Idea

Apr 16, 2021

Ind-group

$$G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow$$

$$G := \operatorname{colim} G_n$$

For simplicity, assume affine and $k = \mathbb{C}$

Pro-group

$$\twoheadrightarrow G_2 \twoheadrightarrow G_1 \twoheadrightarrow G_0$$

$$G := \lim G_n$$

I Ind-variety

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots$$

$X := \operatorname{colim} X_n$ in cat Top.

Ex. (1) $X = \operatorname{colim} X$

(2) $\mathbb{A}^\infty = \operatorname{colim} \mathbb{A}^n$

	variety	ind-variety
regular functions	$k[X]$ discrete topology	$\lim k[X_n]$ topological k-algebra
Zariski top.		U open if $U \cap X_n$ open; Z closed if $Z \cap Z_n$ closed
Structure sheaf	$\mathcal{O}_X : U \mapsto k[U]$	—
Morphism	$f : X \to Y$	$f : X_n \to Y_m$
Comorphism	$f^* : k[Y] \to k[X]$	—
closed embedding		$f : X_n \to Y_m$ closed embedding; $f : X \to Y$ closed; $f : X \to f(X)$ homeo
open subvariety	$U \to X$	colim $U \cap X_n$
closed subvariety	$Z \to X$	colim $Z \cap X_n$
subvariety	$Z \cap U \to U \to X$	colim $Z \cap U \cap X_n$
product	$X \cap Y$	colim $X_n \times Y_n$

local ring	$\mathcal{O}_{X,x} \supset m_x$	$\lim \mathcal{O}_{X_n,x} \supset \lim m_x(n)$
tangent sp.	$T_x(X) := \mathrm{Hom}(m_x/m_x^2, k)$	$\mathrm{colim}\, T_x(X_n)$
derivative	$f: X \to Y$	
	$(df)_x : T_x(X) \to T_{f(x)}(Y)$	—
smooth	φ^p is iso	$\hat{\varphi}^p$ is iso
	$\varphi^p : S^p(m_x/m_x^2) \to m_x^p/m_x^{p+1}$	
vector bundle	$E \to X$	$E_n \to X_n$

I Ind-group

$m : G \times G \to G$ $\qquad$ G : affine ind-variety

$inv : G \to G$

$e : * \to G$

Lie algebra $\quad \mathrm{Lie}\, G := T_e(G)$

$\alpha : G \to H$

induces $(d\alpha)_e : \mathrm{Lie}\, G \to \mathrm{Lie}\, H$

Rep. of G $\quad \rho : G \times V \to V$ $\qquad$ V : ind-v.s.

Rep of $\mathrm{Lie}\, G$ $\quad d\rho : \mathrm{Lie}\, G \times V \to V$

<u>Thm</u> If $\mathrm{ch}\, k = 0$, then G is smooth.

Pro-group

$\varphi_n : G \to G_n$

Def $G : \lim G_n$

G_n : affine alg. gp.

lim : co filtered limit in cat Gp.

- $G_\ell \rightrightarrows \begin{matrix} G_n \\ G_m \end{matrix}$
- $G_\ell \to G_n \rightrightarrows G_m$

Ex. (1) $G = \lim G/N_n$

(2) $\twoheadrightarrow G_2 \twoheadrightarrow G_1 \twoheadrightarrow G_0$

$G = \lim G_n$

(3) $G \times H = \lim G_n \times H_n$

alg. gp

pro-gp

G_n is unipotent

unipotent

$G_n \to H_m$

mor $G \to H$

lim top.

Zariski

top

subgp closed under top.

normal subgp. $H \triangleleft G$

$\varphi_n(H) \triangleleft G_n$

$\lim G_n$

G/H

$\varphi_n(H) = 1$

quotient gp

image $f : G \to G'$

$\mathrm{Im} f = G/\ker f$

/ Pro-Lie alg.

<u>Def</u> $\mathfrak{g} : \lim \mathfrak{g}_n$ $\varphi_n : \mathfrak{g} \to \mathfrak{g}_n$

G_n : affine alg. gp.

$\lim$: cofiltered limit in cat Lie Alg

<u>Ex.</u> $\mathfrak{g} = \lim \mathfrak{g}/\mathfrak{h}$

	Lie alg	pro Lie alg
nilpotent		$\mathfrak{g}_n$ is nilpotent
mor	$\mathfrak{g} \to \mathfrak{h}$	$\mathfrak{g}_n \to \mathfrak{h}_m$
top	discrete	lim top.
ideal	closed under top.	—
sub alg.	$\mathfrak{h} \subset \mathfrak{g}$	$\mathfrak{h} = \lim \varphi_n(\mathfrak{h})$
quotient subalg	$\mathfrak{g}/\mathfrak{h}$	$\lim \mathfrak{g}_n$ $\varphi_n(\mathfrak{h}) = 0$
image	$f : \mathfrak{g} \to \mathfrak{g}'$ $\operatorname{Im} f = \mathfrak{g}/\ker f$	—

Pro Lie alg of Pro gp.

$G = \lim G_n$

$\mathrm{Lie}\, G := \lim \mathrm{Lie}\, G_n$

$G \to H$ induces $\mathrm{Lie}\, G \to \mathrm{Lie}\, H$

$\exp : \mathrm{Lie}\, G_n \to G_n$

induces $\exp : \mathrm{Lie}\, G \to G$

Thm $\left\{ \begin{array}{c} \text{unipotent} \\ \text{gp} \end{array} \right\} \xrightarrow[\sim]{\mathrm{Lie}} \left\{ \begin{array}{c} \text{nilpotent} \\ \text{Lie} \end{array} \right\}$

Thm G : unipotent

$\{ \text{rep of } G \} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{locally nilpotent} \\ \text{rep of } \mathrm{Lie}\, G \end{array} \right\}$