$x \in \mathfrak{h}^*$, $h \in \mathfrak{h}$.

$s_i(x) = x - x(\alpha_i^\vee)\alpha_i \quad \forall i$

$W_Y := \{s_i\}_{i \in Y}$ (Recall $s_i(h) = h - \alpha_i(h)\alpha_i^\vee$, $\forall i$).

<u>Thm:</u> $W'_Y := \{ w \in W \mid \ell(wv) \geq \ell(w),\ \forall v \in W_Y \}$ the set of shortest repn. of right cosets of W_Y

Let L be an integrable highest weight module with highest weight λ

Then, for any $p \geq 0$,

$$H_p(\mathfrak{u}_Y^-, L) \cong \bigoplus_{\substack{w \in W'_Y \\ \ell(w) = p}} L_Y(w^{-1} * \lambda), \text{ as } \mathfrak{g}_Y\text{-modules}.$$

<u>e.g.</u> $Y = \phi \Rightarrow \mathfrak{g}_Y = \mathfrak{h}$

$$\Rightarrow H_p(\mathfrak{n}^-, L) \cong \bigoplus_{\substack{w \in W \\ \ell(w) = p}} L_\phi(w * \lambda), \text{ as } \mathfrak{h}\text{-modules}$$

$$= \bigoplus_{\substack{w \in W \\ \ell(w) = p}} \mathbb{C}_{w*\lambda}$$

<u>e.g.</u> $\mathfrak{g}$: semisimple (f.d.) Thus $\dim L(\lambda) < \infty$, for $\lambda \in D$.

$L(\lambda) \cong L(\lambda)^{**}$. Set $\hat{\lambda} = -w_0\lambda$

$-w_0(w(-w_0\lambda + \rho) - \rho)$

$= w_0 w w_0(\lambda + \rho) - \rho$

Thus,

$$H^p(\mathfrak{n}^-, L(\lambda)) \cong H_p(\mathfrak{n}^-, L(\lambda)^*)^* \cong \bigoplus_{\substack{w \in W \\ \ell(w) = p}} \mathbb{C}^*_{w*\hat{\lambda}} = \bigoplus_{\substack{w \in W \\ \ell(w) = p}} \mathbb{C}_{w_0 w w_0 * \lambda}$$

$$= \bigoplus_{\substack{w \in W \\ \ell(w) = p}} \mathbb{C}_{w*\lambda}.$$

Application 1:

$\mathfrak{g} = \mathfrak{g}(A)$ symmetrizable KM with $A : l \times l$ GCM.

Δ : set of roots of $\mathfrak{g}$

$Y \subseteq \{1, \dots, l\}$

$\Delta_Y = \Delta \cap \bigoplus_{i \in Y} \mathbb{Z}\alpha_i$ and $\Delta_Y^{\pm} := \Delta_Y \cap \Delta^{\pm}$

$$\mathfrak{g}_Y = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_Y} \mathfrak{g}_\alpha$$

$$\mathfrak{u}_Y^- = \bigoplus_{\alpha \in \Delta^- \setminus \Delta_Y^-} \mathfrak{g}_\alpha \quad \text{nil-radical}$$

Then $\mathfrak{g}(A_Y) \overset{i_Y}{\hookrightarrow} \mathfrak{g}_Y \hookrightarrow \mathfrak{g}$ $\quad A_Y = (a_{ij})_{i,j \in Y}$

(note: A_Y is symmetrizable again)

($\mathfrak{h}_Y \subseteq \mathfrak{h}$ smallest s.t.

$\alpha_i^\vee \in \mathfrak{h}_Y$, $\forall i = 1, \dots, l$

$\alpha_i|_{\mathfrak{h}_Y}$: independent $i = 1, \dots, l$.

$\dim \mathfrak{h}_Y = l + \text{corank}\, A_Y$)

$D_Y := \{ \lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}, \ \forall i \in Y \}$

V : $\mathfrak{g}_Y$-module. Then ①. V : integrable $\overset{\text{def}}{\Longleftrightarrow}$ V : integrable as $\mathfrak{g}(A_Y)$-module. ②. V : highest weight module $\overset{\text{def}}{\Longleftrightarrow}$ V : semisimple / $\mathfrak{h}$ and V : highest weight module as $\mathfrak{g}(A_Y)$-module.

$\{$ Irreducible, integrable, highest weight $\mathfrak{g}_Y$-modules $\}$

$= \{ L_Y(\lambda) \mid \lambda \in D_Y \}$.

Application 1: Weyl-Kac character formula.

