

$C^p := \mathrm{Hom}(\Lambda^p \mathfrak{s}, V)$, $C^0 = \mathrm{Hom}(\mathbb{C}, V)$.

with differentials d^p defined by

$$\boxed{i_x(d^p w) = -d^{p-1}(i_x w) + i_x \cdot w} \quad \forall x \in \mathfrak{s},\ w \in C^p,$$

where

$$i_x w \in C^{p-1} \text{ by } i_x w(x_1 \wedge \cdots \wedge x_{p-1}) = w(x \wedge x_1 \wedge \cdots \wedge x_{p-1})$$

$\Rightarrow$ $d^0 w(x) = -x\, w(1)$, $\forall x \in \mathfrak{s}$, and in general

$$d^p w(x_1 \wedge x_2 \wedge \cdots \wedge x_{p+1})$$

$$:= \sum_{i<j} (-1)^{i+j}\, w([x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_{p+1})$$

$$+ \sum_i (-1)^{i+1} x_i\, w(x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_{p+1})$$

for $w \in C^p$, $x_1, \dots, x_{p+1} \in \mathfrak{s}$.

and $\boxed{x \cdot (d^p w) = d^p(x \cdot w)}$, $\forall x \in \mathfrak{s}$, $w \in C^p$.

" $\boxed{C^* \text{ is a complex}}$ " :
$\forall w \in C^p$, $\forall x \in s$ we have

$$i_x d^{p+1} d^p w = -d^p i_x d^p w + x \cdot d^p w$$
$$= d^p d^{p-1} i_x w - d^p (x \cdot w) + x \cdot d^p x$$
$$= 0 \text{ by induction.}$$

Define the Lie algebra cohomology of s with coefficients in V

$$H^p(s, V) := \ker d^p / \operatorname{Im} d^{p-1}, \quad \forall p \geq 1.$$

" $\boxed{H^p(s,V) \text{ is a trivial } s\text{-module}}$ " :

$x \cdot w = i_x d^p w + d^{p-1} i_x w = 0$, for any $w \in \ker d^p$.

e.g. $s = sl_2(\mathbb{C})$ and $V = \mathbb{C}$

$$C^* : 0 \to \operatorname{Hom}(\Lambda^0 s, \mathbb{C}) \xrightarrow{d^0} \operatorname{Hom}(\Lambda^1 s, \mathbb{C}) \xrightarrow{d^1} \operatorname{Hom}(\Lambda^2 s, \mathbb{C}) \xrightarrow{d^2} \operatorname{Hom}(\Lambda^3 s, \mathbb{C}) \to 0$$

Note: $d^0(\gamma)(x) = -x\gamma(1) = 0$, $\forall \gamma \in C^0$, $x \in s$.

Note: $d^1(\gamma)$: $e \wedge f \to -\gamma(h)$, $e \wedge h \to 2\gamma(e)$, $f \wedge h \to -2\gamma(f)$, $\forall \gamma \in C^1$

$\Rightarrow d^1 : C^1 \cong C^2$.

Note: $d^2(\gamma)(f \wedge h \wedge e) = -\gamma(2f \wedge e) - \gamma(2e \wedge f) = 0$

$\Rightarrow d^2 = 0$

$$\Rightarrow H^p(s, V) = \begin{cases} \mathbb{C}, & \text{if } p = 0, 3 \\ 0, & \text{otherwise} \end{cases}.$$

H^1 & H_1 :

$$H^1(S,V) \cong \mathrm{Ext}^1(\mathbb{C},V)$$
$$H_1(S,V) \cong \mathrm{Tor}_1(\mathbb{C},V)$$

$$0 \to I \to U \to \mathbb{C} \to 0$$

$$\Rightarrow 0 \to H_1(S,V) \to I \otimes_S V \to V \to V/IV \to 0$$

$$0 \to \underbrace{\mathrm{Hom}_S(\mathbb{C},V)}_{V^S \cong} \to V \to \mathrm{Hom}_S(I,V) \to H^1(S,V) \to 0.$$

$$\left(\text{So, } H_1(S,M) = \ker(I \otimes_S V \to V),\ H^1 = \mathrm{Der}(S,V)/\mathrm{Der}_{inn}(S,V) \right)$$

e.g. $0 \to H_1(S,\mathbb{C}) \to I \otimes_S \mathbb{C} \xrightarrow{\text{exact}} 0$ since $\mathbb{C} \cong \mathbb{C}/I\mathbb{C}$

But $I \otimes_S \mathbb{C} \cong S/[SS]$, so $H_1(S,\mathbb{C}) = 0$ if S is s.s.

e.g. $\left.\begin{aligned} &\mathrm{Hom}_S(I,V) \cong \mathrm{Der}(S,V) \\ &\qquad V \twoheadrightarrow \mathrm{Der}_{inn}(S,V) \\ &\Rightarrow H^1(S,V) \cong \mathrm{Der}(S,V)/\mathrm{Der}_{inn}(S,V) \end{aligned}\right\}$ in general.

$$H^1(S,\mathbb{C}) \cong \mathrm{Der}(S,\mathbb{C}) \cong \mathrm{Hom}_S(S,\mathbb{C}) \cong \mathrm{Hom}(S/[SS],\mathbb{C})$$
$$\Rightarrow H^1(S,\mathbb{C}) = 0, \text{ for } S: \text{s.s.}$$

Thms :

- $H^1(S,M) = 0 = H^2(S,M)$, if S: s.s., $\dim M < \infty$
- $H^K(S,M) = 0$, if S: s.s., M: simple but $M \neq \mathbb{C}$.

$$H^2(s,M) \cong \mathrm{Ext}(s,M)$$

$$\| $$

$$\{ 0 \to M \to E \to s \to 0 \text{ extension of Lie algebras} \}.$$

Definitions:

(1). (Lie algebra homology of a pair (s,t)):

$t \subseteq s$: Lie algebras

V: s-module. $\Rightarrow$ $\Lambda^p(s/t)$: t-module, $\forall\, p \geq 0$.

Consider: $\Lambda_p(s,t,V) := \Lambda^p(s/t) \otimes V / t\cdot(\Lambda^p(s/t) \otimes V)$ $(p \geq 0)$.

Note: $\partial_{p+1}(x \wedge f) = -x\cdot f - x \wedge \partial_p(f)$ $\quad f \in \Lambda_p$

$\partial_p(x\cdot f) = x\cdot\partial_p(f)$

for any $x \in t$ implying that ∂_p descends to a map $\partial_p : \Lambda_p(s,t,V) \to \Lambda_{p-1}(s,t,V)$, $\forall\, p \geq$

Conclusion:

$$\cdots \to \Lambda_p(s,t,V) \xrightarrow{\partial_p} \Lambda_{p-1}(s,t,V) \to \cdots \xrightarrow{\partial_1} \Lambda_0(s,t,V) \xrightarrow{\partial_0} 0$$

$$\Lambda_0(s,t,V) = \frac{V}{t\cdot V}$$

is a complex.

Define: the Lie algebra homology of the pair (s,t) in to be

$$H_p(s,t,V) := \ker(\partial_p)/\mathrm{Im}(\partial_{p+1}), \quad \forall\, p \geq 0.$$

(2). (Lie algebra cohomology of a pair $(\mathfrak{s},\mathfrak{t})$)

$\mathfrak{t} \leq \mathfrak{s}$ Lie algebras

V: $\mathfrak{s}$-module $\Rightarrow \Lambda^p(\mathfrak{s}/\mathfrak{t})$: $\mathfrak{t}$-module, $\forall p \geq 0$.

Consider $C^p(\mathfrak{s},\mathfrak{t},V) := \mathrm{Hom}_{\mathfrak{t}}(\Lambda^p(\mathfrak{s}/\mathfrak{t}), V) \quad (p \geq 0)$

$$d^p : C^p(\mathfrak{s},\mathfrak{t},V) \to C^{p+1}(\mathfrak{s},\mathfrak{t},V)$$

$$C^0 = \mathrm{Hom}_{\mathfrak{t}}(\mathbb{C}, V) = V^{\mathfrak{t}}$$

Similarly,

$$C^*(\mathfrak{s},\mathfrak{t},V) : 0 \to \underset{\parallel}{C^0} \cdots \to C^p(\mathfrak{s},\mathfrak{t},V) \to C^{p+1}(\mathfrak{s},\mathfrak{t},V) \to \cdots$$

is a complex

Define :

$$H^p(\mathfrak{s},\mathfrak{t},V) := \ker d^p / \mathrm{Im}\, d^{p-1}, \quad \forall p \geq 0.$$

Example:

Set $s = sl_2(\mathbb{C})$, $t = \mathfrak{h}$ (the Cartan subalgebra of s).

Consider $V = \mathbb{C}$.