$\mathfrak{g}$: Symmetrizable KM algebra. (So, $\mathfrak{g} = \bar{\mathfrak{g}}$ & $\bar{\Delta}^+ = \Delta^+$)

Goal: Solve for $a_{\mu\lambda}$ in the equation

$$\operatorname{ch} L(\lambda) = \sum_{\mu \in \mathfrak{h}^*} a_{\mu\lambda} \operatorname{ch} M(\mu).$$

$$\sum_{n\geq 0} (-1)^n \operatorname{ch} \Lambda^n n^- \otimes L(\lambda) = \sum_{n\geq 0} (-1)^n \operatorname{ch} H_n(\Lambda^\bullet n^- \otimes L(\lambda))$$

$$= \sum_{n\geq 0} (-1)^n \operatorname{ch} H_n(n^-; L(\lambda))$$

$$\Rightarrow \operatorname{ch} L(\lambda) = \Big(\sum_{n\geq 0} (-1)^n \operatorname{ch} \Lambda^n n^-\Big)^{-1} \cdot \Big(\sum_{n\geq 0} (-1)^n \operatorname{ch} H_n(n^-, L(\lambda))\Big)$$

$$\Rightarrow \operatorname{ch} L(\lambda) = \prod_{\alpha\in\Delta^+} (1-e^{-\alpha})^{-\operatorname{mult}\alpha} \cdot \sum_{w\in W} (-1)^{\ell(w)} e^{w*\lambda}$$

- Note: $\Big(\sum_{n\geq 0} (-1)^n \operatorname{ch} \Lambda^n n^-\Big)^{-1} = \prod_{\alpha\in\Delta^+} (1-e^{-\alpha})^{-\operatorname{mult}(\alpha)}$

$$= \operatorname{ch} M(0)$$

$$\Rightarrow \operatorname{ch} L(\lambda) = \sum_{\substack{\mu\in\mathfrak{h}^* \\ n\geq 0}} (-1)^n \, [H_n(n^-, L(\lambda)) : \mathbb{C}_\mu] \operatorname{ch} M(\mu)$$

- So, $a_{\mu\lambda} = \sum_{n\geq 0} (-1)^n \, [H_n(n^-, L(\lambda)) : \mathbb{C}_\mu]$

$$= \begin{cases} (-1)^{\ell(w)}, & \text{if } \mu = w*\lambda \\ 0, & \text{otw} \end{cases}$$

Application 2: Blocks of $\mathcal{O}$.

Recall: ① $K^{w.g.} := \mathfrak{h}^* \setminus \bigcup_{\alpha \in \Delta^+_{im}} \{\lambda \in \mathfrak{h}^* \mid \langle \lambda+\rho, \alpha \rangle = \frac{\langle \alpha, \alpha \rangle}{2}\}$

the set of weakly good weights.

② $W(\lambda) := \langle s_\beta \mid \beta \in \Delta^+_{re}, \langle \lambda+\rho, \beta^\vee \rangle \in \mathbb{Z} \rangle$: integral weyl group

(Linkage) ③ Define $\sim^0$ on $K^{w.g.}$ to be the equvi. relation generated by " $[M(\mu) : L(\lambda)] > 0 \Rightarrow \lambda \sim^0 \mu$ ".

Then

$$\mu \sim^0 \lambda \iff \mu \in W(\lambda) * \lambda, \quad \forall \mu, \lambda \in K^{w.g.}$$

Def'n: $\mathcal{O}^{w.g.}$ with obj $\mathcal{O}^{w.g.} = \{M \in \mathcal{O} \mid [M : L(\lambda)] \neq 0 \Rightarrow \lambda \in K^{w.g.}\}$

Thm: $\mathcal{O}^{w.g.} = \bigoplus_{\Lambda^0 \in K^{w.g.}/\sim^0} \mathcal{O}^{w.g.}_{\Lambda^0}$.

Rmk: (Difficulty 1):

: ⊙ To classify blocks of R-fmod for $\dim R < \infty$

⇒ Since modules in R-fmod have fin. length

⇒ To consider $\mathrm{Ext}^1(S, S')$, for simples S, S'

⊙ But modules in $\mathcal{O}^{w.g.}$ may not have composition series.

But $\forall M \in \mathcal{O}$

$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ with M_{i+1}/M_i : quotient of Verma and $\bigcup M_i = M$.

⇒ To consider $\mathrm{Ext}^1(X, Y)$, for highest weight modules X, Y.