Then

$$C^*(s,t,\mathbb{C}) : \quad 0 \to \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C},\mathbb{C}) \xrightarrow{d^0} \mathrm{Hom}_{\mathfrak{h}}(\Lambda^1(s/\mathfrak{h}),\mathbb{C}) \xrightarrow{d^1} \mathrm{Hom}_{\mathfrak{h}}(\Lambda^2(s/\mathfrak{h}),\mathbb{C}) \xrightarrow{d^2} 0$$

$H^0(s,t,\mathbb{C}) = \ker d^0/_0 \cong \mathbb{C}$.

$H^1(s,t,\mathbb{C}) = \ker d^1/\mathrm{im}\, d^0 = 0$ since $\mathrm{Hom}_{\mathfrak{h}}(\Lambda^1(s/\mathfrak{h}),\mathbb{C}) = 0$.

$H^2(s,t,\mathbb{C}) \cong \mathrm{Hom}_{\mathfrak{h}}(\Lambda^2(s/\mathfrak{h}),\mathbb{C}) \cong \mathbb{C}$

In general,

Thm (Delorm): $\mathcal{O} = \mathcal{O}(sl_2)$

$$\mathrm{Ext}^*_{\mathcal{O}}(M,N) \cong H^*(sl_2, \mathfrak{h}, \mathrm{Hom}_{\mathbb{C}}(M,N)).$$

It is natural to find $\mathrm{Ext}^1_{\mathcal{O}}(\mathbb{C},\mathbb{C}) = ?$

e.g. $M = N = \mathbb{C}$

$$\mathrm{Ext}^*_{\mathcal{O}}(M,N) \cong H^*(sl_2,\mathfrak{h},\mathbb{C}) = \begin{cases} \mathbb{C}, & \text{if } * = 0,2 \\ 0, & \text{if } * \neq 0,2. \end{cases}$$

Claim: $0 \to L(0) \to \nabla(0) \to \Delta(0) \to L(0) \to 0 \neq 0$ in $\mathrm{Ext}^2(\mathbb{C},\mathbb{C})$.

To check it: Apply $\mathrm{Hom}_{\mathcal{O}}(-, L(0))$

to $0 \to L(-2) \to \Delta(0) \to L(0) \to 0$

$$\Rightarrow \quad 0 \to \mathrm{Ext}^1_{\mathcal{O}}(L(-2), L(0)) \xrightarrow{\theta} \mathrm{Ext}^2_{\mathcal{O}}(L(0), L(0)) \to 0$$

$$\Rightarrow \quad \underset{\mathbb{C}}{\underset{\|}{\mathrm{Ext}^1_{\mathcal{O}}(L(-2), L(0))}} \cong \mathrm{Ext}_{\mathcal{O}}(\mathbb{C},\mathbb{C})$$

$\theta(0 \to L(0) \to \nabla(0) \to L(-2) \to 0)$

$\|$

$0 \to L(0) \to \nabla(0) \to \Delta(0) \to L(0) \to 0$

is non-zero.

since $\dfrac{\mathrm{rad} P(0)}{\mathrm{rad}^2 P(0)} = L(-2)$.

Delorme: $\forall M, N \in \mathcal{O}$, $\forall n \geq 0$

① $\mathrm{Ext}^*_{\mathcal{O}}(M,N) \cong \mathrm{Ext}^*_{\mathfrak{g},\mathfrak{h}}(M,N) \cong H^*(\mathfrak{g}, \mathfrak{h}, \mathrm{Hom}(M,N))$

② $\mathrm{Ext}^*_{\mathcal{O}}(M(\mu), N) \cong H^*(\mathfrak{n}, N)_\mu$, $\forall \mu \in \mathfrak{h}^*$.

Relatively homological algebra.

Lie alg: $s \supseteq t$. Set $U(s) =: S$, $U(t) =: T$

① $0 \to A \to B \to C \to 0$ (S,T)-exact
if it is a SES of S-modules
Splits as T-modules

② P: S-module is (S,T)-projective
if $\mathrm{Hom}_S(P,-)$ exact on (S,T)-SES

③ Q: S-module is (S,T)-injective
if $\mathrm{Hom}_S(-,Q)$ exact on (S,T)-SES.

④ $\mathrm{Ext}^p_{s,t}(N,M) := H^p(\mathrm{Hom}_S(P_\bullet, M)) \cong H^p(\mathrm{Hom}_S(N, Q_\bullet))$

$$\mathrm{Tor}_p^{s,t}(N,M) := H_p(P_\bullet \otimes_S M) \cong H_p(N \otimes_S P'_\bullet)$$

$P_\bullet \to N \to 0$
$P'_\bullet \to M \to 0$ (S,T)-proj. resolution
$0 \to M \to Q_\bullet$ (S,T)-inj. resolution.

Identification:

§ Lie alg. (co)homo ≡ Relative (co)homo.

Define a resolution of $\mathbb{C}$ as follows:

$s \geq t$ Lie algebras

$D_p = D_p(s,t) := Us \otimes_t \Lambda^p(s/t)$, $\forall p \geq 0$

$$\cdots \to D_2 \xrightarrow{\partial_2} D_1 \xrightarrow{\partial_1} D_0 \xrightarrow{\varepsilon} D_{-1} := \mathbb{C} \to 0$$

$\varepsilon : Us \otimes_t \mathbb{C} \to \mathbb{C}$ the standard augmentation

$$\partial_p (a \otimes \bar{y}_1 \wedge \cdots \wedge \bar{y}_p)$$

$$:= \sum_{i<j} (-1)^{i+j} \partial_p (a \otimes [\bar{y}_i\ \bar{y}_j] \wedge y_1 \wedge \cdots \wedge \hat{\bar{y}}_i \wedge \cdots \wedge \hat{\bar{y}}_j \wedge \cdots$$

$$+ \sum_i (-1)^{i+1} a y_i \otimes \bar{y}_1 \wedge \cdots \wedge \hat{\bar{y}}_i \wedge \cdots \wedge \bar{y}_p .$$

Prop: $D_* \to \mathbb{C} \to 0$ is exact

Suppose that $s \geq t$ and s is finitely s.s. over t.

Coro: $D_* \otimes V \to V \to 0$ is a (Us, Ut)-projective resolution of V, for any s-module V.

"D_p is (Us, Ut)-projective":

$\mathrm{Hom}_s(D_p, -) \cong \mathrm{Hom}_t(\Lambda^p(s/t), \mathrm{Res}^s_t -)$ splits.

on any (Us, Ut)-exact segence.

"$Us \otimes \Lambda^p(s/t) \twoheadrightarrow Us \otimes_t \Lambda^p(s/t) \to 0$ exact

$\Rightarrow Us \otimes_t \Lambda^p(s/t) = D^p$ is finitely s.s. over t." #

Lie alg. (co)homo. $\equiv$ Relative (co)homo.

H^K, H_K Relative

Connection: (Co)homologies & Ext, Tor

Lemma: $s \geq t$ such that s is finitely semisimple over t. Then for any s-module V

$$H_p(s,t,V) \cong \mathrm{Tor}_p^{(s,t)}(\mathbb{C}, V) \cong \mathrm{Tor}_p^{(s,t)}(V^t, \mathbb{C})$$

$$H^p(s,t,V) \cong \mathrm{Ext}^p_{(s,t)}(\mathbb{C}, V)$$

Pf:

$$H_p(s,t,V) \cong H_p\left(\frac{\Lambda^\bullet(s/t)\otimes V}{t\cdot(\Lambda^\bullet(s/t)\otimes V)}\right)$$

$$\cong H_p(\mathbb{C} \otimes_t (\Lambda^\bullet(s/t)\otimes V)) \quad \text{by tensor identity}$$

$$\cong H_p(\mathbb{C} \otimes_s Us \otimes_t (\Lambda^\bullet(s/t)\otimes V)) \cong H_p(\mathbb{C} \otimes_s D_p \otimes V).$$

$$= \mathrm{Tor}^p_{s,t}(\mathbb{C}, V).$$

$$H^p(s,t,V) \cong H^p(\mathrm{Hom}_t(\Lambda^\bullet(s/t), V))$$

$$\cong H^p(\mathrm{Hom}_s(Us \otimes_t \Lambda^\bullet(s/t), V))$$

$$\cong H^p(D_\bullet, V) = \mathrm{Ext}^p_{s,t}(\mathbb{C}, V).$$

#

Example (Exercise 3.2E):

Let s be a free Lie algebra.
Then $H_p(s, M) = H^p(s, M) = 0$, for any $p \geq 2$
and any s-module M.

Proof:

① Suppose s is the free Lie algebra on the set X.
Then for any associative algebra, we have

$$\mathrm{Hom}_{Alg.}(Us, A) \cong \mathrm{Hom}_{Lie}(s, Lie(A))$$

$$\cong \mathrm{Hom}_{Set}(X, Set(A)) \cong \mathrm{Hom}_{\mathbb{C}\text{-mod}}(\mathrm{span}_{\mathbb{C}} X, A)$$

$$\cong \mathrm{Hom}_{Alg.}(\mathbb{C}\{X\}, A) \quad (\mathbb{C}\{X\}: \text{free algebra} \equiv \text{tensor algebra})$$

natural in A. By Yoneda's lemma we have
$Us \cong \mathbb{C}\{X\} (= T(\mathrm{span}_{\mathbb{C}} X))$.