Difficulty ②:

$$\mathrm{Ext}^1\left(\bigcup_{i\geq 0}^{n} M_i, \bigcup_{j\geq 0}^{m} M_j'\right)=0 \overset{?}{\Rightarrow} \mathrm{Ext}\left(\bigcup_{i\geq 0}^{\infty} M_i, \bigcup_{j\geq 0}^{\infty} M_j'\right)=0$$

* But Ext doesn't commute with direct limit.

To solve:
$$\left(\begin{array}{l} \omega: \mathfrak{g}\to\mathfrak{g} \text{ Cartan involution} \\ t: \mathfrak{g}\text{-mod} \leftrightarrow \text{mod-}\mathfrak{g} \\ \sigma = -\omega \text{ anti-automorphism} \\ (\cdot)^\vee : \text{restricted duality given by } \sigma \end{array}\right)$$

① $\mathrm{Ext}^n_{\mathfrak{g}}(X, Y) \cong \mathrm{Tor}^{\mathfrak{g}}_n((X^\omega)^t, Y^\sigma)^*$

② $\mathrm{Tor}^{\mathfrak{g}}_1$ commute with direct limit.

③ By ②,

$$\mathrm{Tor}^{\mathfrak{g}}_n\left(\left(\left(\bigcup_{fin.} M_i\right)^\omega\right)^t, \bigcup_{fin.} M_j'^{\sigma}\right)^* = 0$$

$$\Rightarrow \mathrm{Tor}^{\mathfrak{g}}_n\left(\left(\left(\bigcup M_i\right)^\omega\right)^t, \left(\bigcup M_j'\right)^\sigma\right)^* = 0.$$

$$\Rightarrow \mathrm{Ext}^n_{\mathfrak{g}}\left(\bigcup M_i, \bigcup M_j'\right) = 0.$$

④ $\mathrm{Tor}^{\mathfrak{g}}_{n}(((M(\lambda))^{w})^{t}, M(\mu))$

$\cong \mathrm{Tor}^{\mathfrak{b}}_{n}(((M(\lambda)^{w})^{t}, \mathbb{C}_{\mu})$

$\cong \mathrm{Tor}^{\mathfrak{b}}_{n}(\mathbb{C}_{\mu}^{t}, M(\lambda)^{w})$

$\cong \mathrm{Tor}^{\mathfrak{b}}_{n}(\mathbb{C}_{\mu}^{t}, \mathrm{ind}^{\mathfrak{b}}_{\mathfrak{h}} \mathbb{C}_{-\lambda})$

$\cong \mathrm{Tor}^{\mathfrak{h}}_{n}(\mathbb{C}_{\mu}^{t}, \mathbb{C}_{-\lambda}) \cong H_{n}(\mathfrak{h}, \mathbb{C}_{\mu-\lambda})$

$H_{n}(\mathfrak{h}, \mathbb{C}_{\mu-\lambda})$ is a trivial $\mathfrak{h}$-module.

But $\Lambda\mathfrak{h} \otimes \mathbb{C}_{\mu-\lambda}$ has no trivial weights.

$\Rightarrow \mathrm{Tor}^{\mathfrak{g}}_{n}((M(\lambda)^{w})^{t}, M(\mu)) = 0$

⑤

fix j: $\cdots \to \mathrm{Tor}^{\mathfrak{g}}_{n}(M_i, M_j') \to \mathrm{Tor}^{\mathfrak{g}}_{n}(M_{i+1}, M_j') \to \mathrm{Tor}^{\mathfrak{g}}_{n}(M_{i+1}/M_i, M_j') \to \cdot$

fix i: $\cdots \to \mathrm{Tor}^{\mathfrak{g}}_{n}(M_i, M_j') \to \mathrm{Tor}^{\mathfrak{g}}_{n}(M_i, M_{j+1}') \to \mathrm{Tor}^{\mathfrak{g}}_{n}(M_i, M_{j+1}'/M_j') \to$

$\Rightarrow \mathrm{Ext}^{n}_{\mathfrak{g}}(X, Y) = 0$, for $X \in \mathcal{O}^{w.g}_{\Lambda_1}$, $Y \in \mathcal{O}^{w.g.}_{\Lambda_2}$

with $\Lambda_1 \neq \Lambda_2 \in K^{w.g.}/\sim^{\circ}$.

Similarly,

$$\mathcal{O} = \bigoplus_{\Lambda \in \mathfrak{h}^*/\sim} \mathcal{O}_\Lambda$$

$\sim$ on $\mathfrak{h}^*$ generated by "$[M(\mu) : L(\lambda)] \neq 0 \Rightarrow \mu \sim \lambda$".

Description of $\sim$: see [Kac-Kazhdan, 1978]. Thm 2. of