② We note $X\mathbb{C}\{X\}$ is a free $\mathbb{C}\{X\}$-module.
Thus, $0 \to X\mathbb{C}\{X\} \to Us \to \mathbb{C} \to 0$ is a free resolution of $\mathbb{C}$.

③. $H_p(s, M)$ = The p-th homology of

$$0 \to X\mathbb{C}\{X\} \otimes_s M \to M \to 0 \quad = \langle 0 \rangle.$$

$H^p(s, M)$ = The p-th cohomology of

$$0 \to M \to \mathrm{Hom}_s(X\mathbb{C}\{X\}, M) \to 0 \quad = \langle 0 \rangle.$$

#

Lemma:

$s \supseteq t$ and s is finitely s.s. over t.

V, W are s-modules.

Then

(1). $\mathrm{Tor}_*^{s,t}(W^t, V) \cong \mathrm{Tor}_*(\mathbb{C}, W \otimes V) \cong \mathrm{Tor}_*^{s,t}(V^t, W$

(2). $\mathrm{Ext}_{s,t}^*(V, W) \cong \mathrm{Ext}_{s,t}^*(\mathbb{C}, \mathrm{Hom}(V, W))$

(3). $\mathrm{Ext}_{s,t}^*(V, W^*) \cong (\mathrm{Tor}_*^{s,t}(V^t, W))^*$

e.g. Exercise 3.1 (1). (A special case of (3)):

$$\begin{aligned} H^p(s,t,V^*) &= H^p(\mathrm{Hom}_t(\Lambda^\cdot(s/t), V^*)) \\ &= H^p(\mathrm{Hom}_t(\Lambda^\cdot(s/t) \otimes V, \mathbb{C})) \\ &= H^p((\Lambda^\cdot(s/t) \otimes V / t \cdot (\Lambda^\cdot(s/t) \otimes V))^*) \\ &= H_p(\Lambda^\cdot(s/t) \otimes V)^* = H_p(s,t,V)^*. \end{aligned}$$

since $(X/tX)^* \cong \mathrm{Hom}_t(X, \mathbb{C})$, $\forall X$: t-module

e.g. $H^p(sl(2), k) = H_p(sl(2), k) = \begin{cases} k, & \text{if } p = 0, 2. \\ 0, & \text{if } p \neq 0, 2. \end{cases}$

Rmk: global dimension of Us-mod $\leq \dim s$

In fact, $\mathrm{gl.dim}(Us\text{-mod}) = \dim s$

since $\mathrm{Ext}_s^{\dim s}(\mathbb{C}, \mathbb{C}) = \mathbb{C} \neq 0$.

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$a \geq s \geq t$, a : fin. s.s. $/t$
N : s-mod, fin. s.s. $/t$ $\Rightarrow$ $\mathrm{Ind}_s^a D_p \otimes N$: fin. s.s. $/t$

- $D_p \otimes N \to N \to 0$ induce $\mathrm{Ind}_s^a (D_p \otimes N) \to \mathrm{Ind}_s^a N \to 0$
  $\cong (\mathrm{Ind}_s^a D_p) \otimes N \to \mathrm{Ind}_s^a N \to 0$ (Ua, Us)-proj. resolution of $\mathrm{Ind}_s^a N$.

$\Rightarrow$ The following is a version of Shapiro's lemma :

Lemma: $a \geq s \geq t$ Lie algebras.
a: fintely s.s. over t.
M: t-module
N: s-module and finitely s.s. over a.
Then

$$\mathrm{Tor}_*^{a,t}(M^t, Ua \otimes_s N) \cong \mathrm{Tor}_*^{s,t}(M^t, N)$$

$$\mathrm{Ext}^*_{a,t}(Ua \otimes_s N, M) \cong \mathrm{Ext}^*_{s,t}(N, M)$$

$P^\cdot \to N \to 0$ proj. resolution

Pf: $F(P^\cdot) \to F(N) \to 0$ proj. resolution
$\mathrm{Hom}(FP^\cdot, M) \cong \mathrm{Hom}(P^\cdot, GM)$

$$M^t \otimes_a \mathrm{Ind}_s^a D_* \otimes N \cong M^t \otimes_s D_* \otimes N$$

$$\Rightarrow \mathrm{Tor}_*^{a,t}(M^t, \mathrm{Ind}_s^a N) \cong \mathrm{Tor}_*^{s,t}(M^t, N)$$